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"Nec aranearum sane textus ideo melior quia ex se fila gignunt, nec noster vilior quia ex alienis libamus ut apes." Just. Lips. *Polit.* lib. i. cap. 1. Not.
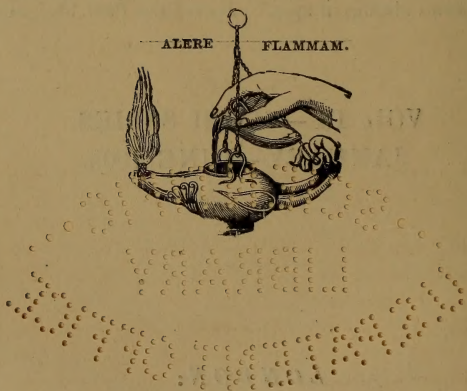

"Meditationis est perscrutari occulta; contemplationis est admirari perspicua .... Admiratio generat quæstionem, quæstio investigationem, investigatio inventionem."—*Hugo de S. Victore.*

———"Cur spirent venti, cur terra dehiscat,
Cur mare turgescat, pelago cur tantus amaror,
Cur caput obscura Phœbus ferrugine condat,
Quid toties diros cogat flagrare cometas,
Quid pariat nubes, veniant cur fulmina cœlo,
Quo micet igne Iris, superos quis conciat orbes
Tam vario motu."

J. B. Pinelli ad Mazonium.

ALERE FLAMMAM.

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PLATES.

ERRATA.

Page 512, lines 24 & 26, *for* 137° C. *read* 139° C.

  „ 579, formula (9), *for* $\dfrac{\partial^2 V}{\partial t^2}$ *read* $\dfrac{d^2 V}{dt^2}$

  „ 582, lines 19 & 20, *for* let Reversal of the light take place, the effect *read* let Reversal of all velocities in the medium take place, one effect

  „ 590, line 5, *for* diagram *read* diatom

I. *On the Boiling-Points of Homologous Compounds.*
By SYDNEY YOUNG, *D.Sc., F.R.S., Trinity College, Dublin*[*].

IT was in 1842 that Kopp first made the statement that a constant difference in chemical composition is accompanied by a constant difference in boiling-point. He at first adopted the value 18° as the rise due to the replacement of the ethyl by the methyl group in organic compounds, but two years later, after making a larger number of comparisons, he concluded that 19° was the true value, and he suggested that this is the constant difference in boiling-point for an addition of CH_2 in any homologous series.

As a result of his own determinations and those of other experimenters, carried out during the following twenty-three years, Kopp was finally induced to give up some of his earlier conclusions. He admitted that isomeric compounds may have different boiling-points, and that the rise due to an addition of CH_2 is not invariably 19°, but he still maintained that in any given homologous series the rise is constant: thus, for homologues of toluene he adopted the value 20°·5, for those of xylene 18°·5, and for those of trimethyl benzene 16°·5.

As a pioneer, Kopp had very great difficulties to contend with when he began his researches; data were scanty and far from accurate, and the substances which could be most easily obtained and, it was thought, readily purified were, unfortunately, those which were the least likely to lead to normal generalizations. Water, the alcohols, and the organic acids all contain a hydroxyl group; and we now know that the physical properties of these substances are abnormal in nearly all respects, owing, probably, to the fact that their

[*] Communicated by the Author, being a development of the first part of the Presidential Address to Section B of the British Association.

molecules tend to associate together ; moreover, the esters, which are formed by the interaction of alcohols and acids, do not behave quite normally, and there is probably molecular association, though to a much smaller extent than with the hydroxyl compounds. There can be little doubt that if Kopp had been able, in the first place, to obtain a considerable number of pure substances of normal behaviour, such as the paraffins or their halogen derivatives, he would not have been led to the erroneous conclusions which he defended with such vigour for so many years.

Taking the normal paraffins as the simplest class of organic compounds, it is found that, instead of the boiling-points rising by equal intervals as the series is ascended, the rise, which is very large for the lowest numbers, becomes smaller and smaller as the molecular weight increases. This fact is, of course, now well known, and various formulæ have been proposed to reproduce these boiling-points.

Thus, Walker has proposed the formula $T = aM^b$, where T is the boiling-point on the absolute scale of temperature, M is the molecular weight, and a and b are constants. Ramage [*] last year suggested that this formula applies only to the CH_2 chain linkage, and that the influence of the terminal hydrogen atoms is considerable in the case of the lowest members, but diminishes as the chain lengthens, and becomes eventually either constant or negligible. In other words, the lower members of the series cannot be regarded as truly homologous, and that is a point which is, I think, important to bear in mind. Ramage suggests a new formula, $T = a[M(1 - 2^{-n})]^{\frac{1}{2}}$, where a is Walker's constant $37 \cdot 3775$, and n is the number of carbon atoms in the molecule. He assumes, however, a constant difference for CH_2 in the case of the alcohols, the aldehydes, and the ketones, but it is doubtful whether the boiling-points of the last two classes of compounds are yet sufficiently well established to allow of any certain conclusions being drawn from them.

It may be useful to regard the rise of B.P. for an increment of CH_2 as being mainly a function of the absolute temperature, and the formula

$$\Delta = \frac{144 \cdot 86}{T^{0 \cdot 0148 \sqrt{T}}}$$

(where Δ is the difference between the boiling-point, T, of any paraffin and that of its next higher homologue) may be provisionally adopted. Taking the boiling-point of methane as $106° \cdot 75$ abs., the values for the higher members agree better with the observed temperatures than those given by Ramage's formula, as will be seen by the following table.

* Cambridge Phil. Soc. Proc. xii. p. 445 (1904).

Table I.

Paraffin.	Boiling-point (abs. temp.).				
	Observed.	Calculated. Ramage.	Difference.	Calculated. Young.	Difference.
CH_4	108·3	105·7	−2·6	106·75	−1·55
C_2H_6	180·0	177·3	−2·7	177·7	−2·3
C_3H_8	228·0	231·9	+3·9	229·85	+1·85
C_4H_{10}......	274·0	275·6	+1·6	272·6	−1·4
C_5H_{12}......	309·3	312·2	+2·9	309·4	+0·1
C_6H_{14}......	341·95	343·9	+1·95	341·95	0
C_7H_{16}......	371·4	372·3	+0·9	371·3	−0·1
C_8H_{18}......	398·6	398·3	−0·3	398·1	−0·5
C_9H_{20} ...	422·5	422·5	0	422·85	+0·35
$C_{10}H_{22}$ ...	446·0	445·2	−0·8	445·85	−0·15
$C_{11}H_{24}$ ...	467·0	466·8	−0·2	467·35	+0·35
$C_{12}H_{26}$ ...	487·5	487·3	−0·2	487·65	+0·15
$C_{13}H_{28}$ ...	507·0	507·0	0	506·8	−0·2
$C_{14}H_{30}$ ...	525·5	526·0	+0·5	525·0	−0·5
$C_{15}H_{32}$ ...	543·5	544·2	+0·7	542·3	−1·2
$C_{16}H_{34}$ ...	560·5	561·9	+1·4	558·85	−1·65
$C_{17}H_{36}$ ...	576·0	579·0	+3·0	574·7	−1·3
$C_{18}H_{38}$ ...	590·0	595·7	+5·7	589·9	−0·1
$C_{19}H_{40}$ ...	603·0	611·9	+8·9	604·5	+1·5

It may be noticed especially that the agreement is much better in the case of those boiling-points which are likely to have been most accurately determined, notably pentane, hexane, and heptane, and that the differences between the calculated and observed boiling-points above 570° abs. are not nearly so marked.

Consider now the replacement of a terminal atom of hydrogen in each normal paraffin by an atom of chlorine, so as to form the homologous series of primary alkyl chlorides. The boiling-points of these chlorides are much higher, and the differences, Δ, are much lower than for the corresponding paraffins, but the gradual fall in the values of Δ as the series is ascended is unmistakable. The same remarks apply to the bromides and iodides, the boiling-points being still higher and the values of Δ lower.

But the point of chief interest is that most of the values of Δ for the halogen derivatives may be calculated without great error from the formula

$$\Delta = \frac{144 \cdot 86}{T^{0 \cdot 0148} \sqrt{T}}$$

which was derived from the boiling-points of the paraffins themselves. The first calculated value of Δ, as will be seen from Table II., is decidedly high in each series (average

difference between calculated and observed values $+2°\cdot 7$), the later ones are rather low in nearly every case.

TABLE II.

R.	T° abs.	Δ.			T° abs.	Δ.		
		Obs.	Calc.	Diff.		Obs.	Calc.	Diff.
	RCl.				R Br.			
CH_3 ...	249·3	36·2	39·85	+3·65	277·5	33·9	36·2	+2·3
C_2H_5 ...	285·5	33·5	35·25	+1·75	311·4	32·4	32·35	−0·05
C_3H_7 ...	319·0	32·0	31·55	−0·45	343·8	30·2	29·15	−1·05
C_4H_9 ...	351·0	28·6	28·55	−0·05	374·0	28·5	26·6	−1·9
C_5H_{11} ...	379·6	26·4	26·15	−0·25	402·5	26·5	24·45	−2·05
C_6H_{13} ...	406·0	27·0	24·15	−2·85	429·0	23·0	22·6	−0·4
C_7H_{15} ...	433·0	24·0	22·4	−1·6	452·0	22·0	21·15	−0·85
C_8H_{17} ...	457·0				474·0			
	RI.							
CH_3 ...	315·8	29·7	31·85	+2·15				
C_2H_5 ...	345·5	30·0	29·0	−1·0				
C_3H_7 ...	375·5	27·5	26·5	−1·0				
C_4H_9 ...	403·0	26·0	24·4	−1·6				
C_5H_{11} ...	429·0	23·0	22·6	−0·4				
C_6H_{13} ...	452·0							

Somewhat similar results are in general obtained with other homologous series of compounds in which molecular association is not believed to occur; for example, with other hydrocarbons, with the ethers, aldehydes, amines, and hydrosulphides. In the great majority of cases the deviations are greatest for the lowest members of a series, and this may perhaps be explained in the manner suggested by Ramage. Each series, therefore, is divided into two groups; the first ending and the second beginning with the lowest member of the series that contains a CH_2 group linked to two carbon atoms. Thus, of the alkyl chlorides, the first group contains CH_3Cl, $CH_3— CH_2Cl$, and $CH_3—CH_2—CH_2Cl$, and the second group begins with propyl chloride, so that all its members contain one or more $C—CH_2—C$ groups.

In the case of the ethers, esters, and other compounds which contain two alkyl groups, a series is regarded as homologous when one radical remains unaltered and the other increases by stages of CH_2. The variable radical only is considered in dividing the series into the two groups; thus, although propionic acid contains a $C—CH_2—C$ group, it remains unchanged in the propionic esters, the first group

of which consists of methyl, ethyl, and propyl propionate, the second beginning with the last-named ester.

TABLE III.

Left half

Formula.	T° abs.	Δ. Obs.	Δ. Calc.	Δ. Diff.
Isoparaffins.				
C_5H_{12}...	300·95	34·05	33·5	−0·55
C_6H_{14}...	335·0	28·3	30·0	+1·7
C_7H_{16}...	363·3			
Olefines. $H_2C=CHR$.				
C_4H_8 ...	170·5	52·3	53·6	+1·3
C_5H_{10}...	222·8	45·2	43·95	−1·25
C_6H_{12}...	268·0	44·5	37·45	−7·05
C_7H_{14}...	312·5			
Olefines. $HRC=CHR'$.				
C_4H_8 ...	274·0	35·5	36·6	+1·1
C_5H_{10}...	309·5	31·5	32·5	+1·0
C_6H_{12}...	341·0	30·0	29·45	−0·55
C_7H_{14}...	371·0			
Polymethylenes.				
C_5H_{10}...	323·5	30·3	31·05	+0·75
C_6H_{12}...	353·8	36·7	28·25	−8·45
C_7H_{14}...	390·5			

Right half

R.	T° abs.	Δ. Obs.	Δ. Calc.	Δ. Diff.
C_6H_5—R.				
CH_3 ...	383·6	25·4	25·85	+0·45
C_2H_5 ...	409·0	22·5	23·95	+1·45
C_3H_7 ...	431·5	21·5	22·5	+1·0
C_4H_9 ...	453·0	21·5	21·1	−0·4
C_5H_{11}...	474·5			
$C_6H_4 \left\{ \begin{array}{c} CH_3 \\ R \end{array} \right.$ (ortho).				
CH_3 ...	414·0	17·5	23·6	+6·1
C_2H_5 ...	431·5	23·0	22·5	−0·5
C_3H_7 ...	454·5			
Ditto (meta).				
CH_3 ...	412·0	19·5	23·75	+4·25
C_2H_5 ...	431·5	18·5	22·5	+4·0
C_3H_7 ...	450·0			
Ditto (para).				
CH_3 ...	411·0	24·0	23·85	−0·15
C_2H_5 ...	435·0	21·5	22·15	+0·65
C_3H_7 ...	456·5			
$C_6H_4 \left\{ \begin{array}{c} C_2H_5 \\ R \end{array} \right.$ (ortho).				
CH_3 ...	431·5	25·5	22·5	−3·0
C_2H_5 ...	457·0			
Ditto (meta).				
CH_3 ...	431·5	23·0	22·5	−0·5
C_2H_5 ...	454·5			
Ditto (para).				
CH_3 ...	435·0	20·5	22·15	+1·65
C_2H_5 ...	455·5	21·0	20·95	−0·05
C_3H_7 ...	476·5			

TABLE III. (continued).

R.	T° abs.	Δ Obs.	Δ Calc.	Δ Diff.
Ethers.	**CH₃—O—R.**			
CH_3 ...	249·35	34·45	39·85	+5·4
C_2H_5 ...	283·8	28·1	35·45	+7·35
C_3H_7 ...	311·9	31·4	32·25	+0·85
C_4H_9 ...	343·3			
C_7H_{15}...	422·8	23·2	23·05	−0·15
C_8H_{17}...	446·0			
Ditto.	**C₂H₅—O—R.**			
CH_3 ...	283·8	23·8	35·45	+11·6
C_2H_5 ...	307·6	29·0	32·75	+3·75
C_3H_7 ...	336·6	27·8	29·85	+2·05
C_4H_9 ...	364·4			
C_7H_{15}...	439·6	22·6	21·9	−0·7
C_8H_{17}...	462·2			
Ditto.	**C₃H₇—O—R.**			
CH_3 ...	311·9	24·7	32·25	+7·55
C_2H_5 ...	336·6	27·1	29·85	+2·75
C_3H_7 ...	363·7	26·4	27·45	+1·05
C_4H_9 ...	390·1			
C_7H_{15}...	460·6	19·4	20·65	+1·25
C_8H_{17}...	480·0			
Ditto.	**C₄H₉—O—R.**			
CH_3 ...	343·3	21·1	29·2	+8·1
C_2H_5 ...	364·4	25·7	27·4	+1·7
C_3H_7 ...	390·1	23·8	25·35	+1·55
C_4H_9 ...	413·9			
C_7H_{15}...	478·7	20·0	19·65	−0·35
C_8H_{17}...	498·7			
Ditto.	**C₇H₁₅—O—R.**			
CH_3 ...	422·8	16·8	23·05	+6·25
C_2H_5 ...	439·6	21·0	21·9	+0·9
C_3H_7 ...	460·6			
C_7H_{15}...	534·9	16·9	16·85	−0·05
C_8H_{17}...	551·8			

R.	T° abs.	Δ Obs.	Δ Calc.	Δ Diff.
Ethers.	**C₈H₁₇—O—R.**			
CH_3 ...	446·0	16·2	21·5	+5·3
C_2H_5 ...	462·2	17·8	20·55	+2·75
C_3H_7 ...	480·0			
C_7H_{15}...	551·8	12·9	16·15	+3·25
C_8H_{17}...	564·7			
Aldehydes.	**R—CHO.**			
H	252·0	41·8	39·6	−2·2
CH_3 ...	293·8	28·2	34·35	+6·15
C_2H_5 ...	322·0	25·0	31·3	+6·3
C_3H_7 ...	347·0	29·0	28·9	−0·1
C_4H_9 ...	376·0	24·9	26·45	+1·55
C_5H_{11}...	400·9	27·1	24·55	−2·55
C_6H_{13}...	428·0			
Amines.	**R—NH₂.**			
CH_3 ...	267·0	24·7	37·55	+12·85
C_2H_5 ...	291·7	31·0	34·6	+3·6
C_3H_7 ...	322·7	25·8	31·25	+5·45
C_4H_9 ...	348·5	27·5	28·8	+1·3
C_5H_{11}...	376·0	26·0	26·45	+0·45
C_6H_{13}...	402·0	25·0	24·5	−0·5
C_7H_{15}...	427·0			
Hydrosulphides.	**R—SH.**			
CH_3 ...	279·0	30·0	36·05	+6·05
C_2H_5 ...	309·0	31·5	32·55	+1·05
C_3H_7 ...	340·5	30·0	29·5	−0·5
C_4H_9 ...	370·5			

In the tables, the compounds which do not contain a C—CH_2—C group, their boiling-points, and the corresponding values of Δ are printed in italics.

In many cases determinations by different observers are by no means concordant, and some of the boiling-points quoted are doubtless inaccurate. The selection of material in such an investigation as this is, indeed, a difficult matter, for if all observed boiling-points, without exception, were included, the errors would in some cases be so large as to obliterate all signs of regularity ; while, on the other hand, the exclusion of particular results is a dangerous and somewhat invidious process. The general plan which has been adopted is as follows :—

(1) All boiling-points which I have myself observed are included.

(2) When an experimenter has made a series of determinations, most of which are undoubtedly accurate, the whole series has been included ; thus, all the observations of Schumann and of Gartenmeister on the esters and those of Dobriner on the ethers have been accepted, while the boiling-points of the higher paraffins, as determined by Krafft, have all been employed in calculating the constants for the general formula.

(3) When there are two or more fairly concordant observations of the boiling-point of a given liquid, and there is no apparent reason for preferring any one of them, the mean has been taken ; but a determination by a worker of known accuracy has always been adopted in preference to others.

(4) When there are two or more discordant observations they have all been rejected unless there is reason to consider any one of them to be accurate.

(5) If in any series the number of trustworthy observations is sufficient to leave no doubt as to the approximate boiling-point of a given member, and the calculated value differs widely from this, it has been rejected. Thus the differences between the calculated and observed values of Δ for the higher ethers are in most cases small, the mean of those given in the tables being $+0°\cdot87$. But the observed boiling-point of ethyl hexyl ether ($134°$ to $137° = 135°\cdot5$) is $31°\cdot1$ lower than that of ethyl heptyl ether as determined by Dobriner, while the calculated value of Δ for the temperature $135°\cdot5$ is only $24°\cdot0$. The rise of $3°$ during the distillation of ethyl hexyl oxide also shows that that ether was imperfectly purified, and it therefore seemed justifiable to omit it from the table.

On the other hand, the boiling-point of di-octyl ether, having been observed by Dobriner, has been included, although it appears to be $2°$ or possibly $3°$ too low.

(6) When the boiling-point is given in such a form as 160° to 161° at 750 mm., the middle temperature, roughly corrected to 760 mm., has been taken.

A few of the data included in the tables are doubtful ; it seems probable, for example, that the observed boiling-points of *n*-propyl ethylene, propylamine, heptamethylene, and methyl ethyl ether are considerably too high, and that those of dioctyl ether, *o*-methyl ethyl benzene, and probably *p*-xylene are too low, while the boiling-points of some of the aldehydes seem doubtful.

In my presidential address to the Chemical Section of the British Association at Cambridge the only data given were the mean differences between the calculated and observed values of Δ, and the number of values from which the means were calculated, each homologous series being, however, divided when necessary into two groups. Since the address was printed a few more boiling-points have been found, and a mistake has been discovered in the data for the ethers. The amended data are given in Table IV.

TABLE IV.

Homologous Series.	Lower Members.		Higher Members.	
	No. of values of Δ.	Mean diff. between calc. & obs. values.	No. of values of Δ.	Mean diff. between calc. & obs. values.
Alkyl chlorides....................	2	+2·75	5	−1·04
„ bromides	2	+1·12	5	−1·25
„ iodides	2	+0·52	3	−1·00
Isoparaffins	...	......	2	+0·57
Olefines $H_2C = CHR$	...	......	3	−2·35
„ $HRC = CHR'$	...	......	3	+0·52
Polymethylenes	...	......	2	−3·85
Toluene, &c., C_6H_5R	1	+0·45	3	+0·68
Xylene, &c., $C_6H_4 \left\{ \begin{matrix} CH_3 \\ R \end{matrix} \right.$	3	+3·40	3	+1·38
Ethyl toluene, &c., $C_6H_4 \left\{ \begin{matrix} C_2H_5 \\ R \end{matrix} \right.$	3	−0·62	1	−0·05
Ethers	12	+5·29	10	+0·87
Aldehydes...........................	2	+1·98	4	+1·30
Amines	2	+8·22	4	+1·68
Hydrosulphides	2	+3·55	1	−0·50
Total	31	+3·50	49	−0·05

It will be seen that, while for the lower members the mean of all the differences between the calculated and observed values of Δ is $+3°\cdot50$, that for the higher members, $-0°\cdot05$, is very small indeed.

It is, however, improbable that all the individual differences in the case of the higher members of each series can be due to experimental error ; the composition and constitution of the compounds must, no doubt, have some influence : thus, the differences for the alkyl halides are uniformly negative. Taking the results as a whole, however, it would seem that the values of Δ for the higher members may be calculated with a fair degree of accuracy from the formula derived from the boiling-points of the normal paraffins.

The Esters.

According to Ramsay and Shields[*] the esters should be classed among the non-associating compounds, but the behaviour of the lower members of the series is slightly abnormal in many respects ; thus the ratios of the actual to the theoretical density at the critical point are high[†], but do not show the usual slight rise with increase of molecular weight, and this would seem to indicate that the lower members are associated to some extent, but that the association diminishes, as in other cases, as the series is ascended. Again, the values

of $\dfrac{dt}{dp} \cdot \dfrac{1}{T}$ at the boiling-points under normal pressure[‡] are

somewhat small for the lower members, and do not show the usual diminution as the series is ascended[§] ; and these facts point to the same conclusion.

As the question of molecular association may be regarded as an open one, and the number of esters examined is very large, I have taken them separately in Table V.

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[*] *Zeit. physik. Chem.* xii. p. 433 (1893).
[†] Phil. Mag. l. p. 303 (1900).
[‡] Trans. Chem. Soc. lxxxi. p. 780 (1902).
[§] It may be noticed that the abnormality in physical properties is most marked in the case of those esters in which both alkyl groups are of normal structure ; the physical properties appear to be less abnormal when one or both alkyl radicals contain an iso-group, thus the values of $\dfrac{dt}{dp} \cdot \dfrac{1}{T}$ are higher and the fall with increase of molecular weight is quite marked.

TABLE V.

Normal Esters.

H—COOR.

R.	$T°$abs.	Δ Obs.	Δ Calc.	Δ Diff.
CH_3 ...	304.9	22.4	33.0	+10.6
C_2H_5 ...	327.3	26.6	30.7	+ 4.1
C_3H_7 ...	353.9	26.0	28.3	+ 2.3
C_4H_9 ...	379.9	23.5	26.1	+ 2.6
C_5H_{11} ...	403.4	23.2	24.4	+ 1.2
C_6H_{13} ...	426.6	23.1	22.8	− 0.3
C_7H_{15} ...	449.7	21.4	21.3	− 0.1
C_8H_{17} ...	471.1			

CH₃—COOR.

R.	$T°$abs.	Δ Obs.	Δ Calc.	Δ Diff.
CH_3 ...	330.2	19.95	30.4	+10.45
C_2H_5 ...	350.15	24.4	28.6	+ 4.2
C_3H_7 ...	374.55	22.95	26.5	+ 3.55
C_4H_9 ...	397.5	23.1	24.9	+ 1.8
C_5H_{11} ...	420.6	21.6	23.2	+ 1.6
C_6H_{13} ...	442.2	22.1	21.8	− 0.3
C_7H_{15} ...	464.3	18.7	20.4	+ 1.7
C_8H_{17} ...	483.0			

C₂H₅—COOR.

R.	$T°$abs.	Δ Obs.	Δ Calc.	Δ Diff.
CH_3 ...	352.7	19.1	28.4	+ 9.3
C_2H_5 ...	371.8	23.5	26.8	+ 3.3
C_3H_7 ...	395.3	23.1	25.0	+ 1.9
C_4H_9 ...	418.4			
C_7H_{15} ...	481.0	18.4	19.5	+ 1.1
C_8H_{17} ...	499.4			

C₃H₇—COOR.

R.	$T°$abs.	Δ Obs.	Δ Calc.	Δ Diff.
CH_3 ...	375.75	17.15	26.5	+ 9.35
C_2H_5 ...	392.9	22.8	25.1	+ 2.3
C_3H_7 ...	415.7	23.0	23.5	+ 0.5
C_4H_9 ...	438.7	19.1	22.0	+ 2.9
C_5H_{11} ...	457.8	20.3	20.85	+ 0.55
C_6H_{13} ...	478.1	20.1	19.7	− 0.4
C_7H_{15} ...	498.2	17.0	18.6	+ 1.6
C_8H_{17} ...	515.2			

C₄H₉—COOR.

R.	$T°$abs.	Δ Obs.	Δ Calc.	Δ Diff.
CH_3 ...	400.3	17.4	24.6	+ 7.2
C_2H_5 ...	417.7	22.8	23.4	+ 0.6
C_3H_7 ...	440.5	18.3	21.9	+ 3.6
C_4H_9 ...	458.8	17.9	20.7	+ 2.8
C_5H_{11} ...	476.7	20.1	19.8	− 0.3
C_6H_{13} ...	496.8	19.8	18.7	− 1.1
C_7H_{15} ...	516.6	16.6	17.7	+ 1.1
C_8H_{17} ...	533.2			

C₅H₁₁COOR.

R.	$T°$abs.	Δ Obs.	Δ Calc.	Δ Diff.
CH_3 ...	422.6	17.0	23.0	+ 6.0
C_2H_5 ...	439.6	18.9	21.9	+ 3.0
C_3H_7 ...	458.5	18.8	20.75	+ 1.95
C_4H_9 ...	477.3			
C_7H_{15} ...	532.4	15.8	17.0	+ 1.2
C_8H_{17} ...	548.2			

C₆H₁₃—COOR.

R.	$T°$abs.	Δ Obs.	Δ Calc.	Δ Diff.
CH_3 ...	445.1	15.0	21.6	+ 6.6
C_2H_5 ...	460.1	19.3	20.65	+ 1.35
C_3H_7 ...	479.4	18.7	19.6	+ 0.9
C_4H_9 ...	498.1			
C_7H_{15} ...	547.6	15.8	16.3	+ 0.5
C_8H_{17} ...	563.4			

C₇H₁₅—COOR.

R.	$T°$abs.	Δ Obs.	Δ Calc.	Δ Diff.
CH_3 ...	465.9	12.9	20.35	+ 7.45
C_2H_5 ...	478.8	18.9	19.6	+ 0.7
C_3H_7 ...	497.7	15.8	18.7	+ 2.9
C_4H_9 ...	513.5			
C_7H_{15} ...	562.8	16.1	15.7	− 0.4
C_8H_{17} ...	578.9			

C₈H₁₇—COOR.

R.	$T°$abs.	Δ Obs.	Δ Calc.	Δ Diff.
CH_3 ...	486.5	14.0	19.3	+ 5.3
C_2H_5 ...	500.5			

TABLE V. (*continued*)

Normal Esters (*continued*).

R—COOCH₃.

R.	T°abs.	Obs.	Calc.	Diff.
H	304.9	25.3	33.0	+7.7
CH₃ ...	330.2	22.5	30.4	+7.9
C₂H₅ ...	352.7	23.05	28.4	+5.35
C₃H₇ ...	375.75	24.55	26.5	+1.95
C₄H₉ ...	400.3	22.3	24.6	+2.3
C₅H₁₁...	422.6	22.5	23.0	+0.5
C₆H₁₃...	445.1	20.8	21.6	+0.8
C₇H₁₅...	465.9	20.6	20.35	−0.25
C₈H₁₇...	486.5			

R—COOC₂H₅.

R.	T°abs.	Obs.	Calc.	Diff.
H	327.3	22.85	30.7	+7.85
CH₃ ...	350.15	21.85	28.6	+6.75
C₂H₅ ...	372.0	20.9	26.8	+5.9
C₃H₇ ...	392.9	24.8	25.1	+0.3
C₄H₉ ...	417.7	21.9	23.4	+1.5
C₅H₁₁...	439.6	20.5	21.9	+1.4
C₆H₁₃...	460.1	18.7	20.65	+1.95
C₇H₁₅...	478.8			

R—COOC₃H₇.

R.	T°abs.	Obs.	Calc.	Diff.
H	353.9	20.65	28.3	+7.65
CH₃ ...	374.55	20.75	26.5	+5.75
C₂H₅ ...	395.3	20.4	25.0	+4.6
C₃H₇ ...	415.7	24.8	23.7	−1.1
C₄H₉ ...	440.5	18.0	21.9	+3.9
C₅H₁₁...	458.5	20.9	20.75	−0.15
C₆H₁₃...	479.4	18.3	19.6	+1.3
C₇H₁₅...	497.7			

R—COOC₄H₉.

R.	T°abs.	Obs.	Calc.	Diff.
H	379.9	17.6	26.1	+8.5
CH₃ ...	397.5	20.9	24.9	+4.0
C₂H₅ ...	418.4	20.3	23.3	+3.0
C₃H₇ ...	438.7	20.1	22.0	+1.9
C₄H₉ ...	458.8	18.5	20.7	+2.2
C₅H₁₁...	477.3	20.8	19.75	−1.05
C₆H₁₃...	498.1	15.4	18.65	+3.25
C₇H₁₅...	513.5			

R—COOC₅H₁₁.

R.	T°abs.	Obs.	Calc.	Diff.
H	403.4	17.2	24.4	+7.2
CH₃ ...	420.6			

R—COOC₆H₁₃.

R.	T°abs.	Obs.	Calc.	Diff.
H	426.6	15.6	22.8	+7.2
CH₃ ...	442.2			

R—COOC₇H₁₅.

R.	T°abs.	Obs.	Calc.	Diff.
H	449.7	14.6	21.3	+6.7
CH₃ ...	464.3	16.7	20.4	+3.7
C₂H₅ ...	481.0	17.2	19.5	+2.3
C₃H₇ ...	498.2	18.4	18.6	+0.2
C₄H₉ ...	516.6	15.8	17.7	+1.9
C₅H₁₁...	532.4	15.2	17.0	+1.8
C₆H₁₃...	547.6	15.2	16.3	+1.1
C₇H₁₅...	562.8			

R—COOC₈H₁₇.

R.	T°abs.	Obs.	Calc.	Diff.
H	471.1	11.9	20.05	+8.15
CH₃ ...	483.0	16.4	19.4	+3.0
C₂H₅ ...	499.4	15.8	18.5	+2.7
C₃H₇ ...	515.2	18.0	17.8	−0.2
C₄H₉ ...	533.2	15.0	16.95	+1.95
C₅H₁₁...	548.2	15.2	16.3	+1.1
C₆H₁₃...	563.4	15.5	15.65	+0.15
C₇H₁₅...	578.9			

TABLE V. (*continued*).

Esters, Variable Radical Normal, Iso-group in Fixed Radical.

C_3H_7—COOR.

R.	T°abs.	Obs.	Calc.	Diff.
CH_3 ...	365·3	17·8	27·3	+ 9·5
C_2H_5 ..	383·1	23·8	25·9	+ 2·1
C_3H_7 ...	406·9			

C_4H_9—COOR.

R.	T°abs.	Obs.	Calc.	Diff.
CH_3 ...	389·7	17·6	25·4	+ 7·8
C_2H_5 ...	407·3	21·6	24·1	+ 2·5
C_3H_7 ...	428·9			

R—$COOC_3H_7$.

R.	T°abs.	Obs.	Calc.	Diff.
H	343·0	18·6	29·25	+10·65
CH_3 ...	361·6			

R—$COOC_4H_9$.

R.	T°abs.	Obs.	Calc.	Diff.
H	370·9	18·4	26·85	+8·45
CH_3 ...	389·3	20·5	25·4	+4·9
C_2H_5 ...	409·8	20·1	23·9	+3·8
C_3H_7 ...	429·9			

R—$COOC_5H_{11}$.

R.	T°abs.	Obs.	Calc.	Diff.
H	396·3	15·7	24·95	+9·25
CH_3 ...	412·0	21·2	23·8	+2·6
C_2H_5 ...	433·2	18·4	22·35	+3·95
C_3H_7 ...	451·6	18·4	21·15	+2·75
C_4H_9 ...	470·0	17·0	20·1	+3·1
C_5H_{11}...	487·0	21·0	19·2	−1·8
C_6H_{13}...	508·0	18·0	18·15	+0·15
C_7H_{15}...	526·0	12·0	17·8	+5·8
C_8H_{17}...	538·0			

When there is an iso-group in the variable radical the observed values of Δ appear to agree more closely with the calculated, whether the fixed radical is normal or contains an iso-group[*]; the comparatively small number of esters belonging to this class are therefore given separately in Table VI.

The number of esters that have been examined is so large that the data may be studied in greater detail than is possible in other cases, and in Table VII. (p. 14) the esters are divided into two classes, depending on the constitution of the variable radical, whether normal or containing an iso-group; each class is divided into the two groups of lower and higher members, the latter all containing one or more $C—CH_2—C$ groups; and each group is arranged in the order of the molecular formulæ, so as to illustrate the influence of molecular weight on the values of Δ.

Although there are many irregularities in the table, it is quite clear (1) that the differences between the calculated and observed values of Δ are much greater for the lower members of each homologous series than for those which

* See footnote § on p. 9.

TABLE VI.

Esters ; Iso-group in Variable Radical.

R.	T° abs.	Δ Obs.	Calc.	Diff.	R.	T° abs.	Δ Obs.	Calc.	Diff.

Fixed Radical Normal.

H—COOR.

R.	T° abs.	Obs.	Calc.	Diff.
C_3H_7 ...	343·0	27·9	29·25	+1·35
C_4H_9 ...	370·9	25·4	26·85	+1·45
C_5H_{11} ...	396·3			

CH₃COOR.

R.	T° abs.	Obs.	Calc.	Diff.
C_3H_7 ...	361·6	27·7	27·6	−0·1
C_4H_9 ...	389·3	22·7	25·4	+2·7
C_5H_{11} ...	412·0			

C_2H_5COOR.

R.	T° abs.	Obs.	Calc.	Diff.
C_4H_9 ...	409·8	23·4	23·9	+0·5
C_5H_{11} ...	433·2			

C_3H_7COOR.

R.	T° abs.	Obs.	Calc.	Diff.
C_4H_9 ...	429·9	21·7	22·55	+0·85
C_5H_{11} ...	451·6			

R—COOCH₃.

R.	T° abs.	Obs.	Calc.	Diff.
C_3H_7 ...	365·3	24·4	27·3	+2·9
C_4H_9 ...	389·7			

R—COOC₂H₅.

R.	T° abs.	Obs.	Calc.	Diff.
C_3H_7 ...	383·1	24·2	25·9	+1·7
C_4H_9 ...	407·3	27·1	24·1	−3·0
C_5H_{11} ...	434·4			

R—COOC₃H₇.

R.	T° abs.	Obs.	Calc.	Diff.
C_3H_7 ...	406·9	22·0	24·1	+2·1
C_4H_9 ...	428·9			

Iso-group in Fixed Radical.

C_3H_7COOR.

R.	T° abs.	Obs.	Calc.	Diff.
C_3H_7 ...	393·75	25·85	25·05	−0·8
C_4H_9 ...	419·6	23·2	23·25	+1·05
C_5H_{11} ...	441·8			

C_4H_9COOR.

R.	T° abs.	Obs.	Calc.	Diff.
C_3H_7 ...	415·2	26·5	23·55	−2·95
C_4H_9 ...	441·7	23·3	21·8	−1·5
C_5H_{11} ...	465·0			

R—COOC₃H₇.

R.	T° abs.	Obs.	Calc.	Diff.
C_3H_7 ...	393·75	21·45	25·05	+3·6
C_4H_9 ...	415·2			

R—COOC₄H₉.

R.	T° abs.	Obs.	Calc.	Diff.
C_3H_7 ...	419·6	22·1	23·25	+1·15
C_4H_9 ...	441·7			

contain one or more $C—CH_2—C$ groups ; (2) that the differences are greater for the esters in which the variable radical is normal than for those in which it contains an iso-group ; and (3) that the differences diminish as the molecular weight rises.

The two negative differences 1°·5 and 3°·0 in Class II. (iso-group in variable radical) depend on a single value of Δ in each case, and but little weight can be attached to them ; the large positive mean difference 2°·69 for the esters $C_{13}H_{26}O_2$ and $C_{14}H_{28}O_2$ is chiefly due to a single value +5·8, which is certainly incorrect.

TABLE VII.

Formula.	VARIABLE RADICAL NORMAL.				ISO-GROUP IN VARIABLE RADICAL.			
	Lower Members.		Higher Members.		Lower Members.		Higher Members.	
	No. of values of Δ.	Mean diff. between calc. & obs.	No. of values of Δ.	Mean diff. between calc. & obs.	No. of values of Δ.	Mean diff. between calc. & obs.	No. of values of Δ.	Mean diff. between calc. & obs.
$C_2H_4O_2$	2	+9·15						
$C_3H_6O_2$	4	+7·57						
$C_4H_8O_2$	5	+7·69	2	+3·82	1	+1·35		
$C_5H_{10}O_2$	6	+7·47	4	+3·50	3	+1·42		
$C_6H_{12}O_2$	8	+5·56	6	+2·02	2	+2·20		
$C_7H_{14}O_2$	5	+3·78	8	+1·18	4	+1·34	1	−3·00
$C_8H_{16}O_2$	3	+5·43	9	+2·00	4	+0·02		
$C_9H_{18}O_2$	4	+5·16	9	+1·50	1	−1·50		
$C_{10}H_{20}O_2$	3	+3·00	8	+0·87				
$C_{11}H_{22}O_2$	...	...	7	+1·11				
$C_{12}H_{24}O_2$	...	...	4	+0·73				
$C_{13}H_{26}O_2$	...	...	4	+2·69				
$C_{14}H_{28}O_2$	...	...	3	+0·90				
$C_{15}H_{30}O_2$	...	...	2	−0·12				
$C_{16}H_{32}O_2$	...	...						
Total ...	40	+6·00	66	+1·60	15	+0·93	1	−3·00

Although the values of Δ cannot be calculated from the formula $\Delta = \dfrac{144\cdot86}{T^{0\cdot0148}\sqrt{T}}$ so accurately for the esters as for the non-associating compounds previously considered, yet it may perhaps be stated that the error rarely exceeds $1°\cdot5$ when there is an iso-group in the variable radical, or when the variable radical, if normal, contains at least six carbon atoms. Thus the observed boiling-point of *n*-hexyl formate is $153°\cdot6$, and the value of Δ calculated from the formula is $22°\cdot8$, giving $176°\cdot4$ as the boiling-point of the next higher homologue. This agrees very well with the observed B.P. of *n*-heptyl formate, $176°\cdot7$, but not with that of *n*-hexyl acetate, $169°\cdot2$. Again, the observed B.P. of methyl caproate (hexoate) is $149°\cdot6$, and the calculated value of Δ is $23°\cdot0$, giving $172°\cdot6$ as the B.P. of the next homologue. The observed B.P. of methyl œnanthylate (heptoate) is $172°\cdot1$, but that of ethyl caproate is only $166°\cdot6$.

Associating Substances.

There can be no doubt that water, the alcohols, and the aliphatic acids must be classed as associating substances, the molecular association occurring in all cases in the liquid state and with the acids in the state of vapour also. The researches of Ramsay and Shields on the surface energy of liquids indicate also that the lower aliphatic cyanides, nitro-compounds, and ketones exhibit molecular association in a smaller degree than the hydroxyl compounds * .

The data for associating compounds are given in Table VIII. (p. 16).

There are many irregularities in this Table, chiefly in the data for the alcohols and acids. Among the differences for the normal primary alcohols there is one negative value, $-2°·0$, but this is preceded by a very high positive value, $+6°35$; the most probable explanation of this irregularity seems to be that the observed boiling-point of octyl alcohol is too low, perhaps $3°$ or $4°$, but of course there may be smaller errors also in the boiling-points of other alcohols.

The very high value, $+5°·75$, in the differences for the acids is not so easily explained, for the values on either side of it are not markedly low ; it appears as though there were several small errors accumulating in this difference. Possibly the observed boiling-points of *n*-valeric, caproic, and œnanthylic acid are all too high, and those of caprylic, pelargonic, and capric acid too low.

The boiling-points of the iso-acids, with the exception probably of those of isobutyric and isovaleric acids are too uncertain to be included in the table. Taking the boiling-points of these two acids as $427°$ and $448°$ abs., the observed and calculated values of Δ would be $21°·0$ and $22°·75$, difference $+1°·75$.

As regards the secondary alcohols, there is a negative difference, $-4°·5$, preceded and followed by large positive differences, $+8°·2$ and $+7°·6$. These irregularities seem to indicate serious errors in the observed boiling-points of the three highest alcohols examined, that of heptyl alcohol being almost certainly too high, but it is probable that the error in the mean difference may be small.

The boiling-points of isobutyl and iso-amyl alcohol are, no doubt, nearly correct, and the difference $2°·05$, between the calculated and observed values of Δ is small, as in other cases where the variable radical contains an iso-group. On

* See also Guye, *Archives des Sciences physiques et naturelles*, 1894 [3], xxxi. pp. 38 & 463.

TABLE VIII.

R.	T° abs.	Δ			R.	T° abs.	Δ		
		Obs.	Calc.	Diff.			Obs.	Calc.	Diff.
Cyanides. R—CN.					**Alcohols. R—OH.**				
					(1) *Normal primary.*				
CH_3 ...	354·6	15·5	28·15	+12·65	CH_3 ...	337·7	13·6	29·75	+16·15
C_2H_5 ...	370·1	21·4	26·9	+ 5·5	C_2H_5 ...	351·3	18·9	28·5	+ 9·6
C_3H_7 ...	391·5	21·9	25·25	+ 3·35	C_3H_7 ...	370·2	19·7	26·9	+ 7·2
C_4H_9 ...	413·4				C_4H_9 ..	389·9	21·1	25·4	+ 4·3
					C_5H_{11} ...	411·0	20·0	23·85	+ 3·85
C_6H_{13}...	449·5	20·2	21·25	+ 1·05	C_6H_{13}...	431·0	18·0	22·5	+ 4·5
C_7H_{15}...	469·7	18·3	20·1	+ 1·8	C_7H_{15}...	449·0	15·0	21·35	+ 6·35
C_8H_{17}...	488·0				C_8H_{17}...	464·0	22·5	20·5	− 2·0
					C_9H_{19}...	486·5	17·5	19·25	+ 1·75
Nitro-compounds. R—NO₂.					$C_{10}H_{21}$	504·0			
CH_3 ...	374·0	13·5	26·6	+13·1	**(2) *Normal secondary.***				
C_2H_5 ...	387·5	16·5	25·55	+ 9·05	C_3H_7 ...	355·45	17·55	28·2	+10·65
C_3H_7 ...	404·0	20·5	24·35	+ 3·85	C_4H_9 ...	373·0	19·0	26·65	+ 7·65
C_4H_9 ...	424·5				C_5H_{11} ...	392·0	17·0	25·2	+ 8·2
					C_6H_{13} ...	409·0	28·5	24·0	− 4·5
					C_7H_{15} ...	437·5	14·5	22·1	+ 7·6
Ketones. CH₃—CO—R.					C_8H_{17} ...	452·0			
CH_3 ...	329·3	24·3	30·5	+ 6·2	**(3) *Primary iso-alcohols.***				
C_2H_5 ...	353·6	21·4	28·3	+ 6·9					
C_3H_7 ...	375·0	25·0	26·55	+ 1·55	C_4H_9 ...	381·05	24·0	26·05	+ 2·05
C_4H_9 ...	400·0	24·5	24·6	+ 0·1	C_5H_{11} ...	405·05	17·95	24·25	+ 6·3
C_5H_{11} ...	424·5				C_6H_{13} ...	423·0	14·0	23·0	+ 9·0
					C_7H_{15}...	437·0			
Acids. R—COOH.									
H	373·6	17·5	26·6	+ 9·1					
CH_3 ...	391·1	22·6	25·25	+ 2·65					
C_2H_5 ...	413·7	21·6	23·65	+ 2·05					
C_3H_7 ...	435·3	22·7	22·15	− 0·55					
C_4H_9 ...	458·0	19·5	20·8	+ 1·3					
C_5H_{11}...	477·5	19·0	19·7	+ 0·7					
C_6H_{13}...	496·5	13·0	18·75	+ 5·75					
C_7H_{15}...	509·5	17·0	18·05	+ 1·05					
C_8H_{17}...	526·5	15·5	17·25	+ 1·75					
C_9H_{19}...	542·0								

the other hand, the two remaining differences, $+6°·3$ and $+9°·0$, are exceedingly large, and one seems forced to conclude that the observed boiling-points of isohexyl and isoheptyl alcohol, especially of the latter, are far too low.

That the observed boiling-points of many of the alcohols are far from accurate seems to be proved by the results given in Table IX.

<div align="center">Table IX.</div>

Differences between the Boiling-points of Isomeric Alcohols.

Formula.	Normal, Primary and Secondary.	Primary, Normal and Iso.	Iso-primary and Normal Secondary.
C_4H_9OH	$+16°9$	$+ 8°85$	$+ 8°05$
$C_5H_{11}OH$	$+19·0$	$+ 5·95$	$+13·05$
$C_6H_{13}OH$	$+22·0$	$+ 8·0$	$+14·0$
$C_7H_{15}OH$	$+11·5$	$+12·0$	$- 0·5$
Mean	$17·35$	$8·7$	$8·65$

One would expect, probably not constancy, but at any rate some sort of regularity in each series, and the violent fluctuations must certainly be due to experimental errors. The conclusion already arrived at, that the observed boiling-point of secondary heptyl alcohol is much too high, and that of isoheptyl alcohol much too low seems to be confirmed.

The mean differences between the calculated and observed values of Δ are given in Table X.

<div align="center">Table X.</div>

Homologous Series.	Lower Members.		Higher Members.	
	No. of values of Δ.	Mean diff. between calc. & obs. values of Δ.	No. of values of Δ.	Mean diff. between calc. & obs. values of Δ.
Cyanides, R—CN	1	$+12°65$	4	$+2°90$
Nitro-compounds, R—NO_2...	2	$+11·10$	1	$+3·85$
Ketones, CH_3—CO—R	1	$+ 6·20$	3	$+2·85$
Acids, R—COOH	2	$+ 5·87$	7	$+1·58$
Alcohols, R . OH :—				
(1) Normal primary	2	$+12·87$	7	$+3·71$
(2) „ secondary......	1	$+10·65$	4	$+4·74$
(3) Iso-primary	1	$+ 2·05$	2	$+7·65$?
Total	10	$+ 9·12$	28	$+3·40$

It will be seen that the differences between the calculated and observed values of Δ are considerably greater for

associating than for non-associating substances ; also that among the associating substances they are greatest for the alcohols and least for the acids, although the factor of association in the liquid state is very high for both classes of compounds.

In order to arrive at an explanation of these facts, the effect of replacing hydrogen by chlorine may first be considered. The boiling-point of hydrogen chloride has not yet been determined, but it must be about $-80°$ C. Thus, by replacing an atom of hydrogen in the hydrogen molecule by chlorine, the boiling-point is raised about $173°$ from $20°{\cdot}4$ abs. to about $193°$ abs., or approximately in the ratio of $1:9{\cdot}5$. On replacing an atom of hydrogen in methane by chlorine, the rise of boiling-point is $141°{\cdot}0$ from $108°{\cdot}3$ to $249°{\cdot}3$, or in the ratio $1:2{\cdot}3$. Ascending the series of paraffins the rise of boiling-point, due to the replacement of a terminal hydrogen atom by chlorine, diminishes rapidly at first and then more slowly, until with octane the rise is only $58°{\cdot}4$ from $398°{\cdot}6$ to $457°{\cdot}0$, the ratio being $1:1{\cdot}147$. Thus the influence of the chlorine atom becomes relatively smaller as the formula weight of the alkyl group increases.

Consider, now, the effect of replacing a hydrogen atom by a hydroxyl group. In the formation of water from hydrogen gas the boiling-point is raised no less than $352°{\cdot}6$ from $20°{\cdot}4$ to $373°$, or in the ratio of $1:18{\cdot}3$; in the case of methane the rise is $229°{\cdot}4$ from $108°{\cdot}3$ to $337°{\cdot}7$, or in the ratio of $1:3{\cdot}12$; with octane the rise is $65°{\cdot}4$, from $398°{\cdot}6$ to $464°$, the ratio being $1:1{\cdot}164$; and with hexdecane it is only $56°{\cdot}5$ from $560°{\cdot}5$ to 617, in the ratio $1:1{\cdot}101$.

It will be seen that in the case of hydrogen the influence of the hydroxyl group is enormously greater, and in the case of methane much greater than that of the chlorine atom in raising the boiling-point, but that on ascending the series of paraffins to octane, the influence of the hydroxyl group diminishes more rapidly until it is little greater than that of the chlorine atom, and it is probable that with hexdecane it would be somewhat less. This is, no doubt, to be explained by the fact that the molecules of water and of the lower alcohols are highly associated in the liquid, but not in the gaseous state, and that, in order to vaporize the liquids, this molecular attraction must be overcome, and the temperature must therefore be raised. The molecular association diminishes, however, as the series of alcohols is ascended, and is probably slight in the case of octyl alcohol. If so, it would appear that the effect of the hydroxyl group—apart from association—in raising the boiling-point is not very different from, and is probably somewhat less than, that of

the chlorine atom, and that the difference between the boiling-points of the lower alcohols and of the corresponding chlorides is entirely due to molecular association in the liquid state.

With the acids there is association in the gaseous as well as the liquid state, and since, according to the tables given by Ramsay and Shields, the factor of association for a liquid fatty acid at its boiling-point is rarely greater, and in most cases is somewhat smaller, than for the corresponding liquid alcohol, the molecular attraction to be overcome on vaporization must be considerably less for the acid than for the corresponding alcohol, and the resulting rise of boiling-point above the normal value must be less.

The results for (1) non-associating liquids; (2) esters; (3) associating substances are given in Table XI.

<div align="center">TABLE XI.</div>

	Lower Members.		Higher Members.	
	No. of values of Δ.	Mean diff. between calc. & obs. values of Δ.	No. of values of Δ.	Mean diff. between calc. & obs. values of Δ.
Non-associating substances	30	$+3 \cdot 50$	49	$-0 \cdot 05$
Esters	55	$+4 \cdot 62$	67	$+1 \cdot 53$
Associating substances......	10	$+9 \cdot 12$	28	$+3 \cdot 40$

The esters come between the non-associating substances and those which are recognized as associating, and the conclusion previously arrived at that there is slight association in the case of the esters is supported.

Considering the data as a whole it may, I think, be concluded that the values of Δ may be calculated from the formula with an error rarely exceeding $1^{\circ} \cdot 5$ and generally less than 1° for the majority of organic compounds which contain one or more $C-CH_2-C$ groups provided that their molecules are not associated in the liquid state; that is to say, for the hydrocarbons and their halogen derivatives, the ethers, aldehydes, amines, hydrosulphides, and the higher esters, cyanides, nitro-compounds, and ketones, probably also for all esters which contain an iso-group in the variable alkyl radical, and possibly for acids and even alcohols of very high molecular weight. The calculated values of Δ are, however, too high for the lowest members of nearly all homologous series, and still more so for substances which are characterized by molecular association.

<div align="center">C 2</div>

II. *The Measurement of the Potential of the Electrodes in Stationary Liquids. The Determination of Changes of Concentration at the Cathode during Electrolysis. By* HENRY J. S. SAND, *Ph.D., M.Sc.**

[Plate I.]

IN recent electrochemical investigations accurate data are, as a rule, to be found regarding current-density, voltage, temperature, and concentration of liquid taken, and very often also regarding the potential of one of the electrodes, defined as the drop of potential on the boundary between it and the electrolyte. The actual concentration of the electrolyte in contact with the electrode is, however, very often uncertain, owing to the removal of substance by electrolysis. Some years ago I was able to demonstrate the great importance of this factor in connexion with the electrolysis of acid solutions of copper sulphate †. When small current-densities were employed in a stationary solution, it was possible to show with a fair amount of accuracy that evolution of hydrogen only took place after the complete removal of copper salt from the liquid in contact with the electrode. On the other hand, when large current-densities were employed, this local depletion took place so rapidly that even the most vigorous stirring could not prevent it. The result obtained in this research could be predicted theoretically when the assumption was made that the process occurring at the electrode itself was of a reversible nature; that is, that on the boundary between electrode and electrolyte, chemical equilibrium must be supposed to remain continually established. The matter may also be stated in the following manner corresponding to a form in which it has recently been put forward by Nernst ‡ as a general principle : differences of chemical potential cannot exist for a finite period of time on the boundary of two phases, that is " between infinitely near points," as this would lead to infinitely great forces and reaction velocities. According to this assumption, the actual velocities of electrochemical reactions, and of all others which occur on the boundary of two phases, are infinitely great as soon as conditions of equilibrium have been overstepped by a finite amount, the observed velocities being simply limited by the rate at which diffusion and convection renew the material removed by the reaction. As conditions of equilibrium in electro-chemical reactions are determined by the electrode-potential,

* Communicated by the Faraday Society. Read before the Society October 25th, 1904.
† Phil. Mag. 6th series, i. p. 45.
‡ *Ztschft. Phys. Chem.* xlvii. p. 52 (1904).

by which name I propose to indicate the drop of electro-motive force on the boundary between electrode and electro-lyte, and the rate of these reactions is proportional to the current strength, this may be expressed in the following manner : " When the electrode-potential at which a reaction begins has been exceeded by a finite amount, the current strength becomes independent of it, being only limited by the velocity of convection and diffusion." The instance just referred to of the deposition of copper conforms to this rule, for it seems very probable that when a sufficient drop of potential is maintained on the boundary between electrode and electrolyte, the rate of deposition of copper only depends on the rate of renewal of the metal-salt at the electrode. Examples of a similar kind have been studied by Brunner *, viz., the rate of reduction of iodine to hydriodic acid with definite cathode-potentials and the rate of liberation of hydrogen from weak solutions of acids under similar conditions.

As has already been indicated, Nernst and Brunner are plainly of opinion that all cases of electrochemical action may be ultimately considered from the same point of view. It is not the object of the present paper to enter very fully into a thermodynamical discussion of the principle involved. It will suffice to point out that the latter seems sufficiently invalidated for *à priori* application to actual cases owing to the impossibility of stating that the atoms chemically reacting on the boundary of two phases are in infinite proximity †. In fact the results of several recent researches, notably those of Haber ‡, stand in opposition to conclusions drawn from the principle under discussion. The object of all these researches is the investigation of the dependence of electrode-potential as defined above on current-density. In Haber's work the reduction of nitrobenzene is examined ; but instead of finding the current-density to be independent of the potential of the cathode, his results point to a logarithmic dependence of the latter on the former. In fact he has · partially explained his results, starting from the not improb-able assumption that the variations of potential are brought about by the absorption in varying concentrations of hydrogen, or other reducing material, by the electrode. In the further elaboration of this view, his results may all be foretold if we

* *Ztschft. Phys. Chem.* xlvii. p. 56 (1904).
† Nernst himself points out that the boundary of two phases is not to be considered mathematically discontinuous, but apparently attaches little importance to this fact.
‡ *Ztschft. Phys. Chem.* xxxii. p. 193 (1900) : also Russ, *Ztschft. Phys. Chem.* xliv. p. 641 (1903).

adopt as an empirical fact that the velocity of reduction at the electrode v depends on the concentration of hydrogen c in it according to the relation

$$v = \kappa c^\gamma,$$

in which κ and γ are constants depending on the conditions of the experiments. In contradiction to this, as will be understood from what has already been said, Nernst's view would require that once the conditions of equilibrium have been overstepped, the above velocity should be infinitely great. As a matter of fact, Haber's results have been doubted by Brunner, who believes that changes of concentration in the liquid adjoining the electrode may be partly responsible for them. To this Haber and Russ have replied, stating reasons for the correctness of the results obtained by the former [*]. In view of the fact that the investigation of the dependence of electrode-potentials on current-density seems one of the most important problems of contemporary electrochemical research, it appears very desirable to have independent methods for the determination of the rate at which changes of concentration occur in the layer of electrolyte adjoining the electrode.

When the diffusion coefficients required for calculating these changes of concentration are not known, it would seem that the most reliable method for arriving at an estimation of their rapidity is afforded by the investigation of the behaviour of stationary liquids free from convection-currents. This was the method adopted by me for the examination of copper solutions. When hydrogen began to be evolved owing to complete depletion of the copper-salt at the electrode, a sudden, very large fall of cathode-potential took place, which caused a corresponding increase of voltage between the two electrodes that could be easily observed. This method is only applicable where very sharp breaks of potential occur, and the anode is not polarizable in the liquid employed. Another method has since been described by Cottrell [†]. In it a constant E.M.F. is maintained between the electrodes; the current, however, must remain so minute, or the resistance of the electrolyte be so small, that practically no E.M.F. is required to overcome it according to Ohm's law.

It is the object of the present paper to describe a method which will allow the electrode-potential in stationary liquids of the most varying kind to be measured, when very much larger currents are employed than those which can be made

[*] *Ztschft. Phys. Chem.* xlvii. p. 260 (1904).
[†] *Ibid.* xlii. p. 385 (1903).

use of according to Cottrell's method, and at the same time to indicate the nature and the scope of the conclusions that can be drawn from the results obtained.

Description of Apparatus and Electrical Arrangements.

It was necessary that the apparatus should fulfil the following conditions :—

(1) The electrode under examination must be placed horizontally either at the top or at the bottom of the apparatus. It should be readily interchangeable and accessible to observation during an experiment, and allow the current to have uniform density.

(2) The electrode not under examination should be as non-polarizable as possible, and no gases must be evolved which might stir the liquid.

(3) The difference of potential between the electrode and the layer of electrolyte in contact with it must be capable of continuous measurement during the course of an experiment.

(4) It was found desirable to have the apparatus arranged so that several successive determinations could be carried out without taking it to pieces.

Figs. 1 and 2 serve to illustrate the apparatus that was designed to meet these requirements. For the sake of brevity we shall call the electrode under observation the cathode, the other electrode through which the working current enters or leaves the liquid the anode, although there is no reason why our cathode should not be used as anode and *vice versa*. Making use of these abbreviations, fig. 1 shows the apparatus with the cathode at the top, that is for liquids that become specifically lighter during electrolysis. In fig. 2 the cathode is at the bottom as used for liquids that become specifically heavier during electrolysis. The positions of the anode are indicated in both figures by A. The current passes up the wide tube B, and then enters the long straight tube C of about 1 cm. bore. The cathode is approached through a hole *d* in the glass plate D, as shown in fig. 1. The electromotive force between the cathode and the part of the electrolyte on the other side of the glass plate D is determined, a liquid connexion being tapped off here through the annular space E and the tubes F to the non-polarizable standard electrode G. We shall indicate by T the place in the electrolyte between the glass plate D and the tube C where the liquid connexion is tapped off to the standard electrode. The electromotive force actually measured on the

Fig 1.
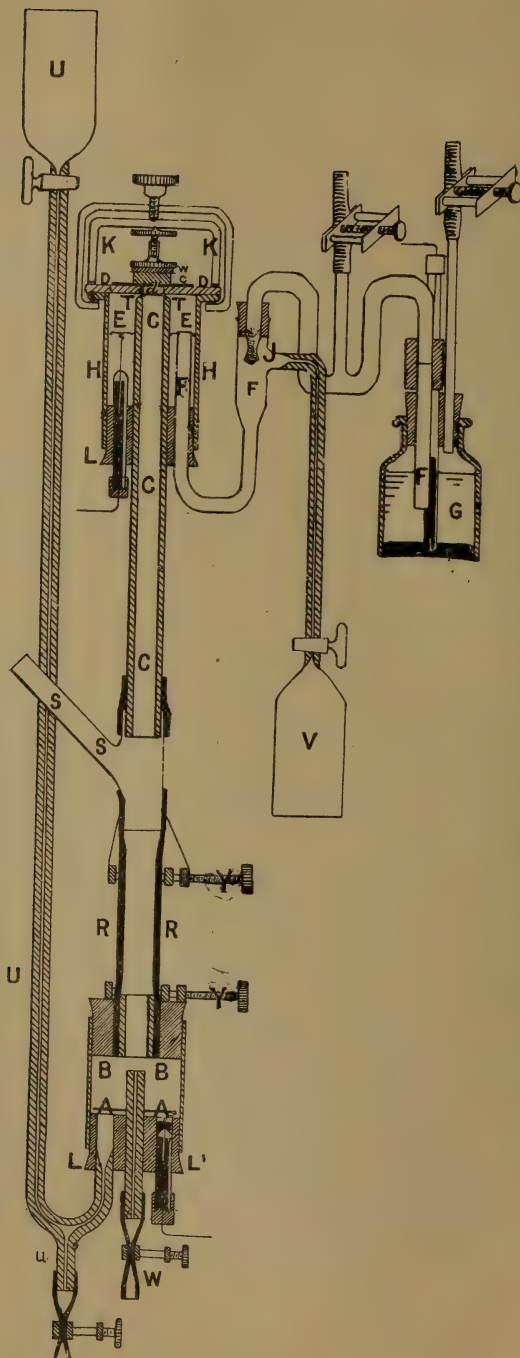
potentiometer is that between the cathode and the standard electrode, and this is composed of the following parts :—

(1) The drop of electromotive force due to polarization between the cathode and the layer of liquid in contact with it.

(2) The difference of potential between the cathode and T calculated according to Ohm's law as the product of the resistance between the two and the current strength.

(3) Any small electromotive force brought about by contact of the electrolyte examined with the decinormal potassium chloride solution of the standard electrode.

(4) The constant E.M.F. of the standard electrode.

The first of these quantities is the one which it is desired to measure, the third was always assumed to be zero, as has been done by other investigators. The fourth may be taken according to Ostwald and Wilsmore to be 0·615 volt for the standard so-called decinormal electrode composed of mercury in contact with a decinormal solution of potassium chloride saturated with mercurous chloride. Being constant for all experiments it needs, however, not be taken into consideration. The second difference of potential mentioned, may as a rule be calculated when the specific resistance of the electrolyte is known. It is better, however, actually to determine the resistance between the cathode and T during the course of an experiment, more especially when there is a possibility of an increase of resistance at the electrode, owing to the formation of badly conducting deposits, gas-bubbles, &c. The electrical method required for this purpose had to be devised in such a manner that no false connexions should be introduced and that it should at the same time be independent of the polarization of the electrode. As far as I am aware, the only method suitable for this purpose is one described several years ago by Nernst and Haagn*. It consists of a modification of the Wheatstone-bridge method for alternating currents, two resistances being replaced by standard condensers, and rough condensers being interposed wherever a false connexion would otherwise be made. Its application to the present case is shown diagrammatically in fig. 3. The two resistances compared are the resistance from the anode to T, and the resistance from T to the cathode, plus a known variable resistance M. These formed two arms of the Wheatstone bridge, the other two being the standard condensers A and B, fig. 3. A special box was made up for this determination containing capacities varying from one-thousandth of a microfarad to half a microfarad. In order to form the bridge

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* *Zeitschft. Phys. Chem.* xxiii. p. 97.

Fig. 2.
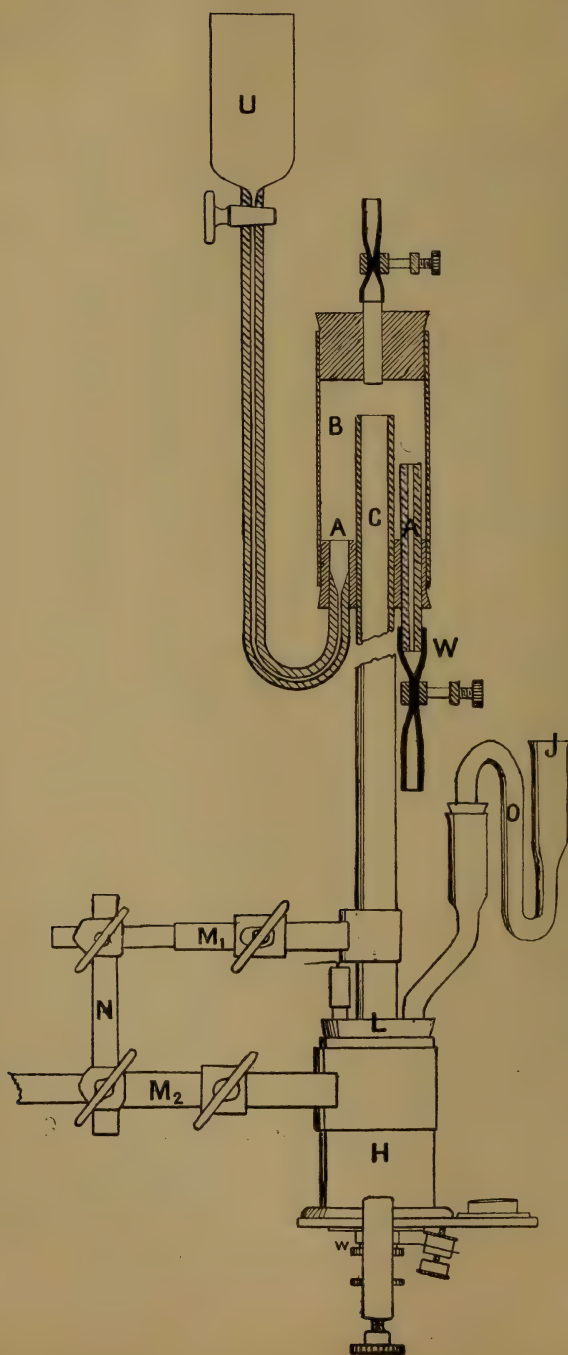
containing the telephone between the junction of the two condensers and T, a second auxiliary electrode P is required in the liquid contained in the annular space *e*, fig. 1. It was found that an electrode of ordinary smooth platinum answered perfectly well for this purpose. The current for these measurements was derived from a small induction-coil, a rough condenser being interposed between it and one of the

Fig. 3.
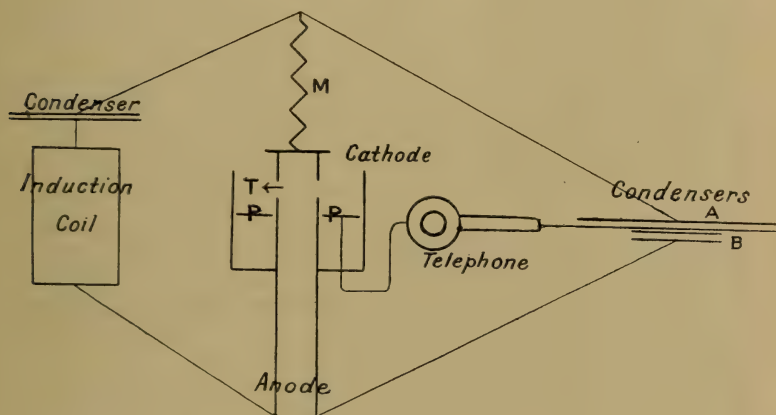
junctions. The method of working consisted in giving the two standard condensers A and B suitable values, say 500 and 20 millemicrofarads, and finding the value m_1 of the variable resistance M taken from a dial-resistance-box that produced a minimum in the telephone. The smaller of the two condensers, B, *i. e.* the one on the anode side, was then altered to its *r*-fold value $20\,r$ millemicrofarads, the other condenser being left unaltered. The new resistance m_2 was now determined, which produced a minimum in the telephone. The required resistance x from the cathode to T is then manifestly derived from the equation

$$\frac{x+m_1}{x+m_2}=\frac{1}{r},$$

its value being

$$x=\frac{m_2-rm_1}{r-1}.$$

The following are details of the apparatus shown in figs. 1 and 2 that must be specially referred to. The cathode and

the glass plate it rests on are both ground perfectly flat. The latter is rectangular in shape, and the edges of the hole d, drilled in it, are perfectly even and free from chips. The plate is put in position in such a manner that the hole d exactly forms the continuation of the tube C; the way in which it is held on the flanged glass cylinder H and the electrode pressed against it by the specially designed clamp K will be understood from the figs. 1 and 2, as also the manner in which electrical connexion is made by means of thin copper foil bent round the rubber washer w (fig. 2). Practically no leakage took place between the electrode and the glass plate, even when no grease was employed to make a joint. Nevertheless, a small amount of liquid always creeps up, and this seems to give rise to convection currents owing to evaporation, more especially when alcoholic solutions are employed. A very small amount of vaseline thickened with a little paraffin-wax was therefore spread round the edge of the electrode before it was placed in position. In order to guard against its being accidentally moved backwards and forwards over the plate, which would have caused the grease to spread over its surface, a cardboard ring c was fixed on the glass plate by means of paraffin-wax, and the electrode rotated inside this ring to place it in position. The electrode could easily be levelled both when used as in fig. 1 and as in fig. 2, by placing a small spirit-level on the glass plate (fig. 2); the retort-stand, to which the apparatus was attached, being fixed on a levelling stand.

The tube C, the end of which was bevelled, was fixed centrally in the flanged cylinder H, and was held firmly in position with regard to the latter, not only by the rubber bung L, but also by means of the two clamps M_1 and M_2 (fig. 2), both of which are fastened to a common short rod N. The clamp M_1 at the same time serves to support the apparatus on its retort-stand. The tube C ends about $\frac{1}{2}$ mm. below the glass plate, thus allowing sufficient room for a liquid connexion between the auxiliary electrodes and the liquid through which the working current is passing.

The two auxiliary electrodes have already been referred to. The place where the junction between the electrolyte examined and the decinormal solution of KCl of the standard electrode is made, is indicated in both figures by J. The method of filling the tubes F and O and keeping the liquid in them by means of cotton-wool plugs before placing the tubes in position will need no further explanation.

Considerable difficulty was at first experienced in arranging the anode to satisfy condition 2 (p. 23). The method finally

adopted consisted in the employment of electrodes of either copper or mercury, the electrolyte under examination being displaced in the vicinity of the electrode by a solution of metal-salt which was run in slowly through the capillary tap-funnel U (figs. 1 and 2). The displaced electrolyte was generally removed through the tube W (fig. 1). The anode solution, employed in conjunction with the copper electrode, when the electrolyte was an acid alcoholic solution, consisted of cuprous chloride dissolved in a mixture of alcohol and hydrochloric acid. When the electrolyte was an alkaline solution containing alcohol, and also in some experiments on alcoholic acid solutions, a mercury electrode was employed in contact with a solution of mercurous nitrate and a little nitric acid. As the latter, however, as a rule, produced a precipitate in contact with the electrolyte under examination, a solution of ammonium nitrate was interposed. The liquids in this case were run under the electrolyte in the following sequence : (1) ammonium nitrate solution ; (2) mercurous nitrate solution ; (3) mercury.

The requirement that the apparatus should allow several successive determinations to be performed without taking it to pieces, was only met satisfactorily when the cathode was at the top, as in fig. 1. For this purpose it was necessary to renew the liquid at the cathode by stirring, without at the same time causing anode solution to come into the cathode compartment. As will be seen from fig. 1, the rubber tube R with screw clips YY and the glass side-tube S were there-fore introduced between the tubes B and C. The method of using them was the following :—When a determination was finished, the side-tube S, which was generally left open during the experiment, was closed, about 5 c.c. of air being left in it. The standard electrode was then removed by undoing the connexion at J, and a rubber stopper with a tap-funnel introduced in its place. The screw clips YY were then closed. A considerable amount of electrolyte was thus forced out of the rubber tube R, and was allowed to escape through one of the taps. The top part of the apparatus was then taken from the retort-stand and moved up and down, the rubber tube R being used as a hinge. Air-bubbles from the side-tube S could thus be admitted to stir the liquid and after-wards be returned to the tube S. When it was necessary to leave the apparatus overnight, the anode solution was removed through the tube u of the tap-funnel U; being replaced by electrolyte run in through the funnel introduced at J. It was thus found possible to keep the cathode compartment perfectly free from anode solution, even when experiments

were spread out over several days. The apparatus was also used for single experiments, the side-tube S and rubber tube R with clips Y Y being then omitted.

For the purpose of filling the apparatus as illustrated in fig. 1, both the funnels U and V must be employed, the top being inverted for filling the upper part through the funnel V. A vent-hole with tap (not shown in the figure) allows the air to escape through the bung L. The last air-bubbles are removed from the cathode through the side-tube S. When in use the apparatus was placed on a non-vibrating shelf.

The electrical arrangements are shown completely in fig. 4.

Fig 4
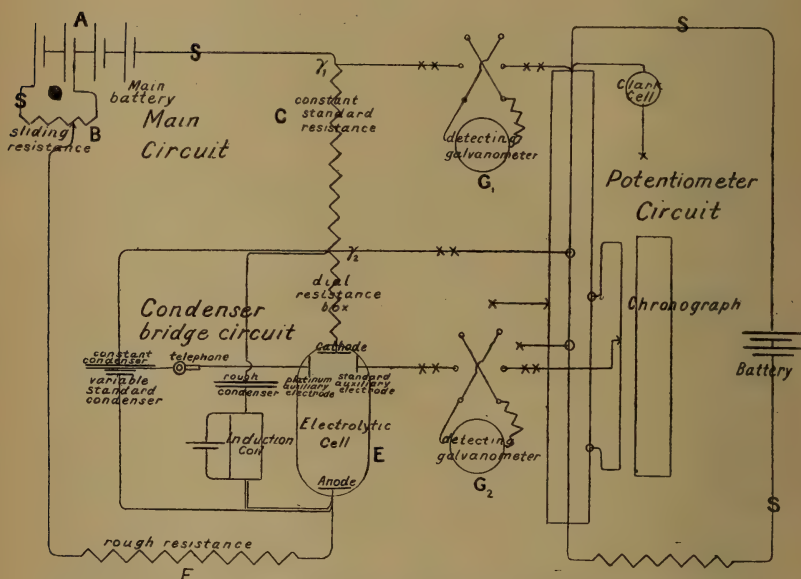
They comprise the main circuit, the condenser-bridge circuit already explained on p. 27, and the potentiometer circuit. The current was derived from a battery of small accumulators A, and was kept perfectly constant by variation within suitable limits of the original electromotive force employed as shown at B, fig. 4. It passed through a known constant resistance C, then through a dial-box, provided for the purpose of introducing when required the resistance M referred to on p. 27. As a rule this box was, however, kept free from resistance. After this the current passed through the electrolytic cell E, and finally through a rough resistance F of about the same magnitude as the latter,

For the purpose of determining the current-strength and the cathode-potential, the voltage between the ends γ_1 and γ_2 of the known resistance C, and between γ_2 and the standard auxiliary electrode, were measured on the same potentiometer circuit. The latter was composed primarily of a slidewire with a variable back resistance. As several points of the potentiometer had to be tapped simultaneously during an experiment, a system of wires along which mercury cups could be moved was arranged in parallel with the slidewire, and the required potentials tapped off from these mercury cups, the electrical position of the latter relative to points of the slidewire being arranged by separate determinations before an experiment.

The following manipulations and measurements were carried out during an experiment. The current was kept constant at a known value, the potentiometer regulated by occasional comparison with a Clark cell, and the resistance between the cathode and T measured now and again as explained on p. 27. At the same time the cathode-potential had to be determined by measurement of the E.M.F. between γ_2 and the standard auxiliary electrode, and the time had to be read. In order to facilitate the last two determinations a chronograph was designed, which contained a slidewire that was shunted parallel to part of the potentiometer circuit. The slider on the latter was moved by hand so as to balance the difference of potential to be measured; its position, however, was automatically recorded on curve-paper moved past it by means of a drum rotated by clockwork.

With regard to the already explained measurement of the resistance from the cathode to T by means of the condenser-bridge-circuit, it must be pointed out firstly that a false connexion is introduced unless the galvanometer G_1 for measuring the current is thrown out of circuit during a determination, and secondly that the rough resistance F is required in the main circuit on account of this measurement in order to prevent the greater part of the current from the induction-coil from finding its way through the battery instead of through the electrolytic cell. The resistance taken from the dial resistance-box must be so small, that its effect on the current during the fraction of a minute it is in circuit need not be considered.

All the points of the various circuits that were ever interchanged were connected by special wires to a central switch-board of mercury cups on which all the connexions were conveniently and rapidly made. All such points are indicated in the figure by xx, other switches being indicated by S.

Results obtained with Aqueous Solutions of Copper Sulphate, Silver Nitrate, and Zinc Sulphate.

The first series of experiments we have to refer to is one carried out with aqueous solutions of metal salts, the substances employed being solutions of copper sulphate, silver nitrate, and zinc sulphate.   As long as metal-salt is left in the layer of liquid touching the cathode, the metal is deposited from such solutions at a cathode-potential, probably either identical with or differing only slightly from the potential at which it is in equilibrium with the solution of its salt.   But when the salt has been removed from the electrode by the current, a sudden drop of the potential of the electrode takes place.   A marked difference is, however, observed in this connexion between solutions of copper sulphate and silver nitrate.   In the case of the former, the cathode-potential drops about one volt, and then becomes fairly constant again.   At the same time the resistance of the layer of liquid around the cathode increases by about 175 ohms per sq. cm., the combined effect being that the difference of potential between the standard electrode and the cathode, as measured in the experiments, increases by 1·2 to 1·3 volts above its original value.   The increase of resistance is found to be due to the formation of a deposit of copper hydroxide arising from hydrolysis of the copper sulphate at great dilutions.   In the case of the silver nitrate, however, when depletion of the silver occurs a drop of the potential of the cathode amounting to only about 0·2 volt takes place, after which the potential approaches its former value again more or less rapidly.   The cathode is then found to have become covered with a mass of silver crystals which grow into the solution.   The curves in fig. 5 show results obtained with a silver nitrate, and with a copper sulphate solution.   As has been shown in the previous communication referred to, the diffusion coefficient of the salts experimented on may be calculated when the time is known which elapses until the concentration of the salt in the layer of liquid at the electrode becomes zero, and this time may be taken as that at which the break in the curve takes place.   The actual points chosen on this break were 5 centivolts in the case of the silver nitrate, and 50 centivolts in the case of the copper sulphate solutions.   The formula employed is—

$$k = \frac{1 \cdot 1284^2}{96540^2} \; \frac{i^2 t n_a{}^2}{c^2},$$

k being the diffusion coefficient of the salt in $\dfrac{cm.^2}{sec.}$ when c is

the concentration of the solution in $\dfrac{\text{mg.-equiv.}}{\text{cm.}^3}$, i the current strength in $\dfrac{\text{milamps.}}{\text{cm.}^2}$, t the time in seconds, and n_a the transport value of the anion. Details of the experiments and results are arranged in the following table.

TABLE I.

A. Copper sulphate solution, concentration $0\cdot204\ \dfrac{\text{mg.-equiv.}}{\text{c.c.}}$.

Electrode: Platinum-iridium of available surface $0\cdot7589$ cm.^{2}
$n_a = 0\cdot63$.

Current-density in $\dfrac{\text{milamps.}}{\text{cm.}^2}$.	Time in Seconds.	Temperature T in °C.	Diffusion Coefficient at T° in $\dfrac{\text{cm.}^2}{\text{sec.}}$.	Diffusion Coefficient at 18° by Formula $k_{18} = \dfrac{k_T}{1 + 0\cdot026\,(T - 18)}$.
$2\cdot109$	790	$15\frac{1}{2}$	$4\cdot58 \times 10^{-6}$	$4\cdot86 \times 10^{-6}$
$1\cdot757$	1,053	16	$4\cdot24 \times 10^{-6}$	$4\cdot45 \times 10^{-6}$
$1\cdot506$	1,566	16	$4\cdot63 \times 10^{-6}$	$4\cdot86 \times 10^{-6}$
$1\cdot042$	3,297	$16\frac{3}{4}$	$4\cdot66 \times 10^{-6}$	$4\cdot81 \times 10^{-6}$

B. Silver nitrate solution, concentration $0\cdot1\ \dfrac{\text{mg.-equiv.}}{\text{c.c.}}$.
Electrode as above, $n_a = 0\cdot528$.

$1\cdot757$	580	$18\frac{1}{2}$	$6\cdot82 \times 10^{-6}$	$6\cdot73 \times 10^{-6}$

According to Wiedeburg [*] the diffusion coefficient of $CuSO_4$ at $18°$ is $4\cdot479 \times 10^{-6}$ for the concentration 0, and $4\cdot226 \times 10^{-6}$ for the original concentration of the preceding table. The values in the table thus appear somewhat high. The value found for the diffusion coefficient of silver nitrate is very much too low if we take as correct the number $k_{1s} = 12\cdot5 \times 10^{-6}\,\dfrac{\text{cm.}^2}{\text{sec.}}$, which has been calculated by Nernst from the migration velocities of the $Ag\cdot$ and the $NO_3{}'$ ion [†], and which agrees with values extrapolated from Scheffer's determinations of diffusion coefficients [‡]. This fact seems to

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[*] Wied. *Ann.* xli. p. 675 (1890).
[†] *Ztschft. Phys. Chem.* ii. p. 400 (1888).
[‡] *Ibid.* ii. p. 628 (1888).

point to the break in the curve occurring before complete depletion of the silver at the electrode has taken place. The matter is, however, undergoing further investigation. See also remarks below regarding deposition-potential of copper.

The numbers given in Table II. may be taken as typical results for the determination of the resistance of the copper sulphate solution between the cathode and the junction T described on p. 27. The experiments 1, *a*, *b*, *c*, *d*, were carried out before the current passed through the solution; experiments 2, *a* and *b*, after the break in the potential of the cathode had taken place.

TABLE II.

No. of Experiment.		Ratio of Condensers.			Resistance in Dial-box.	Resistance calculated by Formula on p. 27.
1	*a*	500	:	20	65	From 1 *a* and 1 *b* 77 ohms.
	b		:	40	207	
	c		:	50	278	From 1 *a* and 1 *c* 77 ohms.
	d		:	70	425	From 1 *a* and 1 *d* 79 ohms.
2	*a*	500	:	50	122	From 2 *a* and 2 *b* 218 ohms
	b		:	100	462	

The potential of the cathode, after allowance had been made for the drop according to Ohm's law between the cathode and the junction T, was, in the majority of experiments on copper sulphate solutions, found to lie about 10 to 20 millivolts below the potential of equilibrium. This matter, as well as the slight retrogression of potential at the beginning of the experiments which is visible in the copper sulphate curves (fig. 5, Pl. I.), similarly the conditions on which the crystallization of the silver depends, is undergoing further investigation. Some of the results referred to, if corroborated, would lead to the conclusion that the deposition of metals from solutions like copper sulphate is not an absolutely reversible process, and this again would prove of importance in its bearing on the different nature of metal deposits obtained from different solutions. In a series of experiments on a decinormal zinc-sulphate solution with an amalgamated copper electrode, similar curves resulted to those obtained with the copper sulphate. Owing, no doubt, to the fact that hydrogen is evolved sometimes after all the zinc has been removed from the solution, and sometimes

before, according to slight differences in the nature of the electrode, the results were somewhat irregular, and have therefore not been reproduced as curves. The values obtained for the diffusion coefficient of zinc sulphate varied between $3\cdot5 \times 10^{-6}$ and $4\cdot5 \times 10^{-6}\,\frac{cm.^2}{sec.}$ They point to an increase of the diffusion coefficient of zinc sulphate at great dilutions, the value obtained by Weber * for solutions of $2\cdot114$ per cent., *i. e.*, $0\cdot26\,\frac{mg.\text{-}equiv.}{c.c.}$, being $2\cdot78 \times 10^{-6}\frac{cm.^2}{sec.}$ In agreement with this we should expect the diffusion coefficient of zinc sulphate not to differ greatly from that of copper sulphate at great dilution, owing to the fact that the mobilities of the zinc and the copper ion are nearly equal at great dilution.

Alcoholic Solutions of Cuprous Chloride.

The next series of determinations was carried out with the object of testing whether the method was suitable for alcoholic solutions. The solution chosen was one of cuprous chloride in a mixture of hydrochloric acid, alcohol, and water, its composition being 1 g. Cu_2Cl_2, 25 c.c. strong hydrochloric acid, 45 c.c. 95 per cent. alcohol, 30 c.c. water. Owing to contact with the air, part of the cuprous chloride became oxidized to cupric chloride. The curves on fig. 6 show the degree of coincidence obtained in successive experiments, three determinations having been carried out with the current-density $1\cdot054\frac{milamps.}{cm.^2}$, and two with the current-density $1\cdot318\,\frac{milamps.}{cm.^2}$. The curves obtained are of interest inasmuch as they show plainly that two successive processes take place : firstly, the reduction of the cupric chloride to cuprous chloride, which begins at potentials lower than those registered in the figure, and goes on to about 45 centivolts ; and secondly, the deposition of copper from the cuprous chloride beginning at about 45 centivolts and going on until the break in the curve is completed, when the potential becomes fairly constant at about 65 to 70 centivolts, and hydrogen comes off.

Acid Solutions of Nitrobenzene.

We now come to the discussion of curves obtained when the solution of an organic substance, namely, nitrobenzene,

* Wied. *Ann.* vii. p. 536 (1879).

was reduced under varying conditions. Two solutions were experimented upon with different electrodes and current-densities, other conditions remaining unchanged as nearly as possible. Stock solutions of accurately known composition were mixed in known proportion, the composition of the resulting liquids working out as follows:—

1 litre acid solution used for experiments of figs. 7, 8, 9, 10 (Pl. I.) contains .... $\begin{cases} \text{5 g. nitrobenzene.} \\ \text{107 g. } H_2SO_4. \\ \text{425 c.c. ethylalcohol.} \\ \text{Remainder water.} \end{cases}$

1 litre alkaline solution used for experiments of fig. 11 contains ... $\begin{cases} \text{5 g. nitrobenzene.} \\ \text{82 g. NaOH.} \\ \text{425 c.c. ethylalcohol.} \\ \text{Remainder water.} \end{cases}$

The anode for the experiments on acid solutions was, as a rule, one of copper in contact with a solution of 30 g. cuprous chloride in 200 c.c. strong hydrochloric acid and 100 c.c. 95 per cent. alcohol. For the experiments on the alkaline solution the anode was mercury in contact with a solution of 100 g. mercurous nitrate and 35 c.c. strong nitric acid in 500 c.c. water, the solution interposed (see p. 29) containing 125 g. ammonium nitrate in 500 cc.

The curves obtained show, firstly, the degree of coincidence yielded by a repetition of experiments, and, secondly, they exhibit both interesting similarities independent of the material of the electrode and characteristic differences due to it. The abscissæ again denote the experimentally found values of the potential of the standard electrode, the potential of the cathode being taken as zero. A small correction may be made for the difference of potential due to the resistance overcome by the current between the cathode and the junction T (p. 23), by taking the value $3\frac{1}{2}$ ohms for this resistance in the case of the acid solution, and 6 ohms for the alkaline solution. All the curves representing the results of experiments on acid solutions exhibit first a rise of potential. After this it recedes, or becomes almost stationary, then increases once more, and finally becomes stationary again. Bubbles of hydrogen may now be observed on the cathode. It may perhaps be premature to attempt to explain definitely the first retrogression of potential in the curves. The most probable cause seems to be a temporary change in the nature of the electrode, resulting from electrolysis. Such changes have been recently shown to occur frequently

by Russ [*], Luther and Brislee [†], and others. As a rule they tend to make cathodes more efficient, *i. e.*, they enable them to accomplish reduction with the expenditure of a smaller amount of energy. The electrodes are also certainly permanently changed under certain conditions in such a manner that they become less efficient. This is very notable with the cathode of smooth platinum when small current-densities are employed. If an electrode is used which has been freshly cleaned with ligroïn, boiling strong nitric acid and boiling caustic-soda solution, the potential of the standard electrode shows a comparatively low value throughout the experiment, the results being similar to those obtained with a cathode of platinum coated with platinum black. On repetition (p. 29) the cathode is, however, found to have become less efficient, the potential being considerably higher than in the first experiment. Further repetitions yield curves almost identical with the second, the potential, however, still becoming slightly higher. When the cathode is cleaned after this, as before the first experiment, it again yields a curve similar to the first. In fig. 10 the first curve and one of its repetitions are shown.

Fig. 7 illustrates results obtained with a copper and a platinum cathode respectively. The experiments with the copper cathode were stopped after the first retrogression of potential. It will be seen that the latter occurs approximately at the same time in the experiments with the copper as in those with the platinum electrode. The difference between the actual values of the potentials is, however, very marked, the standard electrode at the beginning of the copper experiments showing a lower value than at the beginning of the platinum experiments. This corresponds to a greater efficiency, *i. e.*, a smaller expenditure of energy in the case of the copper than in that of the platinum electrode. In accordance, however, with the greater E.M.F. required for the liberation of hydrogen on a copper cathode than on one of platinum [‡], the copper curves attain higher values of potential than the platinum curves at the end of the experiment.

Fig. 9 represents experiments with a copper cathode carried out until evolution of hydrogen takes place.

In fig. 8 results are shown obtained with an electrode of platinum thickly coated with platinum black, the current-density being the same as in the experiments of fig. 7. The

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* Russ, *Ztschft. Phys. Chem.* xliv. p. 641 (1903).
† Luther and Brislee, *Ztschft. Phys. Chem.* xlv. p. 216 (1903).
‡ Caspari, *Ztschft. Phys. Chem.* xxx. p. 89 (1899).

low values of the potential represent the most conspicuous feature of these experiments.   As the electrode is not smooth, the time values are not comparable with those of fig. 7.

Alkaline Solutions of Nitrobenzene.

Fig. 11 represents results obtained with the alkaline solution of nitrobenzene.   As the solution at the cathode becomes heavier during electrolysis in an alkaline electrolyte, owing to accumulation of alkali, the cathode was placed horizontally at the foot of the cell in these experiments (fig. 2, p. 26).   At the points of maximum voltage hydrogen begins to be evolved.   The differences between the experiments with copper and platinum electrodes are similar to those observed in acid solution.   The time which elapses until hydrogen is evolved by a current of given density is considerably smaller than in acid solution.   This points to the conclusion that the diffusion coefficient of nitrobenzene is considerably smaller in alkaline than in acid solution, and possibly also that the reduction does not proceed quite so far before hydrogen is evolved.

Example of Application of Results.

In conclusion, we have to show how experiments carried out according to the method described may be employed in order to arrive at an estimate whether results that have been obtained in electrochemical investigations are due simply to concentration changes at the electrode that take place so rapidly that they cannot be neutralized by stirring, or whether they are really properties of the solution originally made up by the experimenter.   As an example, we shall investigate whether concentration changes can have taken place during the experiments by Haber, referred to in the introduction, with sufficient rapidity to explain his results.

We may approach such questions from two points of view. Firstly, we may make an arbitrary assumption regarding the time that must elapse until artificial stirring can completely renew the layer of liquid touching the electrode.   For example, we might assume that such a renewal takes place completely after the lapse of one second, or of one minute, and then calculate what is the maximum change of concentration that can have taken place.   Secondly, we may assume that owing to stirring the concentration of the electrolyte is always left unaltered at some definite distance, say $\frac{1}{10}$ mm., from the electrode, and then calculate the maximum change of concentration that is conceivable on this assumption.

Proceeding on the first plan, we have the equation[*]

$$C = C_0 - 2F\sqrt{\frac{t}{\pi k}}, \quad . \quad . \quad . \quad . \quad (1)$$

in which C_0 is the original concentration of the substance electrolysed, C its concentration after the time t, k its diffusion coefficient, and F the amount removed by electrolysis at the electrode per unit of surface and time.

From this equation we find the following value for the relative change of concentration at the electrode :—

$$\frac{C_0 - C}{C_0} = \frac{2F}{C_0}\sqrt{\frac{t}{\pi k}}. \quad . \quad . \quad . \quad . \quad (2)$$

Proceeding according to the second plan and indicating the distance from the electrode at which no change of concentration takes place by δ, we have the following relation according to Fick's diffusion law :—

$$F = k\frac{C_0 - C}{\delta},$$

from which we obtain for the relative change of concentration at the electrode the value

$$\frac{C_0 - C}{C_0} = \frac{\delta F}{k C_0}. \quad . \quad . \quad . \quad . \quad (3)$$

When the current is employed solely for reduction we have

$$F = \frac{i}{96540}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

concentrations being expressed in g.-equivalents or mg.-equivalents per c.c., and current-densities in amperes per sq. cm. or milamps. per sq. cm. respectively.

In order to eliminate k from these equations we may calculate its value from the experiments we have described in the present paper. For this purpose we must assume that nitrobenzene has been completely removed from the electrode by reduction to azoxybenzene when hydrogen is evolved. We then find k by applying equations (1) and (4) to our experiments. We thus obtain

$$k = \left(\frac{2i'}{96540\ C_0'}\right)^2\frac{t'}{\pi}, \quad . \quad . \quad . \quad . \quad (5)$$

quantities referring to our experiments being distinguished

[*] Sand, *loc. cit.* p. 53.

by dashes. The values for k derived from this equation are no doubt often vitiated by the fact that hydrogen may come off before complete reduction of the nitrobenzene has taken place, so that the values found for k are too small, and those for $\dfrac{C_0 - C}{C_0}$ too large. This does not matter seriously, however, as we only desire to find an upper limit for the latter quantity. Eliminating F and partly k from the equations (2), (3), (4), (5), we find the following relations :—

$$\frac{C_0 - C}{C_0} = \frac{i}{i'} \frac{C_0'}{C} \sqrt{\frac{t}{t'}}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (6)$$

and

$$\frac{C_0 - C}{C_0} = \frac{i}{96540} \frac{\delta}{k C_0}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7)$$

As an example, by which to test whether concentration changes played an important part in Haber's results, we take numbers from the paper recently published by him in conjunction with Russ[*]. According to Table II., p. 279, when a current of density $\dfrac{4 \cdot 5}{1 \cdot 5 \times 10} \dfrac{\text{milamps.}}{\text{cm.}^2}$ was passed through a vigorously stirred alkaline solution of nitrobenzene of the concentration $0 \cdot 8 \dfrac{\text{mg.-equiv.}}{\text{c.c.}}$, an absolute potential of $0 \cdot 0737$ volts was observed ; whereas when the current had a density of $\dfrac{36}{1 \cdot 5 \times 10} \dfrac{\text{milamps.}}{\text{cm.}^2}$, the potential was $0 \cdot 1171$ volts, the difference being $0 \cdot 0434$ volt. Now, according to the Table VII a. on p. 285, when a current of the density $\dfrac{5}{1 \cdot 5 \times 10} \dfrac{\text{milamps.}}{\text{cm.}^2}$ was employed in a solution of the concentration $0 \cdot 8 \dfrac{\text{mg.-equiv.}}{\text{c.c.}}$, the potential $0 \cdot 0993$ volt was recorded. In order to obtain a rise of potential of $0 \cdot 0434$ volt by reduction of the concentration of nitrobenzene, the current strength being left unchanged, we can infer by interpolation from numbers given in the same table that the concentration must be reduced to $0 \cdot 115 \dfrac{\text{mg.-equiv.}}{\text{c.c.}}$, *i. e.* to 14 per cent. of its

[*] *Loc. cit.*

former value. If the first experiments are to be explained as a result of local reduction of concentration at the cathode, such reduction must therefore amount to at least 86 per cent.

Now, in order to obtain an estimate of the magnitude of depletion at the cathode that actually took place, we refer to one of our experiments represented in fig. 11. Taking, for example, curve 4, we have

$$i' = 1{\cdot}184 \frac{\text{milamps.}}{\text{cm.}^2}, \ t' = 460'', \ C_0' = \frac{5 \times 3}{123} \frac{\text{mg.-equiv.}}{\text{c.c.}} .$$

Employing equation (6), we find that if the liquid touching the cathode is renewed completely after every second by stirring, the lowering of concentration could never go below 1·4 per cent. I have shown formerly[*] that under more unfavourable circumstances, viz., when the cathode was mercury contained in a cup, it is possible to prevent depletion, which would require 0·4 sec. if the liquid were stationary; it seems improbable, therefore, that the lowering of concentration in Haber's experiment went below the above value.

By employing equations (5) and (7), we find that if the concentration of the liquid be maintained continually at its original value by stirring at $\frac{1}{10}$ mm. distance from the cathode, the utmost lowering that could take place would amount to $5\frac{1}{4}$ per cent. Brunner[†] calculates the distance from the electrode to which he was able to keep the concentration unchanged in his experiments to have been $\frac{1}{50}$ to $\frac{1}{20}$ mm., so that, adopting his values, we should conclude that the lowering of concentration did not amount to more than about $1\frac{1}{2}$ per cent. Starting from other curves on fig. 11, we obtain similar values. In any case the results show that we cannot assume changes of concentration amounting to 86 per cent. to have taken place in Haber's experiments, and that the variations of electrode-potential with current-density observed by him can therefore not be explained as the result of concentration changes at the cathode.

III. *On Transfinite Cardinal Numbers of the Exponential Form.* By PHILIP E. B. JOURDAIN, *B.A., Trinity College, Cambridge* *.

AMONG cardinal numbers of the form

$$\mathfrak{a}^{\mathfrak{b}},$$

where $\mathfrak{b}$, at least, is transfinite, the smallest and most interesting is the cardinal number of the number-continuum :

$$2^{\aleph_0} = \nu^{\aleph_0} = \aleph_0^{\aleph_0} = \aleph_1^{\aleph_0}.$$

Cantor has always been of the conviction that

$$2^{\aleph_0} = \aleph_1,$$

and investigations in the theory of manifolds tend to increase one's belief in the truth of this conviction, although hitherto no proof of it has been given. It is very important to prove that $2^{\aleph_0}$ is equal to some Aleph in order to be certain that the number-continuum is not what I have called an "inconsistent aggregate" †.

A failure to prove the above equality by an attempted arrangement of all the real numbers between 0 and 1 in a well-ordered series ultimately led me to the result of § 1, that the cardinal number of all the real numbers which can be represented by fundamental series of which the general term is known as a rational function of its index is $\aleph_0$, which proves that it is impossible to obtain a series of type ω_1 from such numbers, and consequently the impossibility of actually proving that

$$2^{\aleph_0} = \aleph_1$$

in a large class of cases.

This negative result, which is the only definite result I have as yet been able to obtain on the question of the equality

$$2^{\aleph_a} = \aleph_{a+1},$$

where α is any ordinal number, allows, however, a number of conclusions to be drawn in what I have called the " cardinal theory of functions " (§ 2). The result that only a small

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* Communicated by the Author.

† Phil. Mag. January 1904, p. 66. In § 5 (p. 67) of this article I tacitly assumed that the exponential numbers in question belonged to *consistent* aggregates, or manifolds ; for, though this is not rigorously proved to be the case, nothing seems more unlikely than that it should not be so. Further information on the subject of inconsistent aggregates is given below, §§ 6–9.

portion of the whole manifold of analytic functions, for example, are analytically representable by no means implies that general theorems cannot be found which apply to *all* analytic functions, and even in particular to all which are not representable (§ 3) ; so that the concept of function taken by Pringsheim, in the recent *Encyclopädie der mathematischen Wissenschaften* *, as the basis of the general theory of functions, appears to be too narrow.

After a digression on the cardinal theory of functions and on the utility of the concept of the " aggregate of definition " (§ 4), I prove (§ 5) a theorem due to Bernstein on exponential numbers, which includes a result of my own † as a special case, and allows us to find the necessary and sufficient conditions that

$$\mathfrak{a}^{\mathfrak{b}} = \mathfrak{a},$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are any cardinal numbers.

In § 6, I make a few remarks on the extended principle of induction used in § 5, which serves to define the series W of ordinal numbers. The series $(\mathfrak{W})$ such that every well-ordered series is ordinally similar either to $\mathfrak{W}$ or to a segment of $\mathfrak{W}$ extends beyond W (§ 7), and this more exact account of W throws a clearer light on my solution of Burali-Forti's contradiction (§ 8).

Finally, in § 9, I revert to the consideration of the concept of " consistency," with especial reference to investigations of Cantor, Hilbert, and Russell.

1.

Every real number is determined by an enumerable sequence of rational numbers, and hence the cardinal number of the aggregate of real numbers is seen without difficulty to be $2^{\aleph_0}$. But, if this enumerable sequence is

$$u_1, \ u_2, \ \ldots, \ u_\nu, \ \ldots, \qquad \cdots \quad \cdots \quad (1)$$

we must, if we are to be able to determine exactly the real number in question, limit the form of u_ν to be a function obtained by performing the elementary operations a finite number of times on ν and a finite number (n) of given rational numbers ; in symbols

$$u_\nu = f(\nu, \ r_1, \ r_2, \ \ldots, \ r_n).$$

By this limitation, the cardinal number of the aggregate of

the real numbers which can now be represented by the sequence (1) is merely $\aleph_0$. This may be shown as follows.

The function obtained by performing the elementary operations a finite number of times on $(n + 1)$ arguments is a rational function of arguments, the coefficients of which are integers *. Since, then, in each case we only have a finite number (m) of coefficients to choose, and each coefficient can be chosen out of $\aleph_0$ values (the integers), the cardinal number of those functions of m coefficients is

$$\aleph_0^m.$$

Further, we get *all* such functions by giving m all possible finite values in turn; consequently the cardinal number of all these functions is

$$\aleph_0^1 + \aleph_0^2 + \ldots + \aleph_0^m + \ldots,$$

the series being of type ω, and consequently—remembering that each term reduces to $\aleph_0$—the cardinal number in question is

$$\aleph_0 \cdot \aleph_0 = \aleph_0.$$

We may state this result in words as follows: The cardinal number of all the real numbers that we can *actually* determine (that is to say, determine in the sense explained above) is

$$\aleph_0.$$

Accordingly, if, as is the case with some methods that suggest themselves for arranging real numbers in a well-ordered series, we only use such "actually determinable" real numbers, we can never arrange them in a series of type ω_1. For every enumerable manifold can be well-ordered, but the series always breaks off before some number of the second number-class is reached.

Now, this conclusion has applications, which seem to me to be of some importance, in the theory of functions. In the first place, such sequences as (1) enter into Weierstrass' construction of whole transcendental functions with given zeros, Mittag-Leffler's construction of analytic functions whose singularities form an aggregate whose first derivative is enumerable, and the construction of whole transcendental functions which take given values at certain points †. We

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* *Cf.* Harnack, "An Introduction to the Elements of the Differential and Integral Calculus," Eng. trans. p. 67 (1891).

† This construction, which forms an extension of Lagrange's interpolation-formula to whole transcendental functions, is given by me in part of an essay " On the General Theory of Functions," which is to appear shortly in Crelle's *Journal für Math.* It is very simple, and is obtained by the multiplication of a whole function constructed by Weierstrass' theorem with a meromorphic function constructed by Mittag-Leffler's theorem.

conclude then that, although the cardinal number of any of the above classes of functions is $2^{\aleph_0}$, the cardinal number of those functions which we can actually represent is $\aleph_0$. Of course, we make the same stipulation as to representability in the case of the extraneous factor in all these constructions. In Weierstrass' construction this factor is

$$e^{g(z)},$$

where $g(z)$ is any whole function. Thus, we cannot, for example, consider

$$c \cdot P(z)$$

as a constructible function, if $P(z)$ is the product of primary factors and c is *any* real number; for c must be a representable real number.

In the second place, it appears that the postulate of " arithmetical definability," which Pringsheim has introduced as an essential qualification of the functions which can be treated in a general theory of functions, cannot be considered as relevant, for the double reason that it is necessary to take account of functions which cannot be defined by $\aleph_0$ conditions and that even functions which are so definable are not in general " arithmetically representable." The former reason rests on a theorem which constitutes an important part of what I have called " the cardinal theory of functions "; the latter reason rests on a theorem which is easily obtained from what precedes and completes, in a sense, the cardinal theory of functions.

2.

The cardinal theory of functions consists of two parts : The determination of the cardinal numbers of the various aggregates of functions, and the drawing of conclusions, from inequalities between these numbers, as to the non-inclusion of certain aggregates in certain others. Thus, from the results that the cardinal number of all integrable functions is

$$2^{2^{\aleph_0}},$$

while that of all functions representable as limits of sequences of continuous functions is

$$2^{\aleph_0},$$

and

$$2^{2^{\aleph_0}} > 2^{\aleph_0},$$

we conclude that a function, even when it is restricted to be

integrable, is not, in general, representable as the limit of a sequence of continuous functions [*].

This example suffices to substantiate the contention that the requirement of arithmetical definability is unnecessarily narrow for the possibility of a general theory of functions. In other words, there exist propositions in the general theory of functions (on integrable functions, for example) which apply to a much wider class of functions than that of arithmetically definable functions.

Now the class of functions which can be represented as limits of infinite series of continuous functions, or, what is the same thing, of functions to which an " existence-theorem" is applicable, contains, of course, all arithmetically definable functions, but not inversely. For every function of the former class is completely and uniquely determined by the datum of the enumerable sequence of the coefficients of the sequence of polynomials by which it can be replaced, and the cardinal number of the sequences (1) whose general term can be found in the manner indicated, but in which, possibly, a finite number of terms are completely arbitrary, is

$$\aleph_0,$$

and

$$2^{\aleph_0} > \aleph_0.$$

3.

Although there is thus no possibility of actually constructing a greater cardinal number of functions than $\aleph_0$, it by no means follows that definite theorems cannot be found which hold for a greater number. The fact of the existence of a general theory of analytic functions is alone sufficient to disprove this, and, consequently, also for this reason the requirement of the arithmetical definability of functions is too narrow.

Further, it is interesting to see that there is a theorem which holds of actually non-representable analytic functions, due to Borel and Fabry [†]. The series

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_\nu z^\nu + \ldots, \quad . \quad . \quad . \quad (2)$$

where z is a complex variable, represents either the whole of an analytic function or part of one within a circle on whose circumference is at least one singularity. The theorem of

[*] Messenger of Math. Sept. 1903.
[†] *Cf.* Hadamard, ' La série de Taylor et son prolongement analytique,' Paris, 1901, pp. 33–36.

Borel and Fabry now says that, when the sequence

$$a_0, a_1, \ldots, a_\nu, \ldots,$$

is a "série écrite au hasard."—that is to say, a sequence whose general term cannot be given in the manner explained above,—this circle of convergence is a line entirely composed of essential singularities, so that (2) cannot be continued beyond this line, and represents the whole function.

4.

I have regarded the conception of an "aggregate of definition" as an essential part of the cardinal and ordinal theories of functions. By an "aggregate of definition" I understand any aggregate of values among those of the independent variable such that, when the values of the (one-valued) function are given for the points of merely *this* aggregate, the values for all other points in the domain of existence are determined. When the domain of the variable is the continuum of real numbers, the cardinal number of this aggregate, when the function is continuous or analytic, is

$$\aleph_0,$$

and the ordinal types are respectively

$$\eta \text{ and } \omega \text{ or } {}^*\omega.$$

Since, however, a knowledge of the means whereby the value of the function at one point is calculated from its values at other points (which varies for different classes of functions) appears indispensable in addition to a knowledge of the aggregate of definition, and the latter knowledge then follows from the former, it might appear that the "aggregate of definition" is always a superfluous conception. The following example will show that this is not the case.

It has long been known that a one-valued analytic function $f(z)$ reduces to a constant if it has a period smaller in absolute amount than any assignable positive number. This pro-position, without the necessary restriction to one-, or at least finite-valuedness, was treated, without complete justification, as obvious by Jacobi; so that Weber in the reprint[*], edited by him, of Jacobi's memoir, has given a proof of the proposition in question together with an analogous theorem on one-valued continuous functions of a real variable.

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[*] "Ueber die vierfach periodischen Functionen zweier Variabeln, auf die sich die Theorie der Abel'schen Transcendenten stützt," von C. G. J. Jacobi (Crelle's *Journal*, Bd. xiii. (1834) ; Ostwald's *Klassiker d. exakten Wiss.* No. 64, hrsgb. von H. Weber, Leipzig, 1895, pp. 36-39).

Now, these theorems, together with an extension, follow at once from a consideration of the character of the aggregates of definition.

Suppose that a one-valued analytic function, $f(z)$, with a period smaller in absolute amount than any real positive number without being a constant exists, and let $f(z)$ have a finite value for every point z such that $|z| < a$, where a is some positive constant. Then there must be a sequence of points $\{z_\nu\}$ condensing at some point z within the circle and such that $|z_\nu| < a$, such that

$$f(z_\nu) = f(z) = A.$$

Now $\{z_\nu\}$ forms an aggregate of definition, and consequently

$$f(z) = A.$$

Further, if x and $F(x)$ are real, and $F(x)$ is one-valued and continuous, and $F(x)$ has a period of the above nature, it is easy to see that the points where $F(x)$ is equal to some number A lie everywhere dense, and thus form an aggregate of definition of the continuous function. But it is evident that this argument applies also to the case where x and $F(x)$ are complex and the periodicity of $F(x)$ is double. Hence, a real or complex function of a real or complex variable cannot have a (respectively single or double) period smaller in absolute amount than any positive non-zero number *provided only that the function is continuous.*

5.

I now return to the consideration of exponential numbers in general, and prove the theorem of Bernstein * that, if $\aleph_\alpha$ and $\aleph_\beta$ are any two Alephs,

$$\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha \cdot 2^{\aleph_\beta}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the first part † of Bernstein's proof, (3) is proved if

$$\aleph_\beta \geqq \aleph_\alpha,$$

* "Untersuchungen aus der Mengenlehre," *Gött. Diss.*, Halle-a.-S., 1901, pp. 49–50.

† A more general theorem than that of the *first* part was proved by me before I had seen Bernstein's memoir (Phil. Mag. March 1904, p. 302), namely: if $\aleph_\alpha \leqq 2^{\aleph_\beta}$, then $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta}$.

I take this opportunity of correcting two slips in this paper:

p. 303, line 7; *delete* " $\leqq 2^{\mathfrak{a}}$."

line 18; *for* "$\aleph_1^{\aleph_0} = \aleph_1$" *read* "$\aleph_1^{\aleph_0} = 2^{\aleph_0}$."

and in the second part it is supposed that

$$\aleph_\alpha > \aleph_\beta :$$

and an extended form of complete induction, which extends to all Alephs, is used, and is, in essentials, as follows.

By Cantor's definition of an exponential number,

$$\aleph_\alpha^{\aleph_\beta}$$

is the cardinal number of all coverings (Belegungen) of a manifold of cardinal number $\aleph_\beta$ with the elements of a manifold of cardinal number $\aleph_\alpha$, which we will suppose to be arranged in a series of type ω_α. Now every such covering is obtained by the covering of the manifold of cardinal number $\aleph_\beta$ with some (or all) elements of some segment of the above series of type ω_α; and the cardinal number of this segment is less than $\aleph_\alpha$. Hence, each of the coverings first-named is found among the aggregate of all the coverings of the manifold of cardinal number $\aleph_\beta$ with the elements of the various segments, taken in turn, of the series of type ω_α. Now, the cardinal number of all the coverings of a manifold of cardinal number $\aleph_\beta$ with the elements of a segment M_γ of the above series M_{ω_α} of type ω_α is

$$\mathfrak{m}_\gamma^{\aleph_\beta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\mathfrak{m}_\gamma$ is the cardinal number of M_γ. Hence we can state, by the Schröder-Bernstein theorem, that

$$\aleph_\alpha^{\aleph_\beta} \leq \sum_{\gamma < \omega_\alpha} (\mathfrak{m}_\gamma^{\aleph_\beta}), \quad . \quad . \quad . \quad . \quad (5)$$

where the Σ means that the summation is to be extended over all the numbers (4) such that M_γ is a segment of M_{ω_α} (or such that $\gamma < \omega_\alpha$).

On the other hand,

$$\mathfrak{m}_\gamma^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta},$$

and hence the right-hand side of (5) is less than or equal to

$$\sum_{\gamma < \omega_\alpha} (\aleph_\alpha^{\aleph_\beta}) = \aleph_\alpha \cdot \aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}.$$

Comparing this result and (5), we conclude that

$$\aleph_\alpha^{\aleph_\beta} = \sum_{\gamma < \omega_\alpha} (\mathfrak{m}_\gamma^{\aleph_\beta}) . \quad . \quad . \quad . \quad . \quad (6)$$

Hence, if we know a theorem for all Alephs less than $\aleph_\alpha$, we may, by substitution in the right-hand side of (6), find

the same theorem for $\aleph_a$. Such a theorem is

$$\aleph_a^{\aleph_\beta} = 2^{\aleph_\beta} = \aleph_a \cdot 2^{\aleph_\beta},$$

where

$$\aleph_a \leqq \aleph_\beta \; ;$$

and (6) allows us to conclude the validity of the theorem (3), even if

$$\aleph_a > \aleph_\beta.$$

From (3), we may deduce an interesting theorem concerning those cardinal numbers which are unaltered by exponentiation. In fact, from (3) and the laws of multiplication of Alephs *, it follows that if, and only if

$$2^{\aleph_\beta} \leqq \aleph_a, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the right-hand side of (3) reduces to $\aleph_a$.

Thus, that

$$\aleph_a^{\aleph_\beta} = \aleph_a, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

it is necessary and sufficient that (7) should hold. In particular, if, as is probable, we can assert (8) if only

$$\aleph_\beta < \aleph_a,$$

it is necessary and sufficient that

$$2^{\aleph_\beta} = \aleph_{\beta+1}.$$

6.

In the extended principle of induction used above, which may be stated thus : If a certain proposition P holds of $\aleph_0$, and if, when it holds of all Alephs less than $\aleph_a$, it holds of $\aleph_a$, P holds of all Alephs ; the proof of P for $\aleph_a$ is reduced, by (6), to the proof of P for a sum (of cardinal number $\aleph_a$) of numbers for which P is assumed to hold. This method cannot be applied to give a shorter proof of the equality †

$$\aleph_\gamma^\nu = \aleph_\gamma,$$

since we must have previously proved that

$$\aleph_\gamma^2 = \aleph_\gamma$$

in order to prove that the cardinal number of a series of type $\omega_{\gamma+1}$ is greater than that of a series of type ω_γ ‡. But if the exponent, instead of being ν, is transfinite, we can, as we

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* Phil. Mag. March 1904, p. 301.
† *Ibid.* p. 300. This theorem seems, from an indication given by Bernstein (*op. cit.* p. 49), to have been known to Cantor.
‡ *Ibid.* Jan. 1904, p. 74.

can easily convince ourselves, apply the extended method of induction, provided that exponentiation with this transfinite number leaves *some* Aleph unaltered.

This extended principle of induction is very closely connected (through Cantor's "third principle of generation") with the question as to whether the ordinal number of the series of all the ordinal numbers defined by Cantor can be defined without contradiction, and hence with the argument of Burali-Forti [*]. I have returned, then, in the following section, to the considerations which I have advanced in the January (1904) number of this Magazine.

7.

In defining an aggregate which should serve as a criterion whether any given aggregate is "consistent" or "inconsistent" [†], I have used the conception, mentioned by Schönflies, of the (well-ordered) series 𝖂 [‡] such that every well-ordered series is similar to it or to a segment of it.

This series 𝖂 was, now, stated by Schönflies [§] to be similar to the series (W) of all the ordinal numbers, as defined by Cantor by the help of his three generating principles [‖].

This statement appears to me to be incorrect; in fact, I shall now show that we must agree to regard the series of these "Cantor's ordinal numbers" as similar to a *segment*

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[*] *Ibid.* p. 64.

[†] *Ibid.* p. 67, line 18. The wording in the definition of W is to be replaced by the slightly different wording given above.

[‡] We consider in the criterion the *aggregate* which is the field of the generating-relation of the *series* 𝖂.

[§] " Die Entwickelung . . . ." p. 41.

[‖] The purpose of the third principle of generation is sometimes misunderstood. For example, in the in some respects excellent fourth Note (" Sur la théorie des ensembles et des nombres infinis ") on pp. 617-655 of Couturat's book " De l'infini mathématique," Paris, 1896 (see esp. pp. 639–642), the object of this principle is taken to be to enable one to surpass the second number-class, just as the second principle has enabled one to surpass the first. This view seems to agree with that of Schönflies (*op. cit.* p. 48; *cf.* Phil. Mag. March 1904, p. 300); but rather further on, a different, and self-contradictory, view of this object is taken. The third principle shows, namely, the occasion for using the second principle to create a new number *after all* those generated by the application of the first two principles to a fundamental number ω, Ω, . . .

The true view was clearly stated by Cantor in his 'Grundlagen.' The first two principles create an infinite series of ordinal numbers, while the third principle enables us to separate out various number-classes in this series (*cf.* §§ 7, 8).

merely of 𝔐. The ground of this lies in the fact that Cantor's third, or limiting principle, which applies to all ordinal numbers, does not apply to certain well-ordered *aggregates*, which transcend even the series of all the transfinite ordinal numbers of Cantor.

In order to state shortly what is contained in the third principle, it is convenient to single out the first number of each of the number-classes as the "class-characteristic" of all the other numbers of that class. We thus define the "class-characteristic" of any ordinal number α as either α itself, if α is the first number of a number-class ($\alpha = \omega_\gamma$) *, or, if not, the first number (ω_γ) after α which is the first of a number-class.

Then the principle in question can be stated :—

The cardinal number of all the ordinal numbers preceding the class-characteristic ω_γ of a given ordinal number is $\aleph_\gamma$.

Let us now consider whether the series of all the ordinal numbers which are subject to the third principle has a type : in other words, whether the assumption that it has a type leads to a contradiction, as was the case in Burali-Forti's argument. Let the type be β : then β is its own class-characteristic †, say $\beta = \omega_\alpha$. To find the cardinal number of all the ordinal numbers preceding β, we notice that every Aleph less than $\aleph_\beta$ (that is to say, every Aleph whose suffix is less than β) is the cardinal number of some segment of the series of type β, so that the cardinal number in question is at least equal to $\aleph_\beta$. That it is also at most equal to $\aleph_\beta$ is evident from the fact that $\aleph_\beta$ is the next greater Aleph to the series of Alephs of all the segments. Thus the cardinal number of the β ordinal numbers is

$$\aleph_\beta \text{ or } \aleph_{\omega_\alpha},$$

and, since it is not $\aleph_\alpha$, the third principle does not appear to be satisfied.

However ‡, although ω_α can never be equal to α when α is a Cantor's ordinal number, it does not follow that β is not equal to ω_β. And, in fact, this is so, as the following considerations show.

The series of Cantor's ordinal numbers is known to be

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* See the notation in Phil. Mag. March 1904, p. 295.

† For if β is not the first number of a class, there are predecessors of the same class. But every predecessor of β belongs to one of Cantor's number-classes which is itself surpassed by a Cantor's number-class.

‡ My attention was called to this point, which I had overlooked, by a remark of Mr. G. H. Hardy, Fellow of Trinity College, Cambridge.

ordinally similar to the series of Alephs, or, what is the same thing, to the series of class-characteristics :

$$\omega, \; \omega_1, \; \omega_2, \; \ldots \; \omega_\omega, \; \ldots \; \omega_\gamma, \; \ldots.$$

Hence, to every class-characteristic ω_γ of Cantor's ordinal numbers corresponds one, and only one, Cantor's ordinal number γ, and *vice versâ.* Thus, if

$$\beta = \omega_a,$$

a cannot be a Cantor's ordinal number, for, if it were, β would be one too. Further, β or ω_a (if it exists) is the least ordinal number which is greater than all Cantor's ordinal numbers, so

$$\alpha = \beta.$$

Accordingly, if β exists, the third principle is satisfied, in spite of first appearances, by the series of all Cantor's ordinal numbers; and the (Burali-Forti's) contradiction resulting herefrom leads us to deny the existence of β, the type of W.

Now, the series W is well-ordered *, although it cannot have a type, and evidently other well-ordered series (having no types) transcending W, can be formed. So we must conclude that the series W is similar to a segment merely of the series (𝔚) such that every well-ordered series is similar either to it or to a segment of it †.

We can define a series ordinally similar to W by positing one element and then positing successive elements according to Cantor's first and second principles. It results from our considerations that the ordinal number of every element thus formed is subject to Cantor's third principle ; that is to say, we cannot, without contradiction, speak of an ordinal number of an element which follows all those whose ordinal numbers obey the third principle. In other words, we cannot, as seemed possible if we assumed that

$$\omega_\gamma > \gamma$$

always, define ordinal numbers which transcend all Cantor's ordinal numbers. The name of " principle of limitation " may, then, convey the wrong impression that the series W is not, as we shall say in the next section, " absolutely " infinite ‡.

* Phil. Mag. Jan. 1904, pp. 65–66.

† This is the series described, in not quite such accurate terms as the above, in Phil. Mag. Jan. 1904, p. 67, lines 18–19. It follows from the above that 𝔚 can *not* be used as a substitute for W in a criterion of " consistency."

‡ The " absolute " infinity of W was *stated* by Cantor in 1882 ('Grundlagen . . .,' p. 44).

8.

The series of all ordinal numbers may, it seems to me, properly be called an "*absolutely*" infinite series. For, if a well-ordered series has a type, it is, in a certain sense, completed ; while the above series W cannot, as is shown by Burali-Forti's contradiction, have a type.

This seems to be the most promising way of regarding Burali-Forti's contradiction, and the words "absolutely infinite" seem preferable to the equivalent word "inconsistent," which I, in common with Cantor, have used hitherto ; because an "inconsistent" aggregate is not itself contradictory (it exists, in the mathematical sense of the word), but a cardinal number or type of it does not exist. However, I shall, in the next section, enter briefly into the history of the use of this word in the theory of aggregates.

9.

The conception and name of an "inconsistent" aggregate originated with Cantor *, but the only published reference to them occurs in two papers by Hilbert †.

With regard to Hilbert's statements, it does not seem to follow that if the axioms of arithmetic (which are, according to Hilbert, the laws of operation with real numbers and the axiom of continuity) do not contradict one another, then the real number-continuum is "consistent." For it does not appear to be doubtful that the laws of operation with ordinal numbers or Alephs form a system free from contradiction, and yet the aggregate of all ordinal numbers or Alephs is "inconsistent."

Further, Hilbert states that a "similar" method to that pursued by him for the axioms of real numbers, when applied to all Alephs, fails, so the totality of all Alephs is an "inconsistent" aggregate (a mathematically non-existent

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* In a letter to me of January 6th, 1901, Professor Cantor said :— "Ich unterscheide auf's strengste zwischen unendlichen *Mengen* (consistenten Vielheiten) einerseits und den ihnen zukommenden abstracten unendlichen *Zahlen* andrerseits." There was no further explanation of the term " consistency," and I confused it with Schröder's requirement in the conception of a " common manifold " (' Vorlesungen über die Algebra der Logik (exakte Logik),' Bd. i. 1890, pp. 147–148). On finding that the aggregate of all ordinal numbers had no cardinal number, I applied the name " inconsistent " used by Schröder (' Algebra und Logik der Relative,' 1895, p. 4) to this aggregate (Phil. Mag. Jan. 1904, p. 67).

† " Ueber den Zahlbegriff," *Jahresber. d. d. M.-V.* Bd. viii. (1900) pp. 180–184 ; " Mathematische Probleme," *Gött. Nachr.* 1900, pp. 253–297, see especially pp. 264–266.

conception). There is, however, so I contend, no reason for thus denying existence to the totality of Alephs, but only for denying the existence of the cardinal number of this aggregate. This indicates the difference between my conception of "inconsistency" and that of Hilbert.

Cantor * has defined a "consistent" aggregate (consistente Vielheit) as such that the supposition of a collection by the mind of all its elements to one thing leads to no contradiction. Since this collection was considered by Cantor as the essential thing in his definition of "Menge," and hence of cardinal number †, this definition tends to agree with mine, in opposition to Hilbert's. But Cantor's definition is not of the nature of the (nominal) definitions in the symbolic logic of Peano and Russell, but rather a "phrase indicating what is to be spoken of" ‡.

So I replaced Cantor's definition, in my first paper §, by a formal definition, and I contend that the necessary limitation, noticed by Russell ‖, but not discovered by him, in the notion of a "class" is supplied by introducing the postulate of "consistency." For Russell's contradiction seems to arise solely from the use of Cantor's inequality

$$2^{\mathfrak{a}} > \mathfrak{a},$$

where $\mathfrak{a}$ is supposed to be the cardinal number of an *inconsistent* class, such as the class of all propositions ¶.

Although we have thus arrived at the formulation of the restricted concept of "class," the "search with a mental telescope" ** for this concept appears difficult, and Cantor's "definitions" are, I think, to be regarded as attempts in such a search.

The idea of an inconsistent aggregate as an *absolutely* infinite one (§ 8)—a term also used by Cantor—appears to me to be suggestive. For then finite and transfinite aggregates (which are now both subject to mathematical operations) appear, after suitable rearrangement, as segments of an infinite whole (which is not thus subject). And thus the relation of this infinite to the transfinite aggregates has a

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* In a letter of November 4th, 1903, referred to in Phil. Mag. Jan. 1904, pp. 67–70.

† See *Math. Ann.* Bd. xlvi. 1895, pp. 481–482, 497.

‡ Russell, 'The Principles of Mathematics,' vol. i., Cambridge, 1903, p. 304. *Cf.* Russell's definition of a cardinal number as a class, pp. 305, 111–116.

§ Phil. Mag. Jan. 1904, p. 67.

‖ *Op. cit.* p. 20; *cf.* pp. 366–368, 101–107.

¶ This is also the opinion of Prof. Cantor (letter of July 9th, 1904).

** Russell, *op. cit.* preface, p. v.

certain analogy with the relation of the transfinite aggregate
of type ω to the finite aggregates.

10.

I will now sum up the results of my investigations on
the transfinite numbers, published in three papers in this
Magazine.

The main result is that any aggregate, the cardinal number
(or type) of which is not self-contradictory, can be well-
ordered. A closer consideration of the proof given in my
first paper led (in the present paper) to a proof of the
universal validity of Cantor's third principle, and hence of
the non-existence of ordinal numbers and Alephs which
transcend all those defined or indicated by Cantor; and (in
the first and second papers) the main theorem led to final
forms that are to be given to the results of adding and
multiplying any two transfinite cardinal numbers. The
results on exponential numbers are not final, but one of the
theorems on exponential numbers (§ 1 of the present paper)
has been shown to have an important bearing on the theory
of functions.

The Manor House, Broadwindsor, Dorset.
        September 6th, 1904.

IV. *The Molecular Weights of Radium and Thorium
Emanations. By* WALTER MAKOWER, *B.A., B.Sc.*[*]

PART I.
THE MOLECULAR WEIGHT OF RADIUM EMANATION.

1. *Introduction.*

RUTHERFORD and Brooks (Trans. Roy. Soc. Canada,
1901; Chem. News, 1902) have determined the rate
of diffusion of the emanation from radium into air by a
method similar to that employed by Loschmidt in his inves-
tigations on the coefficient of interdiffusion of gases, and
deduce that the molecular weight lies between 40 and 100.
It is important to know the molecular weight with greater
accuracy, and it was with the object of solving this problem
that the present investigation was undertaken. During the
course of the work, Curie and Danne (*C. R.* cxxxvi. p. 1314,
1903) have published some observations upon the rate of
diffusion of the emanation from radium through capillary
tubes of different lengths and diameters, and find for the
coefficient of interdiffusion between the emanation and air

[*] Communicated by Prof. J. J. Thomson.

the value ·100, which gives a molecular weight corresponding with the lower limit given by Rutherford and Brooks (*loc. cit.*). It is possible, however, that the value ·100 is somewhat too high, since no precautions seem to have been taken to insure that temperature conditions were steady. In the experiments to be described below, the emanation was made to diffuse through porous plugs, and its rate of diffusion compared with that of other gases of known molecular weight. Assuming Graham's law, that the rate of diffusion of a gas is inversely proportional to the square root of its density, it is possible from such a comparison to calculate the molecular weight of the emanation.

2. *Description of Apparatus, and Method of Experimenting with the Emanation.*

The air containing the emanation, the rate of diffusion of which was to be determined, was confined over mercury in
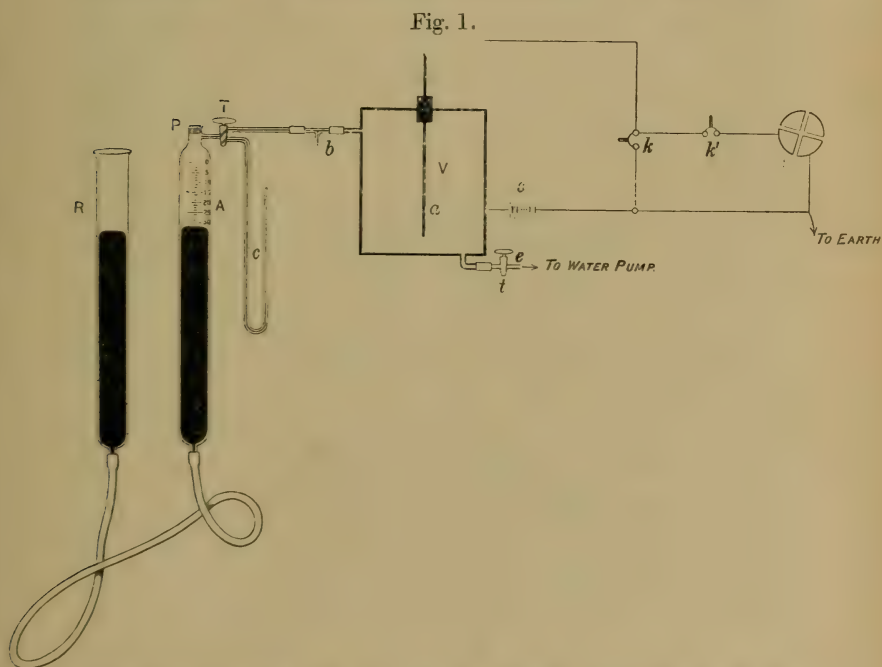
Fig. 1.

the glass vessel A (fig. 1), which will be referred to as the " diffusion-vessel," and which was 3 centimetres in diameter and graduated downwards in cubic centimetres from 0 to 200. The volume of the " dead-space" from the tap T to the first

graduation was measured by completely filling the diffusion-vessel with mercury, and then running out the mercury to the first graduation and weighing; for the first vessel used this volume was found to be 5·0 c.c., and for the second vessel 12·4 c.c. The upper end of the diffusion-vessel was drawn off and closed by a dense porous plug of plaster-of-Paris (P), 1 cm. in diameter and ·5 cm. thick, fitting tightly into the glass*. Just below the porous plug was a three-way capillary stopcock, T, provided with a syphon-pipette for collecting gases over mercury when desired. The lower extremity of the diffusion-vessel was connected by stout rubber pressure-tubing with a cylindrical mercury reservoir R of 3 cm. diameter which could be raised or lowered at will. By means of a short piece of rubber pressure-tubing the diffusion-vessel was connected to a cylindrical brass vessel V, along the axis of which ran an insulated brass wire a connected to earth through the key k, by raising which the wire a and its connexions could be insulated; a difference of potential of about 400 volts was maintained between a and the outside of the brass vessel by a battery of 200 small storage-cells S, which gave a field sufficient to saturate the air in V. All the connexions were carefully shielded from external electrostatic effects by inclosing them in earthed conductors.

The emanation was obtained from a solution of radium bromide in water. By lowering the reservoir R, a stream of air could be carried through the solution and thence through cotton-wool to remove dust, and through a small sulphuric-acid bubbler along the capillary c into the vessel A.

To carry out an experiment, a certain volume of air mixed with emanation was sucked into A and well mixed by raising and lowering the reservoir R several times in succession. The volume of air in A was then read to ·1 c.c. and a known volume carried into V. It was important not to force any air through the porous plug during this operation, and therefore to prevent this the pressure of the air in V was reduced by means of a water-pump attached at e, after which the stopcock t was closed. The stopcock T was then turned so as to allow a certain volume of gas to pass into V, the mercury reservoir R being meanwhile raised and the level of the mercury in A and R finally adjusted to equality. Care was always taken that the pressure in V had been so far reduced, that at the end of this operation the pressure was still below atmospheric. By turning the tap t communication

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* The plaster-of-Paris was inserted while still moist, and after it had been compressed was allowed to set in position.

was again established between the air in V and the out-
side, and the pressure inside thus again brought to atmo-
spheric. After a short known interval (usually 1 minute)
the conductivity of the air in V was determined thus :—
At a definite instant the key *k* was removed so as to insulate
the rod *a* and its connexions; after a known time (τ seconds)
the key *k′* was removed and the key *k* replaced, and the
deflexion of the electrometer-needle read as soon as it had
come to rest, and taken as a measure of the quantity of
emanation which had been carried into V. *Immediately*
after disconnecting the key *k′* the tap *t* was again turned
so as to connect V to the water-pump, and a rapid current
of air drawn through it by opening the side tube *b*, which
had remained closed by a rubber cap during the first part
of the experiment. In this way the emanation was com-
pletely removed from V in a few minutes, after which
the radioactivity " induced " on the walls of the vessel
decayed according to the usual exponential law, becoming
negligible before the next experiment was performed. After
a known time the operations described above were repeated,
and the electrometer deflexion in τ seconds again recorded.
Since it was impossible to transfer exactly the same volume
of air at the beginning and end of an experiment, a correction
had to be applied to allow for this difference; a further
small correction had to be applied for any slight change of
sensitiveness of the electrometer during the experiment.

It was also necessary to allow for the natural decay of the
radioactivity of the emanation in the time during which the
diffusion had been proceeding. The correction to be applied
on this account was always small, and in the preliminary
experiments has been neglected; the method of applying it
when necessary will be discussed when giving the final
observations.

In most of the experiments, the scale of the electrometer
was uniform; in one set of observations, however, it was
found necessary to calibrate the scale and apply suitable
corrections to the observations (Plug I. Table III.).

If y is the concentration of the emanation in the diffusion-
vessel at any instant, then the change of concentration dy in
the time dt due to diffusion is given by the equation

$$dy = -\frac{\lambda}{V} y \, dt, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which V is the volume of the gas diffusing, and λ is a
constant depending on the nature of the porous plug.

If y_0 is the value of y at the beginning of the experiment,

i. e. when $t = 0$, we obtain from equation (1), by integrating, the equation

$$\log\left(\frac{y_0}{y}\right) = \frac{\lambda}{V} t,$$

whence

$$\lambda = \frac{V}{t} \log_e\left(\frac{y_0}{y}\right) = \frac{V}{\cdot 4343\, t} \log_{10}\left(\frac{y_0}{y}\right). \quad . \quad . \quad . \quad (2)$$

The value of λ thus defined is taken as a measure of the rate of diffusion of the emanation through the plug.

3. *Rates of Diffusion of Gases of known Molecular Weight.*

To determine the molecular weight of the emanation, its rate of diffusion was compared with that of hydrogen, oxygen, carbon dioxide, and sulphur dioxide through the same plug. A pure sample of the gas to be experimented with was sucked into the diffusion-vessel, and after a known time (t minutes) some of it was withdrawn and analysed to find out how much had escaped through the porous plug. When no statement to the contrary is made, the gas was diffusing out into the atmosphere.

In deducing the value of λ from these observations, a slight modification had to be introduced into the method of calculation given above; for, since the rates of diffusion of the gases on the two sides of the plug were different, the volume of the gas (at constant pressure) contained in the diffusion-vessel varied during an experiment. It was important to maintain the pressure equal on the two sides of the plug to prevent any gas escaping by *effusion*, and this was done by occasionally adjusting the level of the reservoir R.

As above, equation (1)

$$\frac{dy}{dt} = - \frac{\lambda}{V} y,$$

y being in this case measured volumetrically.

Here V is not constant, but varies during an experiment. Since the variation was small, it was taken as a linear function of the time, so that

$$V = V_0(1 + \alpha t)*,$$

where V_0 is the volume at the beginning ($t = 0$), and α is a constant to be determined by observing V_1, V_0, and t.

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* This assumption is true except for the case of hydrogen; but as this gas was employed only in preliminary experiments the same assumption was taken to be sufficiently nearly correct.

Inserting this value of V in equation (1) we have

$$\log_e y = -\frac{\lambda}{V_0} \int_{t_1}^{t} \frac{dt}{1 + \alpha t}.$$

whence

$$\log_e \frac{y_0}{y} = \frac{\lambda}{\alpha V_0} \log_e (1 + \alpha t).$$

and since $\mid \alpha t \mid\ < 1,$

$$\log_e \frac{y_0}{y} = \frac{\lambda}{\alpha V_0} \left[\alpha t - \frac{\alpha^2 t^2}{2} + \frac{\alpha^3 t^3}{3} - \frac{\alpha^4 t^4}{4} + \dots \dots \right]$$

$$= \frac{\lambda t}{V_0} \left[1 - \frac{\alpha t}{2} + \frac{\alpha^2 t^2}{3} - \frac{\alpha^3 t^3}{4} + \dots \dots \right],$$

whence

$$\lambda = \frac{1}{\left[1 - \frac{\alpha t}{2} + \frac{\alpha^2 t^2}{3} - \frac{\alpha^3 t^3}{4} + \dots \dots \right]} \cdot \frac{V_0}{t} \log_e \frac{y_0}{y}. \quad . \quad (3)$$

4. *Preliminary Experiments.*

The earlier experiments were made to compare the rate of diffusion of the radium emanation with that of hydrogen through the plug of plaster-of-Paris which will be referred to as plug I. Although the results obtained cannot be employed to calculate the molecular weight of the emanation, a short account of them will be given since they lead to some interesting conclusions regarding the nature of the plugs used, as well as throwing light on the conditions to be attended to in performing density determinations by the present method.

As a result of the first experiments, the values of λ for hydrogen and the emanation (mixed with air) were found to be 1·77 and ·138 respectively. Assuming Graham's law of diffusion, these numbers would lead to the conclusion that the molecular weight of the emanation is 328—a value which is more than three times as great as the upper limit given by Rutherford and Brooks (*loc. cit.*). Further experiments were therefore undertaken to explain the cause of this discrepancy.

It was thought possible that the density of the gas with which the emanation was mixed might influence the rate of diffusion of the emanation, and therefore to test this hypothesis experiments were made in which hydrogen was substituted for air. The emanation was carried into the diffusion-vessel by a stream of hydrogen, and in order to have this gas

on both sides of the porous plug a glass tube was attached
by rubber over the plug, and a slow current of hydrogen from
a Kipp maintained through it. In this case the value of λ
for the emanation was found to be ·277, *i. e.* about twice as
great as when the emanation was mixed with air.

To test this point still further a series of experiments was
undertaken in which the pressure of the air with which the
emanation was mixed could be varied, precautions being
taken to insure that the pressures on the two sides of the
porous plug were always equal. The value of λ was found
to vary considerably with the pressure, as will be seen from
Table I.

TABLE I.—Plug I.

Emanation mixed with Air.	
Pressure in centimetres mercury.	λ.
76·0	·138
37·6	·181
36·5	·177
18·6	·222
15·8	·246
15·6	·307
9·2	·350
5·4	·737
5·3	·577

These results were confirmed by using another diffusion-
vessel of 250 c.c. capacity and a new plug (Plug II.) of
porous porcelain attached to the diffusion-vessel by sealing-
wax. Similar results were also obtained when the emanation
was mixed with hydrogen at different pressures.

There are two ways of explaining this change in the value
of λ with the density of the gas with which the emanation is
mixed. We may either suppose that the gas really exerts
some influence on the apparent molecular weight of the
emanation, say by condensing round its molecules ; or it
may be that the pores of the plugs employed are large com-
pared with the mean free path of the gas molecules. In this
case, the number of collisions of a molecule of emanation with
a gas molecule while escaping through the plug would not
be small compared with the number of collisions with the
substance of the plug; and therefore the value of λ for the
emanation would depend upon the nature of the gas with

which it is mixed, and still more upon the number of molecules of such gas present. To test which of these hypotheses is correct, experiments were made to see whether the rate of diffusion of carbon dioxide through the plugs varied with its pressure. The results of these experiments are given in Table II.

TABLE II.—CO_2.

Plug I.		Plug II.	
Pressure in centimetres mercury.	λ.	Pressure in centimetres mercury.	λ.
76·0	·234	76·0	1·58
10·0	·765	8·8	3·28
9·4	·687	7·9	2·76
		5·9	3·74

From the above table it appears that a variation of λ with the pressure, similar to that observed with the radium emanation, occurs with carbon dioxide. We are therefore led to the conclusion that the variation in the density of the emanation is only apparent, and that the pores of the plugs used were comparable in size with the mean free path of gas molecules. Subsequent experiments were all carried out at *atmospheric pressure*, the upper surface of the porous plug being in contact with air. In this way, all experiments were performed under similar conditions, and could therefore be compared.

A further point is brought out by a study of Table II. It will be noticed that the values of λ for hydrogen and carbon dioxide (at atmospheric pressure) are not in the inverse ratio of the square roots of the molecular weights of these gases. We cannot, therefore, assume this law to calculate the molecular weight of the emanation from a comparison of its rate of diffusion with that of hydrogen, a fact which gives an explanation of the very high value for the molecular weight obtained above on this assumption. The cause of the failure of Graham's law as applied to these observations is not certain but the most probable explanation is that when a gas heavier than air is diffusing *upwards* through the plug and air *downwards* into the diffusion-vessel, a layer accumulates on the

lower surface of the plug, preventing diffusion from proceeding as rapidly as theory demands. When a light gas such as hydrogen is used, the air which enters the diffusion-vessel through the plug would tend to fall through the hydrogen, and the error due to the formation of a layer of air is greatly reduced if not entirely obviated. That the failure of the ordinary law of diffusion was not due to any peculiarity of the plugs themselves was proved by testing them in the manner described by Graham (Transactions Roy. Soc. Edinburgh, vol. xii. 1834). The rubber connecting the diffusion-vessel with the adjustable mercury reservoir was detached, and the open end of the vessel placed under water contained in a tall cylinder; the diffusion-vessel was then filled with hydrogen and allowed to stand until all the hydrogen had been displaced by air, water being added from time to time to the cylinder to keep the level inside the diffusion-vessel equal to the level outside. The volume at the end of the experiment was to the original volume of hydrogen as the square root of the density of hydrogen is to the square root of the density of air, within the limits of experimental error. It was of course impossible to adopt this method for the emanation, since it depends on measurements of the volumes of gases escaping and entering through the plugs.

A series of experiments was therefore next made to find out *empirically* how the rate of diffusion of a gas through the plugs used depended on its density, and for the purpose oxygen, carbon dioxide, and sulphur dioxide were used. Unfortunately, before taking these observations the porosity of plug I. was found to have changed*, so that the results given below cannot be compared with those obtained above with the same plug. Plug II. was removed from the second diffusion-vessel and replaced by a similar but somewhat thicker one (plug III.). Some observations were, however, first taken with plug II., and these are quoted below.

5. *Final Experiments.*

Preparation and Analysis of Gases.— *Oxygen.* This gas was prepared by heating pure dry potassium permanganate contained in a horizontal glass tube closed at one end and drawn off at the other, which was connected to a mercury manometer by a T-piece, the other limb of which was

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* Owing to an accident the stopcock of the diffusion-vessel was broken and a new one attached. It is possible that during the operation of making the glass joint near the plug, the change mentioned may have occurred.

attached by rubber to the diffusion-vessel (fig. 2). A small quantity of soda-lime was placed near the open end of the tube, and separated from the potassium permanganate by a plug of glass-wool, so that the oxygen as it was generated had to pass over it.

Fig. 2.
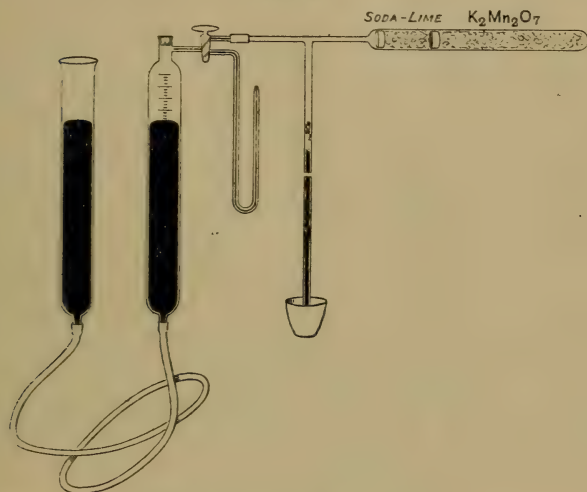
Before starting an experiment, the apparatus was exhausted and filled with oxygen by heating the potassium permanganate in a Bunsen burner; this oxygen was then pumped out and the apparatus refilled with oxygen, after which the diffusion-vessel could be filled with pure oxygen whenever required.

To find the composition of the gas in the diffusion-vessel at the end of an experiment, a suitable volume was withdrawn and transferred to a gas burette, where its volume was measured. The gas was then transferred to a wide glass tube inverted over mercury, where the oxygen was removed in the usual manner by means of phosphorus vapour, and the loss of volume determined.

Carbon dioxide.—To prepare this gas, the tube containing the potassium permanganate was removed and replaced by a similar one filled with pure dry sodium bicarbonate, from which carbon dioxide is readily obtained by gently heating. As in the experiments with oxygen, the apparatus was twice exhausted and filled with carbon dioxide before starting.

The analysis of the gas left at the end of an experiment

was performed by treating a known volume of it with a concentrated solution of caustic potash and measuring the diminution in volume.

Sulphur dioxide.—This gas was obtained directly from a siphon containing the liquid, care being of course taken to remove the air from the connecting tubes. The gas left in the diffusion-vessel at the end of an experiment was analysed by treating a known volume with potassium bichromate solution and measuring the diminution of volume produced.

As already stated, the composition of the gases left at the end of the diffusion experiments was determined volumetrically. The estimation was carried out by means of a gas burette of 60 c.c. capacity of the type described by Travers ('Study of Gases,' p. 66), and graduated downwards in fifths of a cubic centimetre, so that the volume of the gas in the burette could be estimated to $\cdot 1$ c.c. with certainty. To avoid errors due to variations of temperature, the burette was enclosed in a water-jacket and the temperature read off on a thermometer immersed in the water. The burette was carefully calibrated by filling with water and running out by successive stages of 2 c.c. and weighing. About 20 c.c. of the gas to be analysed was usually transferred from the diffusion-vessel to the burette, so that its composition could be determined to one per cent.

6. *Observations and Results.*

In calculating the values of λ from the observations with *oxygen*, account has to be taken of the fact that the upper surface of the plug is in contact with air which contains 21 per cent. of oxygen, so that oxygen is diffusing *into* as well as *out of* the diffusion-vessel throughout the experiment. To allow for this it is necessary to slightly modify equation (3).

Since the concentration of the oxygen in the air in contact with the upper surface of the plug remains constant, the quantity dy of oxygen which, on the whole, escapes from the diffusion-vessel is given by the equation

$$dy = -\frac{\lambda}{V_0(1+\alpha t)}\left(y - \frac{21}{100}y_0\right)dt,$$

whence

$$\lambda = \frac{1}{\left[1 - \frac{\alpha t}{2} + \frac{\alpha^2 t^2}{3} - \ldots\right]}\frac{V_0}{t}\log_e\left(\frac{\frac{79}{100}y_0}{y - \frac{21}{100}y_0}\right)$$

It is the values of $\frac{79}{100}y_0$ and $\left(y - \frac{21}{100}y_0\right)$ which are tabulated in the columns headed y_0 and y respectively (Tables III., IV., and V.).

For carbon dioxide and sulphur dioxide V_0 and y_0 are the same volumes.

In the experiments with the emanation y and y_0 were determined electrically as described above. The values of y, given in the tables, have been corrected for the natural decay of the emanation in the interval during which diffusion was taking place. From the results of Rutherford and Soddy (Phil. Mag. April 1903) the activity of the emanation falls to half its value in 3·71 days; the mean of y and y_0 was taken, and the loss of activity of this quantity in the time t calculated and added to the observed value of y. To make the method of calculation clear, one of the observations taken with plug I. is given in full :—

Time

10.16 A.M. .. Sensitiveness of electrometer in scale-divisions per volt, 86. Diffusion-vessel filled to 67·8 c. c.

10.27 A.M. .. 34·3 c. c. carried into testing-vessel, leaving 33·5 c.c. in the diffusion-vessel.

10.28 A.M. .. Deflexion of electrometer-needle in 20 seconds, corrected for lack of uniformity of scale, 224.

12.33 P.M. .. Sensitiveness of electrometer in scale-divisions per volt, 82. Volume of gas in diffusion-vessel 33·7 c. c.

12.41 P.M. .. 27·7 c. c. carried into testing-vessel.

12·42 P.M. .. Deflexion of electrometer-needle in 20 seconds, corrected for lack of uniformity of scale, 105.

Deflexion corrected to 86 scale-divisions per volt, and to volume 34·3 c. c. carried into testing-vessel

$$= 105 \times \frac{86}{82} \times \frac{34·3}{27·7} = 136.$$

Corrected for natural decay of emanation, 139.
Time during which diffusion had been proceeding, 134 minutes.

$$\lambda = \frac{33·6}{134 \times ·4343} \ \log_{10} \frac{224}{139} = ·120.$$

In the last column of Table III. are tabulated the values of $\lambda\sqrt{M}$, in which M is the molecular weight of the gas employed. In Tables IV. and V., however, the values of λ obtained are first multiplied respectively by ·1362 and ·312 to reduce them to equality with the corresponding quantities for plug I., and the values of $·1362\lambda\sqrt{M}$ and $·312\lambda\sqrt{M}$ respectively tabulated in the last columns. This

reduction makes a comparison of the results obtained with the three plugs less troublesome.

If Graham's law of diffusion were strictly applicable to these experiments, the value of $\lambda\sqrt{M}$ should be constant whatever gas is employed. From an inspection of the results, however, this is seen not to be the case; the value of $\lambda\sqrt{M}$ decreases as λ decreases. If, however, the values of $\lambda\sqrt{M}$ as ordinates be plotted against the values of λ as abscissæ, the points so obtained corresponding to the values of oxygen, carbon dioxide, and sulphur dioxide are found to lie on a straight line not parallel to the axis of λ (as should be the case if Graham's law were true), but inclined at a small angle to it (fig. 3). By linear extrapolation the value of $\lambda\sqrt{M}$

Fig. 3.
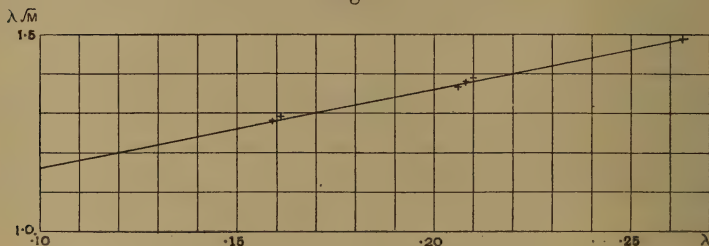
for the emanation is found to be 1·23 for plug I.; the value of ·1362 $\lambda\sqrt{M}$ for plug II. 1·21, and the value of ·312$\lambda\sqrt{M}$ for plug III 1·205. These values are given in brackets in the tables. By dividing the values of $\lambda\sqrt{M}$ for the emanation by the corresponding values of λ and squaring, the value of the molecular weight of the emanation can be calculated. The experiments with plugs I., II., and III. lead respectively to the values 85·5, 97, and 99. On the assumption that the emanation is a monatomic gas, it probably occupies the vacancy in the periodic table in the fluorine group between Mo (96·0) and Ru (102). As this is the first vacancy in the periodic table, it is impossible to assign a place in the periodic table to the emanation on the assumption that it is polyatomic; the present experiments, therefore, confirm the view that the emanation is monatomic.

TABLE III.—Plug I.

Gas.	V_0 in c.c.	V in c.c.	t in minutes.	y_0.	y.	λ.	λ mean.	$\lambda \sqrt{\overline{M}}$.
O_2	36·6	37·7	77	28·85	16·7	·264		
O_2	35·5	36·7	88	28·05	15·0	·257	·263	1·49
O_2	33·0	34·0	75	26·1	14·35	·267		
CO_2 ...	34·3	37·0	75	34·3	21·85	·214		
CO_2 ...	29·3	31·9	63	29·3	19·3	·203	·206	1·37
CO_2 ...	31·9	34·6	81	31·9	19·4	·204		
CO_2 ...	34·8	38·0	86	34·8	21·6	·202		
SO_2 ...	34·3	39·8	90	34·3	23·4	·157		
SO_2 ...	32·0	36·2	70	32·0	23·2	·158	·1595	1·28
SO_2 ...	35·0	41·3	100	35·0	22·6	·166		
SO_2 ...	36·2	42·5	95	36·2	24·75	·157		
Emana-tion	36·0	36·1	149	209	121	·132		
,,	33·5	33·7	134	224	139	·119		
,,	34·1	34·2	119	186	117	·133	·133	(1·23)
,,	33·0	33·0	125	234	139	·137		
,,	33·9	34·0	173	167	84	·135		
,,	34·8	35·1	186	214	99·5	·143		

TABLE IV.—Plug II.

Gas.	V_0.	V.	t.	y_0.	y.	λ.	λ mean.	$\lambda \times ·1362$.	$\lambda \sqrt{\overline{M}} \times ·1362$.
O_2	85·6	89·4	40	67·7	23·5	1·89			
O_2	92·3	94·5	35	72·9	35·2	1·94	1·93	·263	1·49
O_2	94·3	96·4	41	74·5	32·2	1·95			
CO_2 ...	153·2	159·8	35	153·2	106·5	1·63			
CO_2 ...	153·4	164·2	50	153·4	97·5	1·44			
CO_2 ...	111·5	118·4	32	111·5	70·7	1·64	1·54	·210	1·39
CO_2 ...	129·3	136·9	30	129·3	91·5	1·54			
CO_2 ...	122·8	131·4	40	122·8	78·5	1·43			
Emana-tion	104·9	105·4	98	255	109	·910			
,,	104·5	105·5	97	200	88	·884			
,,	102·2	103·0	85	231	105	·943	·906	·123	(1·21)
,,	98·4	99·2	85	295	137·5	·884			
,,	99·4	99·9	104	306	118	·911			

Table V.—Plug III.

Gas.	V_0.	V.	t.	y_0.	y.	λ.	λ mean.	λ×·3120.	λ√M ×·3120.
O_2	44·9	46·7	64	35·5	10·8	·852			
O_2	64·4	66·5	54	50·8	25·3	·845	·843	·263	1·49
O_2	68·3	70·3	56	54·0	27·6	·831			
CO_2 ...	65·8	72·2	62	65·8	36·0	·670			
CO_2 ...	61·5	66·6	50	61·5	37·0	·650	·666	·208	1·38
CO_2 ...	63·0	69·4	60	63·0	34·3	·678			
SO_2 ...	55·6	64·3	53	55·6	34·95	·526			
SO_2 ...	57·4	65·8	54	57·4	36·2	·525			
SO_2 ...	62·4	72·6	66	62·4	38·4	·495	·517	·161	1·29
SO_2 ...	51·1	59·4	50	51·1	31·6	·529			
SO_2 ...	50·3	57·3	43	50·3	33·5	·508			
Emana-tion	49·4	50·2	85	272	138	·395			
tion	53·3	53·8	91	206	107	·384	·389	·121	(1·205)
,,	52·6	53·1	88	222	117	·383			
,,	48·6	49·3	98	274	123·5	·395			

Part II.

The Molecular Weight of Thorium Emanation.

7. *Introduction and General Method.*

On account of the rapid rate of decay of thorium emanation, it is not possible to measure its rate of diffusion by the method used for radium emanation and described above. Rutherford ('Radioactivity,' pp. 233, 234) has, however, described a method by which the diffusion-coefficient into air may be determined, and obtains the value $K = ·09$, from which he concludes that "the two emanations do not differ much from one another in molecular weight." In the following pages an account is given of some experiments in which the rate of diffusion of thorium emanation was compared with that of radium emanation, and as a result of which the above conclusion is fully confirmed. The method used depended on maintaining, through a vessel with porous walls, a constant current of air mixed with emanation, the concentration of which was measured before and after passing through this vessel, which will be referred to as the diffusion-vessel.

To make the quantity of emanation which passed through the walls of the diffusion-vessel as great as possible, it was necessary to make the area of the porous walls great compared with the volume of the vessel, which was therefore constructed as follows :—two plates of porous porcelain

about 100 square centimetres in area (V, fig. 4) were placed parallel and attached to each other by sealing-wax round their edges at a distance of about two millimetres apart ; a layer of soft wax over the sealing-wax served to make this joint air-tight. At two opposite corners, glass tubes were attached so that a current of air could be passed through the vessel. By weighing the quantity of mercury required to fill the vessel, its volume was found to be 24·6 c. c.

If t is the average time taken by a particle of gas to pass through the diffusion-vessel, then, if when a current of radium emanation is passed through, its concentrations before entering and after leaving the diffusion-vessel are y_1 and y_2 respectively,

$$y_2 = y_1 e^{-\lambda t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where λ is a measure of the quantity of emanation which has escaped by diffusion in the time t.

In the case of thorium emanation, however, since the rate of decay is so much more rapid than for radium emanation, the concentration would change in the time taken to pass through the diffusion-vessel, even though none escaped by diffusion. In this case the concentration (z_2) just after emerging from the diffusion-vessel is given by the equation

$$z_2 = z_1 e^{-(K+\lambda')t} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where λ' is a measure of the quantity of thorium emanation which has escaped by diffusion in the time t, z_1 is the concentration of the emanation just before entering the vessel, and K is a constant depending on the natural rate of decay. The air-current was so adjusted that t had the same value as in the experiment with radium emanation.

In order to determine the ratio $\dfrac{\lambda}{\lambda'}$, it was necessary to measure y_1, y_2, z_1 and z_2. It was further necessary to determine the value of Kt ; this was done by replacing the porous vessel by a non-porous glass one of the same volume, and passing a current of thorium emanation through it. If, as before, z_1 is the concentration of the emanation on entering the glass vessel, and z_2' is its concentration on leaving it,

$$z_2' = z_1 e^{-Kt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

From equations (4), (5), and (6)
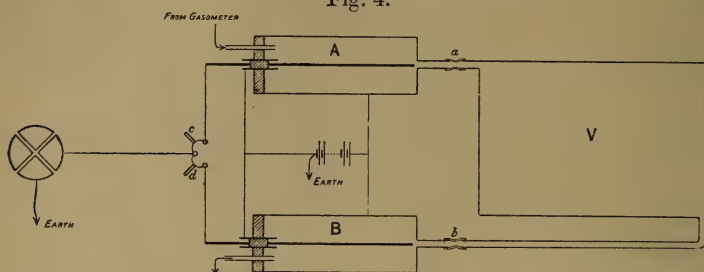
$$\frac{\lambda}{\lambda'} = \frac{\log \dfrac{y_1}{y_2}}{\log \dfrac{z_1}{z_2} - \log \dfrac{z_1}{z_2'}} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The method of measuring the quantities involved will appear from the following account.

8. *Experiments with Radium Emanation.*

The general arrangement of the apparatus is shown in
fig. 4. A current of air from a large gasometer was passed

Fig. 4.

through a solution of radium in water, through a drying-
tube, and through a plug of glass-wool to remove dust,
into the cylindrical testing-vessel A of 56·3 c. c. capacity,
along the axis of which ran an insulated rod reaching nearly
the whole length of the cylinder ; thence the current of air
passed through the porous vessel V into a second testing-
vessel B, and out into the open air. The air-current could
be regulated to any desired value by placing weights on the
gasometer ; the rate at which the gasometer fell was read on
a scale and afforded a measure, in arbitrary units, of the air
current passing. The vessel V was attached to the two
testing-vessels by the rubber connexions a and b respectively,
and could therefore be removed and replaced when necessary.

The air current was allowed to flow until the concentration
of the emanation in the two testing-vessels had become
constant, when the ionization currents in A and B were
measured by means of an electrometer in exactly the same
way as that described above for the experiments on the
molecular weight of radium emanation. By means of keys
c and d the central rods of A and B could be connected to
or disconnected from one pair of quadrants of the electro-
meter, so that the saturation ionization current in either
could be measured separately. There was no need to allow
for the activity excited on the walls of the testing-vessels
since the quantity present is proportional to the concentration
of the emanation.

In spite of the fact that the ebonite plugs, used to insulate
the central rods, were divided into two parts by metal guard-
rings connected to earth, it was found that the rods charged
up slightly on disconnecting from earth, although there
was no emanation in the testing-vessels. The rate at which
they acquired this charge was always measured before

starting an experiment and subtracted from the leaks subsequently obtained.

The vessels A and B were made as nearly as possible alike, so that the ionization currents through them were not very different when they contained equal quantities of emanation. Before proceeding to the diffusion experiments, however, a comparison of the leaks produced in A and B by a given ionizing agent was necessary. This was made by detaching the vessel V and replacing it by a glass tube so that no emanation could escape during its passage from A to B. A current of emanation was then allowed to pass until the leaks through the two vessels had become constant, when the series of observations given in Table VI. were taken. The numbers show considerable variations owing, probably, to irregularities in the air-current flowing. The error produced by these fluctuations is, however, greatly reduced by taking the mean of several observations extending over a long time. It will be noticed that for the same ionizing agent the leak in A is greater than the leak in B, in the ratio 100 : 96·2. In all subsequent experiments, therefore, the observed leak in B was increased in this ratio (see Table X.).

TABLE VI.—Comparison of the Two Leak Vessels.

The Current of Emanation was started at 11.11 A.M.

Time.	Leak for 30 seconds, in scale-divisions of the electrometer.			
	Vessel A.	Mean.	Vessel B.	Mean.
12.27 P.M.	169		...	
12.30 ,,	...		159	
12.33 ,,	167		...	
12.36 ,,	...		173	
12.39 ,,	177		...	
12.42 ,,	...		168	
12.45 ,,	165		...	
12.48 ,,	...	171	164	164·5
12.51 ,,	168		...	
12.54 ,,	...		164	
12.57 ,,	175		...	
1.00 ,,	...		165	
1.03 ,,	175		...	
1·06 ,,	...		158	
1.09 ,,	170		...	
1.12 ,,	...		165	

Experiments were next made to determine the value of λt. The porous vessel (V) was replaced and a current of about 170 c.c. per minute passed through the apparatus. The results of these experiments are given in Table VII.

TABLE VII.—Diffusion of Radium Emanation.

Gasometer fell ·564 mm. per minute.

The Current of Emanation was started at 12.51 P.M.

| Time. | Leak for 30 seconds, in scale-divisions of the electrometer. | | | |
	Vessel A.	Mean.	Vessel B.	Mean.
3.03 P.M.	166		...	
3.06 ,,	...		88	
3.09 ,,	168		...	
3.12 ,,	...		92	
3.18 ,,	176		...	
3.21 ,,	...		93	
3.24 ,,	180		...	
3.27 ,,	...		96	
3.30 ,,	179		...	
3.33 ,,	...		99	
3.36 ,,	168	171	...	89
3.39 ,,	...		87	
3.42 ,,	172		...	
3.45 ,,	...		88	
3.51 ,,	176		...	
3.54 ,,	...		90	
3·57 ,,	169		...	
4.00 ,,	...		83	
4.03 ,,	170		...	
4.06 ,,	...		82	
4.09 ,,	166		...	
4.12 ,,	...		80	
4.15 ,,	164		...	
4.18 ,,	...		87	

9. *Experiments with Thorium Emanation to determine*

$$\log \frac{z_1}{z_2} \text{ and } \log \frac{z_1}{z_2'}.$$

The radium solution was removed and replaced by a wide lead tube containing thorium oxide wrapped in filter-paper over which a current of air could be passed into the vessel

A. through V into B, and out into the open air, as in the previous experiment. It is important to notice that in this case the ratio of the rates at which the central rods in A and B charge up is not equal to $\frac{z_1}{z_2}$. For, let the concentration in A at a point in a plane at right angles to the axis and at a distance x from the ebonite plug be z ; then, since the emanation decays according to an exponential law in passing from one end to the other of A,

$$z = z_0 e^{-Kx}$$

where z_0 is the concentration on entering A.

Now since the central rod connected to the electrometer reaches nearly the whole length of the testing-vessel, the ionization current in A is proportional to

$$S \int_0^X z\,dx = \frac{Sz_0}{K}(1 - e^{-KX})$$

where S is the cross-sectional area of A and X is its length.

Similarly for B the ionization currrent is proportional to

$$\frac{Sz_2}{K}(1 - e^{-KX}).$$

Hence

$$\frac{\text{The ionization current in A}}{\text{The ionization current in B}} = \frac{z_0}{z_2}.$$

To find z_1 it is necessary to measure the decrease in activity of the emanation in passing through the cylinder A. This was done by replacing the diffusion-vessel by a glass one of nearly the same volume, and measuring the ionization current in A and B as before. Knowing the volumes of A and of the glass vessel connecting A and B, it was possible to calculate the fall in concentration due to the natural decay of the thorium emanation in passing through A. It was also possible to calculate the fall in concentration in passing through the glass vessel, from which the value of $\log \frac{z_1}{z_2'}$ could be found.

In these experiments the rate of flow of air was somewhat less than in the previous experiments, on which account an appropriate correction was added to the observed value of z_2'. The results of the experiments with thorium emanation are given in Tables VIII. and IX.

Table VIII.—Diffusion of Thorium Emanation.

Gasometer fell ·562 mm. per minute.
The Current of Emanation was started at 11.58 A.M.

| Time. | Leak for 30 seconds, in scale-divisions of the electrometer. | | | |
	Vessel A.	Mean.	Vessel B.	Mean.
12.40 P.M.	268		...	
12.43 ,,	...		115	
12.54 ,,	283		...	
12.57 ,,	...		120	
1·04 ,,	277		...	
1.07 ,,	...	283	128	119
1.11 ,,	301		...	
1.14 ,,	...		118	
1.17 ,,	270		...	
1.20 ,,	...		118	
1.23 ,,	298		...	
1.26 ,,	...		116	

Table IX.—Natural Decay of Emanation.

Gasometer fell ·465 mm. per minute.
Air-current started at 12.38 P.M.

| Time. | Leak for 30 seconds, in scale-divisions of the electrometer. | | | | |
	Vessel A.	Mean.	Vessel B.	Mean.	Mean corrected to rate of flow ·562 mm. of gasometer per min.
2.21 P.M.	265		...		
2.24 ,,	...		200		
2.27 ,,	257		...		
2.30 ,,	...		203		
2.33 ,,	249	259	...	202	211
2.36 ,,	...		194		
2.39 ,,	265		...		
2·42 ,,	...		201		
2.45 ,,	261		...		
2.48 ,,	...		211		

10. *Calculation of* $\dfrac{\lambda}{\lambda'}$.

In Table X. are given the mean ionization currents in A and B.   For simplicity of calculation the ionization currents in A are reduced to 100.

TABLE X.

	Ionization current in A. I_A.	Ionization current in B. I_B.	$I_B \times \dfrac{100}{96 \cdot 2}$.
Radium emanation. No diffusion-vessel.	100	96·2	100
Radium emanation. Diffusion experiment.	100	52·1	54·2
Thorium emanation. Diffusion experiment.	100	42·1	43·7
Thorium emanation. No diffusion-vessel.	100	81·2	84·4

From Table X. the following values can at once be obtained :—

$$y_1 = 100, \quad y_2 = 54 \cdot 2,$$
$$z_0 = 100, \quad z_1 = 88 \cdot 8, \quad z_2 = 43 \cdot 7, \quad z_2' = 84 \cdot 4.$$

Substituting in equation (7)

$$\frac{\lambda}{\lambda'} = \frac{\log \dfrac{100}{54 \cdot 2}}{\log \dfrac{88 \cdot 8}{43 \cdot 7} - \log \dfrac{88 \cdot 8}{84 \cdot 4}} = \frac{\cdot 266}{\cdot 286} = \cdot 931.$$

It thus appears that the molecular weight of thorium emanation does not differ much from that of radium, though the former would seem to be somewhat smaller.

I am indebted to Professor J. J. Thomson for many suggestions, and I would thank him for the interest which he has taken in the progress of this investigation, which was carried out at the Cavendish Laboratory.

V. *On the Magnetic Change of Resistance in Iron, Nickel, and Nickel-Steel at Various Temperatures.* By W. E. WILLIAMS, *B.Sc., Research Fellow of the University of Wales* [*].

THE influence of temperature on the change of resistance with magnetization in iron and nickel has been studied previously by Dr. Knott and also by the present writer.

Dr. Knott [†] has measured the change of resistance in nickel at various temperatures up to 400° C. for fields from

0 to 34 c.g.s. units. In a previous paper communicated to the Philosophical Magazine [*] the present writer gave an account of measurements of the change of resistance in iron and nickel at the temperatures 0° and 100° C. for fields up to 800 units.

In the experiments described below the range of temperature has been extended up to 665° C., and the change of resistance in iron and nickel has been measured at temperatures within this range for fields up to 900 units.

This range of temperature includes the " critical point " of nickel (380° C.), and some very interesting results have been obtained at temperatures near this point. In particular it has been found that for some temperatures the change of resistance is negative at high fields.

Through the courtesy of Dr. C. E. Guillaume I have also been able to measure the change of resistance in various specimens of nickel-steel, and the results obtained are given below.

Apparatus.

The magnetic field was obtained by means of a coil, one metre long, wound with eight layers of no. 14 copper wire. This coil gave a field of 40 units per ampere, the maximum current used being 23 amperes. The heating was effected by means of the electric current. A coil of platinoid wire was wound non-inductively on a hard glass tube, and covered with two or three layers of asbestos paper, and the desired temperature was obtained by passing a current through this coil. The wires whose changes of resistance were to be measured were placed inside this glass tube, which was then fitted inside the magnetizing coil. The coil was provided with a water-jacket through which a current of cold water was kept flowing in order to prevent the heat reaching the wire of the coil.

The temperature was measured by means of a platinum thermometer placed alongside the iron or nickel wire. The wires were hard-soldered to copper leads and bent double so as to bring the two junctions close together and thus avoid any thermoelectric effect. The platinum wire was provided with double leads in the usual way, and the leads to the iron or nickel wire were made of the same gauge of wire, so that the effect of temperature on their resistance could be allowed for.

The change of resistance was measured by means of a slide-

wire bridge. In order to obtain a large step on the bridge
a thick copper wire was used, and two sliding keys were
made with the connexions entirely of copper, so as to avoid
any thermoelectric effects in the galvanometer circuit. One
of these keys was used to balance when the magnetic field
was " off " and the other when it was " on." By the use of
two keys in this way the comparison of resistances can be
carried out in a few seconds, which it is necessary to do as
the resistance in zero field is continually changing owing to
slight changes in the temperature of the wire. The iron or
nickel wire under investigation was connected to one of the
inner gaps of the bridge, an adjustable resistance being
placed in the other, and two auxiliary coils of about 5 ohms
each were placed in the outer gaps of the bridge. This
arrangement has the advantage that the constant for reducing
the bridge-readings to the form $\frac{d\mathrm{R}}{\mathrm{R}}$, R being the resistance
of the wire, is independent of R to a first order of approxima-
tion, and it is not, therefore, necessary to calibrate the bridge
separately for different resistances of the wire.

The arrangements described above worked very satis-
factorily for nickel and iron up to a temperature of about
500° C. The change of resistance in nickel vanishes at about
400° C., but the change in iron persists to a much higher
temperature.

In attempting to measure the change of resistance in iron
at these temperatures several difficulties were met with, and
a slightly different arrangement of apparatus was adopted.
In order to have space for a thick covering of asbestos around
the heating coil, a large magnetizing coil was used having an
internal diameter of 6 cms. The heating coil was of nickel
and was wound on a porcelain tube and covered with asbestos
powder. The nickel wire is non-magnetic at temperatures
above 400° C., and therefore does not interfere with the
magnetic field inside the coil.

At temperatures above 500° C. the iron wire if in contact
with air oxidizes rapidly, and hence the resistance of the
wire continually increases. For this reason the wire must
be heated either in nitrogen or in a vacuum, the use of
hydrogen being debarred owing to its action on the platinum.
The arrangement adopted for this purpose is shown in fig. 1.
The heating-coil was placed inside a glass tube which was
about a foot longer than the coil. The connecting-wires
were led out through a cork in the top of the tube and the
joint was made air-tight with wax.

Another difficulty arises from the impossibility of keeping the heating-current quite constant.   In order to obtain the highest temperatures, it was necessary to pass a current of about 15 amperes through the heating-coil.   This was

Fig. 1.
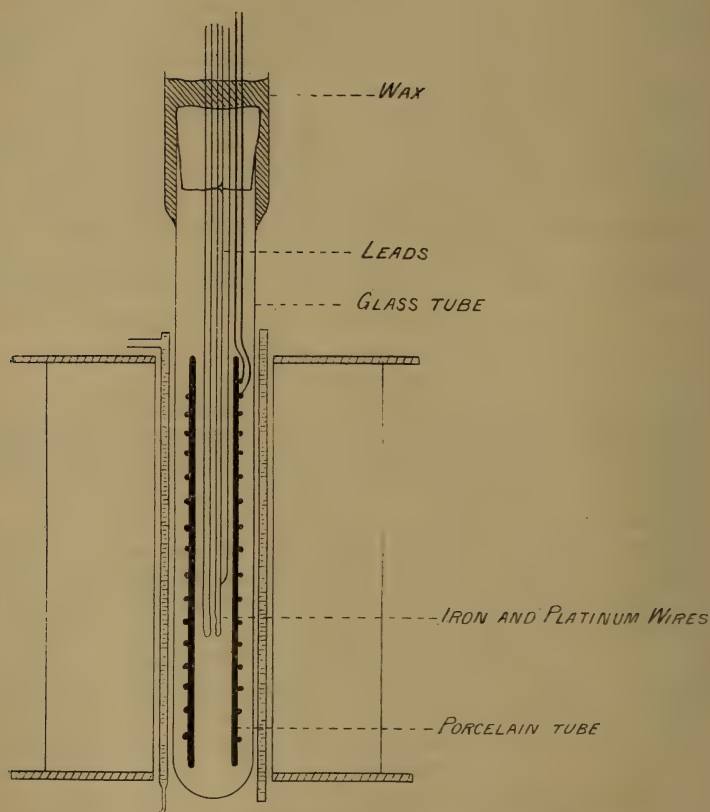
Apparatus for Measuring Change of Resistance at High Temperatures.

obtained from a secondary battery of thirty cells, and as it is necessary to keep the current on for a considerable time, the battery is run down and the current steadily decreases.   As the heating effect is proportional to the square of the current, a comparatively small variation in the current causes a very considerable change in the temperature, making it impossible to obtain satisfactory measurements.   This difficulty was

overcome by using a platinum wire to compensate the heating effect on the iron, the same wire serving for this as for the measurement of temperature. The platinum wire was connected in the arm of the bridge next the iron, instead of the usual balancing resistance. As the temperature-coefficient of platinum is much smaller than that of iron, the compensation thus effected could only be partial. By placing, however, a resistance in parallel with the iron wire, and another in series with it in the arm of the bridge, the effect of the heating of the iron on the total resistance of the arm could be made as small as we pleased, and by adjusting these resistances it could be made exactly equal to the change in the other arm containing the platinum.

The condition for balance under change of temperature is

$$\frac{I\,S^2}{(I+S)^2} = \frac{K'}{K}\frac{P\,.\,A}{B},$$

I = resistance of the iron,
P =   ,,   ,,   platinum,
K = temp. coefficient of iron,
K'=   ,,   ,,   of platinum,
S = shunt on iron,
A, B = resistance of auxiliary coils.

In thus diminishing the effect of temperature on the resistance of the arm we are also diminishing the effect of the magnetic change, and the bridge is to this extent less sensitive, but this disadvantage is far more than counterbalanced by the greater accuracy with which the readings can be taken when this arrangement is adopted. The exact formula for reduction of the readings in this case is :

$$da = da'\frac{(I+S)^2}{S^2\,I}\frac{P\,.\,A}{B} = da'\frac{K}{K'},$$

where da' is the actual reading and da the corresponding reading if there were no extra resistances.
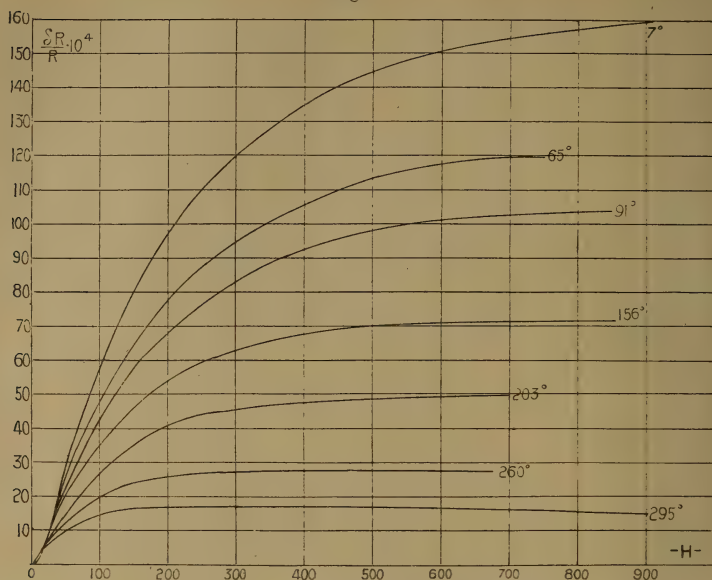
RESULTS.

Nickel.—The results obtained for nickel are shown by the curves in figs. 2, 3, 4. It will be seen that as the temperature is increased the change of resistance diminishes, the diminution being proportionately greater at high fields, so that the highest part of the curve becomes a straight line parallel to the axis of H.
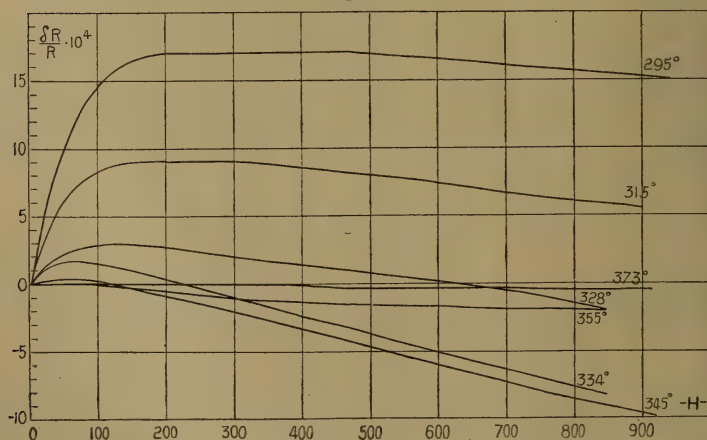
Fig. 2.

Change of Resistance of Nickel at Temperatures 7° to 290° C.

Fig. 3.
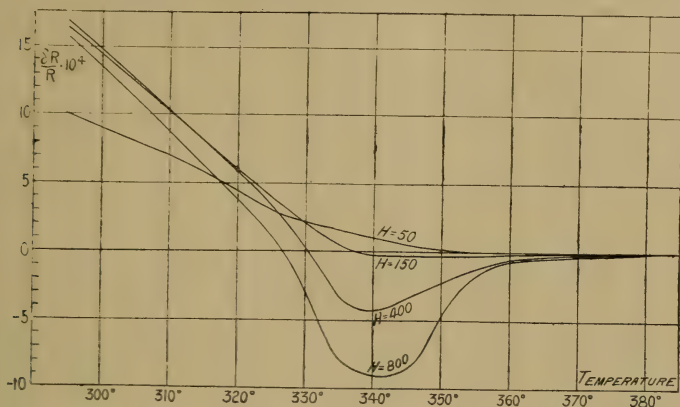

Change of Resistance of Nickel at Temperatures 295° to 373° C.

For temperatures above 300° the curve bends down and finally crosses the axis of H, denoting that the resistance is decreased by magnetization. The curves for these temperatures are shown on a large scale in fig. 3, and in fig. 4 the

change of resistance at four fields, 50, 150, 400, 800, is
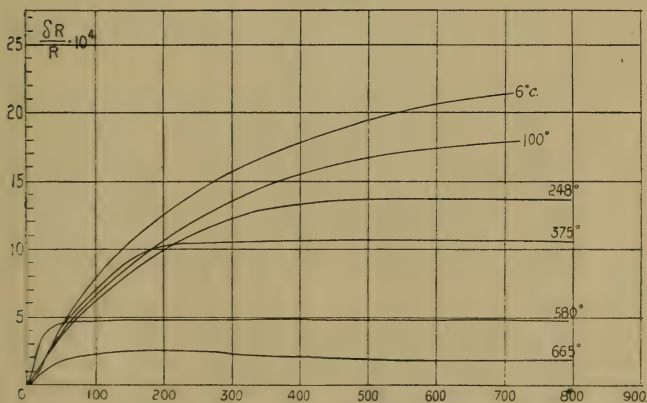plotted against temperature. The decrease of resistance

Fig. 4.

Change of Resistance of Nickel at Temperatures 300° to 380° C.

reaches a maximum at 340° C., at which temperature the
curve lies almost wholly beneath the axis, above this tempera-
ture the change of resistance diminishes and finally vanishes
at 380° C.

Fig. 5.

Change of Resistance of Iron at Temperatures 6° to 665° C.

Iron.—The results obtained for iron are shown by the
curves in fig. 5. The results are very similar to those
obtained for nickel, but there is a very marked increase at
low fields for the temperature 580° C.

Nickel - Steel. — Eight specimens of nickel - steel were

G 2

examined, the percentages of nickel being 24; 27·9, 35, 40, 44, 48·7, 57, 70. The change of resistance for longitudinal magnetization was measured in the same way as for iron and nickel. The results are shown in fig. 6. All the specimens

Fig. 6.
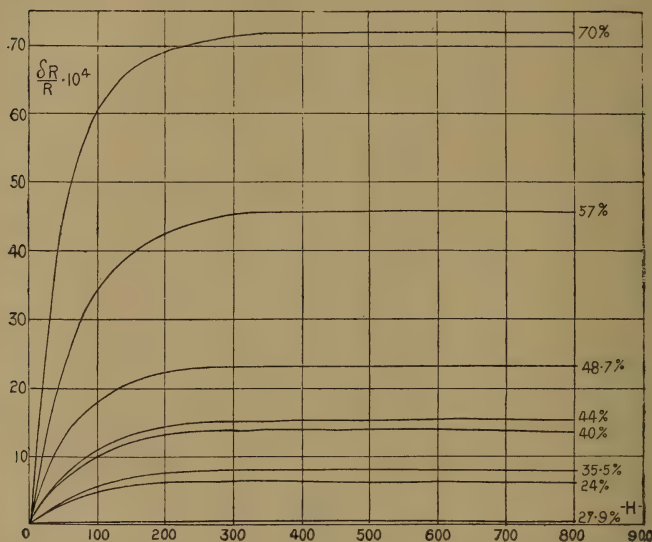
Change of Resistance in specimens of Nickel-Steel containing different percentages of Nickel.

were annealed except the one containing 24 per cent. nickel. This specimen is "irreversible," and after annealing the change of resistance was too small to be measured. The curves for all the different specimens are very similar to each other. The change of resistance reaches a limiting value at a field of about 300, and remains constant up to the highest field used.

The change of resistance for transverse magnetization was also measured for five of the specimens. The wires were wound into flat spirals and placed between the poles of an electromagnet. The change of resistance was measured by means of the slide-wire bridge as described above. The field was measured before the introduction of the wire by means of a ballistic coil.

The results obtained are shown in fig. 7, the curves for iron and nickel obtained with the same apparatus being added for purposes of comparison. In the specimen containing 27·9 per cent. nickel the resistance is apparently diminished

for all fields, but in the other four specimens the resistance is increased at low fields and diminished at high fields. The change of resistance is greatest in the specimen 70 per cent., and it will be noticed that the initial increase is much greater in this specimen than in nickel.

Fig. 7.

Change of Resistance of Iron, Nickel, and Nickel-Steel for Transverse Magnetization.
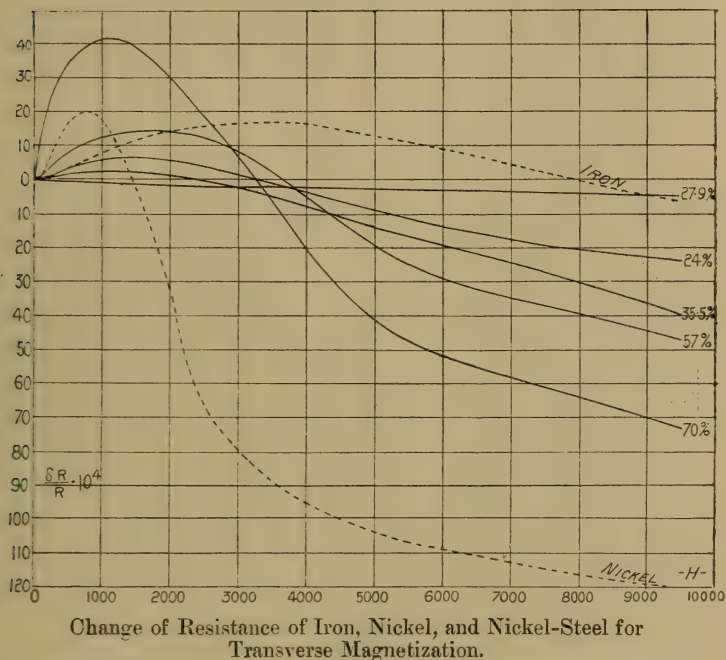
All the experiments described above were carried out in the Physical Laboratory of the University of Glasgow ; and I desire to express my best thanks to Prof. Gray for placing the necessary apparatus at my disposal, and also for much valuable help and advice given in the course of the work.

VI. *On Crystallographic Projections.*
By HAROLD HILTON [*].

1. LET crystallographic axes OA, OB, OC meet at A, B, C a sphere of unit radius with centre O whose equation referred to any three mutually orthogonal axes through O is $x^2 + y^2 + z^2 = 1$. Let P be the pole of the crystal-face whose indices are u, v, w ; and let the points

A, B, C, P be (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3), (l, m, n) respectively. The projections A_1, B_1, C_1 of A, B, C from O on $z = 1$ are the points $\left(\dfrac{l_1}{n_1}, \dfrac{m_1}{n_1}, 1 \right)$ &c. Hence if X, Y, Z are the areal coordinates of the point $(x, y, 1)$ when $A_1 B_1 C_1$ is the triangle of reference in the plane $z = 1$,

$$\Delta X = n_1(x l_1' + y m_1' + n_1') \text{ &c. ;}$$

where l_1' is the cofactor of l_1 in the determinant $[l_1 m_2 n_3] = \Delta$, and so on.

Therefore, by solving,

$$x = \frac{l_1 X}{n_1} + \frac{l_2 Y}{n_2} + \frac{l_3 Z}{n_3}, \quad y = \frac{m_1 X}{n_1} + \frac{m_2 Y}{n_2} + \frac{m_3 Z}{n_3}.$$

Now $lx + my + nz = 0$ meets $z = 1$ in the line $lx + my + n = 0$,
i. e. $\dfrac{X}{n_1}(l l_1 + m m_1 + n n_1) + \ldots + \ldots = 0.$

Denote the angles between OA, OB, OC and the normal to the plane of projection (the axis of z) by α, β, γ, and the angles POA, POB, POC by λ, μ, ν. Then the gnomonic projection of the great circle in which a plane through O perpendicular to OP meets the sphere is the line

$$X \sec \alpha \cdot \cos \lambda + Y \sec \beta \cdot \cos \mu + Z \sec \gamma \cdot \cos \nu = 0,$$

i. e.

$$\frac{u X}{a \cos \alpha} + \frac{v Y}{b \cos \beta} + \frac{w Z}{c \cos \gamma} = 0$$

(where $a : b : c$ are the axial ratios), or $u \xi + v \eta + w \zeta = 0$, taking $a \cos \alpha \cdot \xi = X$ &c.

2. The axis of the zone containing $(u \, v \, w)$ and $(u' \, v' \, w')$ meets the plane of projection in the point

$$u \xi + v \eta + w \zeta = u' \xi + v' \eta + w' \zeta = 0.$$

It follows that the zonal axis [U V W] meets the plane of projection in the point $\dfrac{\xi}{U} = \dfrac{\eta}{V} = \dfrac{\zeta}{W}.$

3. Similarly, if A', B', C' are the poles of the faces (1 0 0), (0 1 0), (0 0 1) and A_1', B_1', C_1' their projections, the projection of the zone [U V W] is $U \xi + V \eta + W \zeta = 0$, where

$$\xi = X a \sec \alpha \cdot \cos AOA' \text{ &c.,}$$

and the projection of the pole of the face (u, v, w) is

$$\frac{\xi}{u} = \frac{\eta}{v} = \frac{\zeta}{w},$$

$A_1' \, B_1' \, C_1'$ being the triangle of reference.

4. These results have many important applications ; for example, the anharmonic ratio of the four cozonal faces $(u_1 v_1 w_1)$, $(u_2 v_2 w_2)$, $(u_3 v_3 w_3)$, $(u_4 v_4 w_4)$ is that of the four concurrent lines $u_1 \xi + v_1 \eta + w_1 \zeta = 0$ &c., *i. e.* that of the four

lines $v_1\eta + w_1\zeta = 0$ &c., which is

$$(v_1 w_3 - v_3 w_1)(v_2 w_4 - v_4 w_2) \div (v_1 w_4 - v_4 w_1)(v_2 w_3 - v_3 w_2).$$

5. It is interesting to extend these theorems to the projection from any point Q onto a plane perpendicular to O Q. The extension is as follows :—

The projection of the line of intersection of a plane through O parallel to the face $(u\ v\ w)$ with the plane A B C is the line $u\xi + v\eta + w\zeta = 0$, and the projection of the zonal axis [U V W] passes through $\dfrac{\xi}{U} = \dfrac{\eta}{V} = \dfrac{\zeta}{W}$; where

$$a(d + \cos\alpha)\xi = X,\ \ b(d + \cos\beta)\eta = Y,\ \ c(d + \cos\gamma)\zeta = Z,$$
$$\text{and } OQ = d;$$

the vertices A_1, B_1, C_1 of the triangle of reference being the projections of A, B, C, and X, Y, Z being areal coordinates.

This may be proved thus :—

Let Q be the point $(0, 0, -d)$, and project from Q onto the plane $z = 0$ (any parallel plane would do). The plane ABC is $D_1 x + D_2 y + D_3 z + \Delta = 0$, where D_1, D_2, D_3, Δ are the determinants $[m_1\ n_2\ 1]$, $[n_1\ l_2\ 1]$, $[l_1\ m_2\ 1]$, $[l_1\ m_2\ n_3]$. The plane joining the intersection of $lx + my + nz = 0$ with the plane ABC to Q is

$$(\Delta - dD_3)(lx + my + nz) + dn(D_1 x + D_2 y + D_3 z + \Delta) = 0,$$

which meets $z = 0$ in the line

$$(\Delta - dD_3)(lx + my) + dn(D_1 x + D_2 y + \Delta) = 0.$$

Now A_1 is the point $\left(\dfrac{l_1 d}{n_1 + d}, \dfrac{m_1 d}{n_1 + d}, 0\right)$ &c. ; and therefore if the areal coordinates of the point $(x, y, 0)$ referred to $A_1 B_1 C_1$ as the triangle of reference in the plane $z = 0$ are X, Y, Z,

$$dDX = (n_1 + d)(L_1 x + M_1 y + N_1 d) \text{ &c. ;}$$

where L_1, M_1, N_1 are the cofactors of l_1, m_1, $n_1 + d$ in the determinant $[l_1\ m_2\ n_3 + d] = D$ and so on.

Solving, we have

$$\frac{x}{d} = \frac{l_1 X}{n_1 + d} + \frac{l_2 Y}{n_2 + d} + \frac{l_3 Z}{n_3 + d},\ \ \frac{y}{d} = \frac{m_1 X}{n_1 + d} + \frac{m_2 Y}{n_2 + d} + \frac{m_3 Z}{n_3 + d}.$$

Substituting in

$$(\Delta - dD_3)(lx + my) + dn(D_1 x + D_2 y + \Delta) = 0,$$

we get

$$(\Delta - dD_3)\left(\frac{X}{n_1 + d}(ll_1 + mm_1) + \dots + \dots\right)$$
$$+ n\left(\frac{X}{n_1 + d}(dl_1 D_1 + dm_1 D_2 + n_1 + d\Delta) + \dots + \dots\right) = 0,$$

or

$$\frac{X \cos \lambda}{d + \cos \alpha} + \frac{Y \cos \mu}{d + \cos \beta} + \frac{Z \cos \nu}{d + \cos \gamma}$$

$$= \left(\frac{n}{d D_3 - \Delta}\right)\left(\frac{dX}{n_1 + d}(l_1 D_1 + m_1 D_2 + n_1 D_3 + \Delta) + \ldots + \ldots\right) = 0.$$

Hence the first part of the theorem follows as in § 1. The second part is proved as in § 2, if we notice that the projection of the intersection of any two planes passes through the projection of the intersection of the lines in which these two planes meet the plane ABC.

6. Similarly, the projection of the line of intersection of the zone [U V W] with the plane A′ B′ C′ is the line $U\xi + V\eta + W\zeta = 0$, and the projection of the normal through O to the face $(u\, v\, w)$ passes through $\dfrac{\xi}{u} = \dfrac{\eta}{v} = \dfrac{\zeta}{w}$, where $(d + \cos \alpha)\xi = Xa \cos AOA'$ &c. ; the vertices of the triangle of reference being the projections of the poles of $(1\,0\,0)$, $(0\,1\,0)$, $(0\,0\,1)$.

7. The most interesting cases are those in which $d = 0, 1$, or ∞, corresponding to a gnomonic, stereographic, or orthogonal projection. It may be readily shown that the theorems still hold in these special cases.

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VII. *The Chemical and Geological History of the Atmosphere.* By JOHN STEVENSON, M.A., F.I.C.[*]

III.

Variations in the Amount of Atmospheric Carbonic Acid, with special reference to the subject of Geological Climate.

IN the two former articles of this series (Phil. Mag. Sept. 1900, p. 312; Oct. 1900, p. 399, and Oct. 1902, p. 435), in which questions relating to the history of free oxygen and the composition and extent of the primitive atmosphere were discussed, we concluded that the atmosphere was almost certainly many times more extensive in very primitive times than it is now; that it probably contained no free oxygen; that it most probably contained very large quantities of carbonic acid; and that it possibly may have contained large quantities of hydrogen and hydrocarbon gases either in addition to, or instead of, the carbonic acid gas.

Assuming that the latter were the principal gases originally

[*] Communicated by the Author.

present, we saw that they would be converted into water and carbonic acid through the agency of vegetation. There is even a possibility that they may have been directly absorbed by ancient forms of vegetation, and in the subsequent heating by volcanic action of the vegetable remains in presence of rocks containing oxide of iron, carbonic oxide, carbonic acid and water would be produced. Carbonic acid, whether produced in this way or present from the first in the atmosphere, would also be absorbed by vegetation, this process of absorption being accompanied by the decomposition of the carbonic acid and the production of free oxygen, though it should be noted that there would not be permanent free oxygen in the atmosphere as long as free hydrogen existed. The free oxygen produced from carbonic acid by vegetable growth being in the nascent condition, would immediately combine with the hydrogen present in the atmosphere producing water, as inferred from Dr. Phipson's experiment which was referred to in a previous article. It is possible, no doubt, that the hydrogen which disappeared in the course of the experiment was absorbed directly by the plant; but its oxidation by nascent oxygen seems a highly probable explanation of the effect observed; and in any case it is obvious that we could not expect free oxygen and free hydrogen to exist long together in the atmosphere in considerable quantities. After all the free hydrogen had disappeared, permanent free oxygen, *i. e.* free oxygen as a permanent constituent of the atmosphere, would begin to appear, and the quantity would gradually increase until it reached a limit determined principally by the amount of oxidizable matter within reach and the readiness of this oxidizable matter to combine with oxygen under prevailing conditions. On the other hand, the quantity of carbonic acid in the atmosphere would probably diminish steadily during the early geological epochs, until a certain limiting percentage determined by a variety of conditions was reached. Large quantities of carbonic acid are annually produced and added to the atmosphere through the operation of various agencies or reactions, and large quantities are removed or decomposed, principally through the growth of vegetation and the weathering of rocks; that is to say, through the decomposition of silicates by carbonic acid and water, resulting in the formation of carbonates and free silica, or of rocks containing a higher proportion of silica than the original rocks. This latter process probably operated with very great effect in early geological epochs, when the proportion of carbonic acid in the atmosphere was greater than it is now; and we also infer that

there was probably a long period during which the forces or agencies which removed carbonic acid from the atmosphere were stronger than those which produced or returned it.

However, it is quite possible that after a certain point was reached in this abstracting process, the amount of carbonic acid in the atmosphere may have become variable; that is to say, the quantity may have alternately increased and diminished according to the relative activity of the forces or reactions which produce or evolve carbonic acid, and the counteracting forces or reactions which decompose carbonic acid or otherwise remove it from the atmosphere.

The view that the amount of atmospheric carbonic acid varies or may vary within very wide limits is not a new one ; at least the view that the percentage was at one tiem very different from what it is now is by no means new. Hugh Miller and probably most of his contemporaries were of opinion that the amount of carbonic acid in the atmosphere at the beginning of the Carboniferous epoch was very much greater than it is now, the difference being represented roughly by the amount of coal deposited during and since that period, and therefore amounting to many times, possibly one hundred or even several hundred times, the amount of our present atmospheric carbonic acid. Probably, however, the view usually taken by the earlier geologists on the question was not so much that the amount of atmospheric carbonic acid varies from time to time, as simply that the amount was very much greater in very early times than it is now. That is not quite the view taken in the present article. We propose to give reasons in support of the view that the amount of carbonic acid in the atmosphere has varied very considerably by increasing at certain times as well as by decreasing at others, during a very large part of geological history—possibly enough in early epochs such as the Cambrian and Silurian, as well as in more recent times.

It is obvious that among the forces which produce or evolve carbonic acid a prominent place is occupied by volcanic action. It has long been known that carbonic acid is evolved from volcanoes and also from the ground in volcanic districts even where the volcanoes are no longer active. This volcanic or telluric carbonic acid (as it was called by Prof. Dittmar) is probably for the most part produced by the heating of rocks containing limestone or other carbonates along with sandstone (silica), or with rocks containing a high percentage of silica to such an extent that the carbonates are decomposed and silicates are produced (or silicates with a larger percentage of base than the original

silicates are produced), the reaction in both cases being accompanied by the evolution of carbonic acid. Another way in which carbonic acid may be evolved through volcanic action, is by heating rocks containing oxide of iron in presence of coal or carbonaceous matter to such an extent that metallic iron is produced, and carbonic acid and carbonic oxide are evolved.

Now it is highly probable that throughout geological history, volcanic action has varied considerably in its degree of activity. At any rate, we know for a certainty that it has varied to a very great extent in special areas or localities; and it is quite probable that on the earth, taken as a whole, volcanic activity has varied very considerably in degree in different geological epochs. Even if the amount of volcanic action as inferred from the production of volcanic rocks and other heating effects should have been fairly constant on the earth as a whole, yet the amount of carbonic acid produced or evolved may have varied very considerably according to variations in the composition and distribution of the rocks subjected to heating. The question therefore naturally arises as to whether such variations may have taken place to such an extent as to affect materially the amount and percentage of atmospheric carbonic acid.

In order to approach this question, it is obviously necessary that we should ascertain with a sufficient degree of approximation (1) the amount of carbonic acid at present existing in the atmosphere ; (2) the amount of carbonic acid annually produced or evolved into the atmosphere by the principal agencies which have this effect; (3) the amount which is annually decomposed or removed from the atmosphere by other agencies; (4) the possible and probable variations that take place in the degree of activity of these various agencies ; and (5) the methods in and extent to which these agencies react on each other.

As regards the amount of carbonic acid in the atmosphere, this may be taken in round figures as $2 \cdot 2 \times 10^{12}$ tons (rather more than two million million tons). The most recent and reliable determinations of atmospheric carbonic acid give a figure considerably lower than those which formerly were accepted. Prof. Letts, of Belfast, investigated the whole subject very carefully a few years ago, and found that the average amount of carbonic acid in the air at Queen's College, Belfast, was only $2 \cdot 93$ volumes in 10,000 volumes of air, or $0 \cdot 0293$ per cent. by volume ; and this figure is more likely to be over than under the average for the whole earth, as the air of cities is richer in carbonic acid than ordinary sea and

country air. Müntz and Aubin of Paris, who made or collected a large number of the most reliable determinations made in various parts of the world, found that the average percentage of carbonic acid in the air of the Northern Hemisphere (as calculated from the data supplied by the material collected) was 0·0282 by volume, and that the average percentage in the Southern Hemisphere was 0·0272. The mean of these two percentages is 0·0277, which may therefore be taken as the average for the atmosphere as a whole. This figure, when reduced to percentage by weight, becomes 0·0423, carbonic acid gas being 1·528 times heavier than ordinary air. At this percentage the total quantity of carbonic acid in the atmosphere (the weight of which, as we have already seen, may be taken as 5200×10^{12} tons) becomes $2·2 \times 10^{12}$ tons, as stated above.

Of the various agencies which affect the amount of carbonic acid in the atmosphere, either by adding to or removing from it, probably the most important at the present time are the growth of vegetation, on the one hand, and the oxidation of vegetable and other organic remains, on the other. So great must be the amount of carbonic acid annually decomposed by vegetation, that it is doubtful if the atmosphere contains enough carbonic acid to suffice (without assistance) for 50 years' growth of vegetation at the present rate. This can be readily shown by a simple calculation. Let us take a very moderate estimate of the rate of growth of vegetation on the land surface of the earth, viz. 1 ton of dry wood or hay &c. per acre per annum, this being only one-half of Liebig's estimate for ordinary meadow, forest, and agricultural land, as stated in a previous article; and let us leave the vegetation that grows in the sea out of account altogether. Now 1 ton of dry wood may be regarded as containing roughly 0·4 ton carbon, and to obtain 0·4 ton of carbon 1·466 tons of carbonic acid would have to be decomposed. The production of 1 ton of dry wood &c. per acre per annum therefore requires the decomposition of 1·466 tons carbonic acid per acre per annum, which is equivalent to 939 tons per square mile, or $46,950 \times 10^{6}$ tons for the whole land surface of the earth, which may be taken in round numbers as 50,000,000 square miles. This figure ($46,950 \times 10^{6}$ tons) is roughly the 47th part of the total weight of atmospheric carbonic acid as given above.

A very considerable quantity of carbonic acid must also be removed from the atmosphere through the decomposition of certain rocks by water and carbonic acid, a reaction which has already been referred to. It is not very easy to

form an approximate estimate of the quantity of carbonic acid annually removed from the atmosphere in this way, even though it is comparatively easy to form an estimate of the total quantity of carbonate of lime and other carbonates that are annually carried into the sea by rivers, the difficulty being in estimating the proportion of these carbonates that is derived from limestone and other carbonates already present in the soil and rocks subject to denudation, and the proportion that is formed through the decomposition of silicates by water and carbonic acid. However, it can be easily shown that the quantity of carbonic acid removed in this way must be very much smaller than the quantity annually removed by the growth of vegetation. Prof. Dittmar (article on Sea Water in Encyc. Britt.) estimated that the total carbonate of lime introduced into the ocean annually by all rivers amounts to $1·34 \times 10^9$ tons (or the $\frac{1}{119000}$ part of the carbonate of lime already present in the ocean, which he estimated at 160×10^{12} tons). The amount of carbonic acid present in $1·34 \times 10^9$ tons carbonate of lime is 590×10^6 tons, a quantity which is equal to about $\frac{1}{86}$ of the quantity annually decomposed by vegetation as calculated above. A certain amount of carbonate of lime is no doubt also formed by the direct action of the sea on rocks containing silicates, and the free carbonic acid which takes part in this reaction may be regarded as being removed from the atmosphere since the sea and the atmosphere react on each other ; but the quantity removed in this way is probably not so great as is required to form the carbonates annually carried into the sea by rivers. Also, as already remarked, it is only a portion of the carbonic acid present in the carbonates carried into the sea by rivers that can be regarded as having been derived directly from the atmosphere ; and therefore we are entitled to infer that the total quantity of carbonic acid which is removed from the atmosphere through the weathering of rocks is very much less than the quantity which is decomposed through the growth of vegetation.

Of the agencies which produce or evolve carbonic acid the most important, as already stated, is the oxidation of vegetable and other organic remains. This oxidation takes place in one or other of three different ways : (1) by animal respiration, (2) by active combustion with evolution of light and heat, and (3) by eremacausis or slow decay. All of these are important as methods in which carbonic acid is produced in large quantities, and as a result of the total oxidation effected in these ways, by far the greatest proportion of vegetable and animal remains is oxidized to water and carbonic acid. Still it

is obvious that a considerable proportion of organic remains escapes oxidation, and is preserved for a considerable time, at any rate, in the form of peat or coal. It is not easy to form an estimate of the proportion or percentage of vegetable remains that is preserved in this way; but when it is observed that about 6,000,000 acres in the United Kingdom, or nearly 8 per cent. of the total surface of the country, are covered by peat-bogs, it is obvious that the proportion must be of considerable importance. No doubt the rate of growth of vegetation in or on peat-bogs may be much less than the average rate for the whole country, and the proportion of the earth-surface, taken as a whole, which is covered by peat-bogs may be very much less than 8 per cent.; but still it is quite possible that the amount of vegetable remains which escapes oxidation in this and in other ways may be as much as 1 per cent. of the total quantity of such remains, and probably it is at least 0·1 per cent. That is to say, there is more carbonic acid annually removed from the atmosphere by the growth of vegetation than is returned to it by the oxidation of vegetable and animal remains, the difference being probably at least 0·1 per cent. of the amount removed, and possibly as much as 1 per cent. or even more.

Of the other agencies which add carbonic acid to the atmosphere, the most important is probably volcanic action, or the heating of certain rocks under such conditions as to cause the evolution of carbonic acid; but still we have very little quantitative information regarding it. Boussingault considered that the volcano Cotopaxi evolved annually more carbonic acid than a whole city like Paris (which he calculated to evolve about 3,000,000 cubic metres of carbonic acid daily); and Lecoq calculated that the mineral springs of Auvergne gave off annually 7000×10^{6} cubic metres of carbonic acid gas, an amount rather less than $\frac{1}{10}$ of the volume produced by the annual combustion of the coal employed throughout the whole of Europe. Bischof estimated that the quantity of carbonic acid evolved in the Brohl Thal in the Eifel district of Rhenish Prussia amounted to 5,000,000 cubic feet, or 300 tons of gas per day. These are not perhaps very large quantities in themselves; but still there are many other volcanoes and volcanic regions in the world, and the total quantity of carbonic acid evolved from subterranean sources may be very large indeed. It is also known from the Report of the 'Challenger' Expedition that carbonic acid is evolved in some places at the bottom of the sea in very considerable quantities. Further, it is even possible that carbonic acid may be evolved from the ground in regions which are

not usually regarded as being volcanic. It is well known that "ground air," that is, air lying close to the ground or enclosed within the upper layers of the soil, is usually richer in carbonic acid than the atmosphere in general. This higher percentage of carbonic acid in ground air is probably for the most part due to eremacausis or oxidation of organic matter within the soil; but possibly it may not be all due to this cause, some of it may come from a deeper source. The carbonic acid that occurs so frequently in wells and pit-workings is not likely to be all produced by eremacausis in the upper layers of the soil, it is more likely to have a deeper source even though the source should not be of a volcanic nature.

There are still one or two other ways deserving of notice in which carbonic acid is produced and added to the atmospheric supply. The process of fermentation (natural and artificial) is one of these, and possibly it is of some quantitative importance; but still it may be regarded as a phase of eremacausis, or at least as a stage in the oxidation of certain substances, and therefore it hardly needs to be discussed separately. The combustion of ordinary coal should also be mentioned, as coal is now mined and consumed on a very large scale, the amount now raised being about 800,000,000 tons per annum. Taking the average amount of carbon in coal as 80 per cent., the burning of the above quantity of coal will cause the addition of about 1700×10^6 tons carbonic acid to the atmosphere annually. This amount is large enough to have an important bearing on the question as to whether the percentage of carbonic acid in the atmosphere is increasing at present or not; but still it is obvious that the whole period during which coal has been mined on a large scale is not very great (from a geological standpoint); and we may quite well leave the raising and burning of coal out of consideration for the present as a noteworthy factor in the history of the atmosphere. It is therefore evident that we must look upon the carbonic acid of telluric or subterranean origin as being sufficient not only to make good the loss of atmospheric carbonic acid which is caused by the weathering of rocks, but also to make good the difference between the amount that is decomposed by the growth of vegetation and the amount that is restored to the atmosphere by the oxidation of vegetable and other organic remains. It is somewhat unfortunate that there is so little information regarding the amount of telluric carbonic acid annually evolved, but there is no difficulty in understanding that it may be or could be very large when we consider the amount of limestone and other carbonates existing in the earth, and the way in which they

may be affected by volcanic action and caused to give up their carbonic acid. Dr. Sterry Hunt calculated that there was probably enough limestone on the earth to yield a quantity of carbonic acid equal to 200 times the volume of our present atmosphere. This quantity would be 700,000 times greater than the amount of carbonic acid at present existing in the atmosphere. It would be about 35,000,000 times greater than the amount annually decomposed by vegetation as calculated above, and of course would be very many times greater than the difference between the amount annually decomposed by vegetation and the amount annually restored by the oxidation of organic remains. It is therefore quite clear that there are ample stores to draw upon, and a simple calculation will show that quite a limited area of country, when subjected to volcanic action, may yield a large quantity of carbonic acid.

Ordinary limestone has the specific gravity 2·6, and therefore weighs 162 lbs. per cubic foot. It contains, if pure, 44 per cent. by weight of carbonic acid. A cubic mile of limestone therefore contains 4686×10^6 tons carbonic acid, a quantity which is roughly equal to $\frac{1}{10}$ of the amount that we have calculated is annually removed from the atmosphere by the growth of vegetation, and 470 cubic miles of limestone contain an amount of carbonic acid equal to the total carbonic acid of the atmosphere, viz. $2·2 \times 10^{12}$ tons. Now the average aggregate thickness of limestone and other carbonates all over the earth must be about 1·64 miles in order to contain the quantity of carbonic acid estimated by Dr. Sterry Hunt; and therefore, on the average, a surface area of 286 square miles contains beneath it in the form of limestone and other carbonates as much carbonic acid as there is in the whole of the atmosphere. A region 100,000 square miles in extent will contain, on the average, about 350 times as much as the above quantity; and if subjected to volcanic action, we may expect that a large proportion of this carbonic acid will be evolved in the free condition. There is therefore no difficulty in understanding that there is enough carbonic acid of volcanic or telluric origin to maintain the balance or equilibrium between the amount of carbonic acid removed from the atmosphere and the amount restored to it by the various agencies already referred to. Further, it is obvious from considerations regarding the erratic or intermittent character of volcanic action generally, and also from considerations regarding variations in the composition and distribution of rocks subjected to volcanic action, that there will most probably be great variations in the amount of telluric

carbonic acid annually evolved. If, then, there has been on the average during recent geological epochs just enough carbonic acid of telluric origin to maintain the balance referred to above, there must have been times when much more was evolved than was required to make good the balance, and there would also be times when there was not enough evolved. Also, from considerations regarding the length of time that volcanic activity prevails or continues in any particular district, we are entitled to infer that a period, during which more telluric carbonic acid than is required for the balance is added to the atmosphere, may last for a long time—long enough to cause a very substantial increase in the amount of carbonic acid present in the atmosphere. Further, it is possible that periods during which there is a deficiency may also last long enough to cause a serious diminution in the amount of atmospheric carbonic acid. It is thus evident that in this way very considerable variations may have taken place in the percentage of atmospheric carbonic acid during recent geological epochs; and it is quite possible that these variations may account for the climatic changes of the occurrence of which there is an abundant and striking body of evidence, and the causes of which have caused much discussion among geologists and astronomers.

As far back as 1845 or so it was pointed out by Ebelmen (as quoted by Dr. Sterry Hunt) that the greater weight of an atmosphere charged with carbonic acid would increase the temperature due to solar radiation at the earth's surface, and greatly modify atmospheric phenomena. Later on, Tyndall by his researches on radiation showed that certain gases have a very considerable heat-absorbing power; so that a small quantity of one or more of such gases present in the atmosphere might have a very considerable effect on climatic conditions. He even suggested that questions relating to the climate of different geological epochs might find their solution in the presence in the atmosphere of moderate quantities of a gas like carbonic acid.

More recently the subject has been investigated by E. Lechner of Vienna and S. Arrhenius of Stockholm. Lechner announced as the result of his investigations (Phil. Mag. ser. 5, vol. ii. 1881, p. 76) that carbonic acid rather than aqueous vapour is the constituent of the atmosphere which absorbs the sun's radiation ; and he was able by his method of working to calculate the percentage of carbonic acid in the atmosphere, finding it 3·27 parts in 10,000 parts

by volume, which is not far from the results obtained by chemical analysis.

Prof. Arrhenius has made an elaborate investigation of the subject by a special method, the results of which he gave in a paper published in the Philosophical Magazine for April 1896, p. 297. From conclusions drawn from the data obtained by means of his experiments, he calculated the extent to which the carbonic acid in the atmosphere would have to vary in order to bring about the temperature of the Tertiary and the Ice Ages respectively. He found that the temperature in the Arctic regions would rise about 8° or 9° C. if the carbonic acid were increased to 2·5 or 3 times its present value; and that in order to get the temperature of the Ice Age between the 40th and 50th parallels of latitude the carbonic acid in the air should sink to 0·62–·5 of its present value (lowering the temperature by 4° or 5° C.).

Assuming that these results are reliable, the question arises as to whether it is possible or probable that the amount of carbonic acid in the atmosphere may have varied to the above extent; that is, whether it may at one period have been two or three times as much as at present, and at another period only about half of the present quantity.

If we suppose that the amount of carbonic acid annually restored to the atmosphere by the oxidation in various ways of organic remains amounts to 99·9 per cent. of the amount annually removed by the growth of vegetation and the weathering of rocks, it is evident that 0·1 per cent. of the amount would have to be added from subterraneous sources to maintain the balance; or, following the estimate above given regarding the growth of vegetation, there would be annually required $46·9 \times 10^6$ tons of carbonic acid of telluric origin. Now if we suppose this to be increased through any cause by 50 per cent. of itself, or $23·4 \times 10^6$ tons, there would be a net addition of $23·4 \times 10^6$ tons per annum to the carbonic acid of the atmosphere; and if we suppose this addition or extra quantity to be continued steadily for many years, in 141,000 years the total carbonic acid of the atmosphere would be $5·5 \times 10^{12}$ tons, or $2\frac{1}{2}$ times its present amount, and in 188,000 years it would be $6·6 \times 10^{12}$ tons, or 3 times its present amount. Similarly, if the amount of subterranean carbonic acid should fall off by the above quantity, causing a deficiency of $23·4 \times 10^6$ tons per annum, and if this deficiency should continue for a long term of years, then in 47,000 years the amount of atmospheric carbonic acid would be reduced to $1·1 \times 10^{12}$ tons, or one-half of its present value.

The above figures are probably well within the limits

imposed by the conditions of the problem (so far as these are known): that is to say, they probably do not err by being put too low, and indeed variations in the percentage of atmospheric carbonic acid to the extent indicated may have taken place in a much shorter time. For, as we have observed already, it is quite possible that the amount of carbonic acid annually restored to the atmosphere by the oxidation of organic remains in various ways may not be more, and may even be less, than 99·0 per cent. of the amount annually removed by the growth of vegetation and the weathering of rocks; in which case we would infer that the normal or average amount of carbonic acid of telluric origin annually added to the atmosphere is not less than 1 per cent. of the amount annually removed by vegetation &c. Now if we suppose, as before, that the average annual amount of telluric carbonic acid is increased by one-half of itself, and that the extra quantity continues to be evolved steadily for a long time, then in 19,000 years the percentage of atmospheric carbonic acid would be increased to 3 times its present value. On the other hand, if the annual yield of telluric carbonic acid were diminished, and remained diminished, by one-half of its normal quantity, then the total amount of atmospheric carbonic acid would be reduced by one-half in 4700 years. Further, since it is quite possible that in periods of great volcanic activity the amount of carbonic acid evolved may be twice or even several times greater than the average or normal amount evolved, and also since it is possible that the amount evolved in very quiescent times may be much less than half the average amount, it is evident that very great changes in the amount of atmospheric carbonic acid may be brought about in considerably shorter periods than 19,000 or 4700 years.

We admit, of course, that in making calculations of this kind, some allowance might have to be made for the effect of an increase or diminution in the quantity of atmospheric carbonic acid on the luxuriance of vegetation, and the activity of other agencies which tend to remove carbonic acid from the atmosphere. It is highly probable that in the event of a large increase in the percentage of atmospheric carbonic acid, there would be an increase in the luxuriance of vegetation on the earth taken as a whole (other conditions being assumed to be as favourable as they are at present), on account of the (hypothetical) increase in temperature itself, and possibly also on account of the greater quantity of carbonic acid readily available for respiration by the leaves of vegetables. The latter is a point of considerable interest and deserves investigation. O. Wollny, who has done a

great amount of work in connexion with the question of local variations in the percentage of carbonic acid and allied subjects, once observed (as quoted by Prof. Letts in his work on the Carbonic Anhydride of the Atmosphere) that the "crown air" of a tall pine-tree in a plantation in the Karsten forest showed a very low percentage of carbonic acid, so low that he was not willing to accept the result; and he was of opinion that some of the carbonic acid had combined with the lead of the long tube used in the experiment, the sides of the tube having become moist. Whether this experiment should have any positive value or not, it is quite credible that on a very calm bright day the percentage of carbonic acid in the crown-air of a tree in full foliage may become much reduced by the process of respiration, and therefore the growth of the tree may be less rapid than it would be if the carbonic acid were kept up to or near its normal percentage by the circulation of the air, or otherwise. At the same time, the limits imposed on the luxuriance of vegetation in those parts of the world where it is poor seem not to have much direct reference to the amount of carbonic acid in the atmosphere, but rather to the conditions regarding temperature, sunlight, rainfall or water-supply of some kind, and fertility of the soil. It is quite possible, however, that in regions where vegetation is very luxuriant, a certain amount of restriction is imposed by the limited quantity of atmospheric carbonic acid, especially if the weather should frequently be very calm. In the forests of the Amazons and other luxuriant forests, where the struggle for life on the part of vegatation is to a great extent a struggle for light, the amount of growth is largely conditioned by the amount of foliage that can reach the sunlight; and if the percentage of carbonic acid in the air at the tops of the trees should be much reduced in calm weather by the respiration of the leaves, it is very probable that the growth of the trees will be thereby to some extent affected.

However, it is highly probable that considerations of this kind do not after all affect the main question before us very seriously; for we have to do not so much with the total amount of vegetable growth per annum or the total amount of carbonic acid annually decomposed and otherwise removed from the atmosphere, but rather with the difference between the amount annually removed by vegetation and otherwise and the amount annually restored by eremacausis and other modes of oxidation. Now it is quite reasonable to suppose that if the luxuriance of vegetation were to be very greatly increased, the amount of carbonic acid arising from the

oxidation of vegetable remains would be very greatly increased also, and that to an extent probably not much less than the ratio of the increase in the luxuriance of vegetation itself. The difference between the amount of carbonic acid annually decomposed by vegetation and the amount annually restored by eremacausis &c. would no doubt be greater in such a case than it is at present ; but it would not necessarily be enormously greater, and indeed it is not at all likely that it would go beyond the limits that a large increase in the supply of telluric carbonic acid (but still an increase only to the extent limited by the general conditions of the problem and the probabilities of the case) would be quite able to overtake.

We may thus consider that a sufficiently clear case has been made out for the variability of the amount of atmospheric carbonic acid within very wide limits, limits wide enough to make the theory that the great climatic changes of geological history are due to variations in the amount of atmospheric carbonic acid quite a practicable, if not also a highly probable one. An interesting point that now suggests itself is whether the theory can be tested by an appeal to known geological facts; that is to say, whether there is evidence to the effect that there was great and unusual volcanic activity during or just before warm geological epochs, and a marked decrease in volcanic activity during cold or glacial epochs. In this connexion it is at once recalled that there was great volcanic activity in Scotland during the Tertiary epoch or part of it; but still it would not do for us to lay much stress on the evidence of a high degree of volcanic activity in one locality only. In order to make this inquiry of any practical use, it would be necessary for us to have some knowledge of the general conditions as regards volcanic activity over the whole world during each epoch; the reason for this being that carbonic acid evolved in any particular region is quickly diffused through the whole atmosphere of the earth. The percentage of carbonic acid in the atmosphere is practically the same in all latitudes and longitudes; and probably this uniformity has always held good no matter what the exact percentage for the time being may have been. For the same reason also the theory of the course of climatic change that is based on variations in the percentage of atmospheric carbonic acid is general or universal in character ; that is, if climatic changes are brought about in this way at all, they must take place in all parts of the world and in both hemispheres alike, no matter where the seat of special volcanic activity (in the case of an increase in the amount of carbonic acid) may be. This feature

of the theory is probably a distinct advantage to it when compared with other theories of climatic change ; but still the theory need not be regarded as antagonistic to other theories, especially to those which have reference to changes in local conditions. There can be no doubt that variable local conditions, such as the height of the land above sea-level, the distance from the sea, and the direction of prevailing winds and ocean currents, have a very great influence on the climate of any particular locality. However, it is fairly well recognized that variations in local conditions like the above cannot be regarded as at all sufficient to account for the facts observed in connexion with the climatic changes of geological history, and a more generally operating cause must be sought for, such as we have in the variability of the carbonic acid of the atmosphere. Though I do not venture to make any definite statement about any observed connexion between volcanic activity and geological climate, I hope that geologists will state their views on the subject; and if it should be considered that there is not enough evidence to draw a definite conclusion, I hope that by and bye a sufficient amount of evidence will be obtained. In any case, as already remarked, the large amount of coal deposited in the earth is a strong argument in favour of the variability of the amount of atmospheric carbonic acid ; and as it is usually considered that the periods in which large quantities of coal were deposited were warm periods, it is obvious that this also to some extent supports the view that there is a connexion between the amount of atmospheric carbonic acid and the general temperature of the atmosphere and surface of the earth.

VIII. *Investigation of the Variations of Magnetic Hysteresis with Frequency. By* Thomas R. Lyle, *M.A., Professor of Natural Philosophy in the University of Melbourne* *.

[Plate II.]

IN the following paper are given some results obtained by my wave-tracer, of which a description has been published in the ' Philosophical Magazine ' for November 1903.

The work described, which is of a preliminary nature, was in great part performed more than a year ago, but a severe illness has prevented me until now both from preparing it for publication and from continuing as intended the same

work in a more accurate way. It is hoped that the experimental results given below will be of sufficient interest to merit publication now.

1. The experiments were made on two rings of laminated annealed iron, in one of which the radial breadth of the iron was considerable relative to its mean radius. These rings were magnetized by alternating currents of different strengths and periods ; both the magnetizing-current wave and the magnetic-flux wave were quantitatively determined by the wave-tracer, using the galvanometer method described in the paper already quoted, and the wave-forms so obtained were subjected to harmonic analysis.

The experiments were divided into series in which the period and wave-form of the magnetizing current were kept as nearly constant as possible throughout any one series, while its strength was varied. The analytic expressions for the associated current and flux waves for a few series are given in tabular form, and some of their more interesting amplitude and phase relations are shown by means of curves.

From the analytic expressions for each pair of associated waves the total iron loss (I, say) per cubic centimetre per cycle was calculated, and it was found, when the magnetizing current is approximately sinusoidal, that I is given with considerable accuracy by the formula

$$I = (\cdot 00186 + \cdot 000026 \ n) \ \mathfrak{B}^{1\cdot57}$$

for ring I.,

and by

$$I = (\cdot 001684 + \cdot 0000272 \ n) \ \mathfrak{B}^{1\cdot57}$$

for ring II.,

where n is the number of periods per second and $\mathfrak{B}$ (called the effective induction) is $\sqrt{2}$ times the root of mean square of $d\mathrm{B}/dt$, for all values of n, and for all values of the induction between 1000 and 12,000.

When from the total iron loss I per cm.[3] per cycle the sum of the statical hysteresis (U, say) previously obtained by Ewing and Klaasen's method, and $\mathrm{\ddot{E}}$ the value that theory assigns to eddy-current loss, was subtracted, a considerable quantity (I − U − E) remained, which increased both when the frequency and when the flux-density increased. This quantity, called by Fleming the kinetic hysteresis, has been obtained for each experiment, and is given in the tables that are to follow, and curves are also given which show how it varies with the frequency and flux-density. That such a

source of loss exists when iron is subjected to alternating magnetizing forces has already been shown by Steinmetz and by Siemens by other means.

An interesting case of transformation and what is called reflexion of energy is drawn attention to and discussed. If the E.M.F. impressed on the magnetizing circuit on the iron ring be sinusoidal, the flux-wave produced contains third, fifth, &c. harmonics. These higher flux-harmonics induce currents in the magnetizing circuit which are dissipated as heat in it. Thus we have a transformation of electric energy due to alternating currents of frequency n to energy of currents whose frequencies are $3n$, $5n$, &c., which is reflected back into the magnetizing circuit. This the author believes is a hitherto unnoticed source of transformer loss.

2. Fig. 1 (Pl. II.) shows the arrangement of the apparatus. The magnetizing current was obtained from a four-pole rotary converter T supplied with direct current from storage-cells. By means of rheostats placed both in the armature and field-circuits, the speed could be varied and adjusted. One end of the spindle of the commutator C was directly connected to the spindle of this converter, and on the other end was a screw thread which worked a tangent wheel. On this wheel an ebonite stud is fixed which momentarily breaks the circuit of an electric chronograph once every revolution, thus recording the time of every 200 periods.

The magnetizing current, drawn from the slip-rings of the rotary converter, passed through a regulating resistance, a Kelvin balance, the primary of the air-circuit transformer M, and the primary coil on the iron ring B.

One end of the secondary of M is joined to one end of that of B, and the junction connected to one of the fixed brushes of the commutator. The other ends of the secondaries are connected, as shown in fig. 1, to the three-way key K, from the lever of which connexion is made to the other fixed brush. By this means either secondary can be joined to the commutator, and thence from the movable brushes of the latter through a reversing key and a high resistance to the galvanometer.

A Clark's cell with a megohm is arranged so that it may be used at any time for the purpose of determining the reducing factor of the galvanometer. The deflexions of the latter were so nearly proportional to the currents producing them over the part of the scale used that no calibration was necessary.

3. The details of the rings, called ring I. and ring II. respectively, are as follows :—

	Ring I.	Ring II.
No. of laminæ	18	31
Internal diameter (r_1)	7·6 cm.	15·223
External diameter (r_2)	11·58 cm.	16·477
Mean thickness (x)	·0475 cm.	·0443
Section of magnetic circuit (a)	1·701 cm.²	·861
Length of magnetic circuit (l)	30·12 cm.	49·79
Specific resistance of iron at 12° C. (ρ)	12590	13600
No. of primary turns (n_1)	164	406
No. of secondary turns (n_2)	5 or 10	10

The laminæ were well annealed, the oxide removed, and they were insulated from each other by oiled paper. The mean thickness was determined from weight, area, and specific gravity. The values obtained for the specific resistance seemed high, so the determination was checked.

4. The method by which the current and flux waves were determined was practically the same as that described in the paper already quoted, except that readings of the galvanometer which give the ordinates for a definite phase were only taken for every six degrees on the divided circle which carries the movable brushes of the commutator. This gave 15 ordinates per half wave, and from these, without plotting, the first, third, and fifth harmonics were easily obtained by an arithmetical method of analysis when the seventh and higher harmonics could be neglected. Whether or not the latter assumption was legitimate would appear during the analysis, and in all cases in which the amplitude of the seventh harmonic was greater than one per cent. of that of the first, it was determined, and though its values are not given in the tables that are to follow, its effect was allowed for.

5. In order to explain the procedure by which the results arrived at were deduced, the galvanometer readings for a pair of associated waves will be given, and the treatment to which these readings were subjected will be indicated.

The following table gives four times the galvanometer deflexions for a pair of associated current and flux waves, with the corresponding divided-circle readings. The latter, when doubled, give the corresponding phase-angles, as one complete revolution of the divided circle corresponds to two full waves, the rotary converter and the commutator both being four-pole. The readings for the current-wave are represented by γ, and those for the flux-wave by β. The other necessary details of the experiment are also given.

Experiment with Ring I.

Resistance in γ circuit $= r_1 = 5030 \, \omega$.
    ,,   in β circuit $= r_2 = 15030 \, \omega$.
Period by chronograph $= T = \cdot 0307$ sec.
Kelvin-balance reading of current C at start $= \cdot 1873$ amp.
    ,,   ,,   ,,   at finish $= \cdot 1868$ amp.
Reducing factor of galvanometer $= \lambda = 1 \cdot 287 \times 10^{-8}$ amp.
M of air-circuit transformer $= 0 \cdot 00061$ henry.
Secondary turns on ring $= n_2 = 5$.

Divided Circle.	0	6	12	18	24	30	36	42	48	54	60	66	72	78	84
4γ	103	338	583	795	1003	1160	1270	1323	1314	1239	1069	872	638	395	147
4β	−636	−561	−450	−296	−82	183	449	661	774	804	795	785	760	729	690

Each of these numbers is the sum of four separate readings of γ or of β, two of these being at corresponding points on the positive and negative halves of each wave, and the other two being a similar pair got after switching the galvanometer.

Analysing the above, we obtain

$$4 \gamma = 1293 \sin \omega t - 41 \sin 3 \, (\omega t - 7) + 5 \sin 5 \, (\omega t - 23 \cdot 4)$$
$$4 \beta = 868 \sin (\omega t - 51 \cdot 76) + 120 \sin 3 \, (\omega t - 63 \cdot 1) + 23 \sin 5 \, (\omega t - 69 \cdot 2).$$

6. The factors (c and f say) to be applied to the different values of γ and of β in order to obtain the corresponding values of current (C) and of flux (F) are (see Phil. Mag. *loc. cit.*)

$$c = \frac{\lambda r_1 T}{4 M}, \quad f = \frac{\lambda r_2 T}{4 n_2}, \quad \ldots \quad (1)$$

which become

$$c = \cdot 00008145, \quad f = 29 \cdot 7$$

for the experiment being discussed when the figures given in § 5 are substituted.

These factors can also be obtained from the Kelvin-balance reading C of the magnetizing current; for if $\bar{\gamma}$ be the R.M.S. of the γ readings

$$C = c \bar{\gamma}$$

and

$$\bar{\gamma}^2 = \tfrac{1}{2} \{ \gamma_1{}^2 + \gamma_3{}^2 + \gamma_5{}^2 + \ldots \}$$

where γ_1, γ_3, γ_5, &c. are the amplitudes of the first, third, fifth, &c. harmonics of γ, and these have already been obtained by analysis (§ 5).

The factor f can then be deduced from c, as equations (1) above give us the relation

$$f = \frac{r_2}{r_1} \frac{M}{n_2} c.$$

By this method we obtain for the experiment in hand

$$c = \cdot 00008178, \qquad f = 29 \cdot 8,$$

which agree satisfactorily with the values obtained above by the other method.

The latter method of obtaining c and f is the one usually adopted.

The factors (h and b say) to be applied to γ and β in order to obtain H and B, where H is the M.M.F. round the ring divided by its mean circumference l, and B is the total flux F divided by the iron cross-section α, are

$$h = \frac{4\pi n_1 c}{l}, \qquad b = \frac{f}{\alpha}.$$

Substituting the values of n_1, l, and α given in the details of ring I. in § 3, and those of c and f just obtained, we find that

$$h = \cdot 0056, \qquad b = 17 \cdot 52 ;$$

so that

$$H = \cdot 0056\gamma, \quad B = 17 \cdot 52\beta.$$

Hence the final expressions for the pair of associated waves investigated are

$$H = 1 \cdot 81 [\sin \omega t - \cdot 0317 \sin 3(\omega t - 7) + \cdot 004 \sin 5(\omega t - 23 \cdot 4)]$$
$$B = 3802 [\sin (\omega t - 51 \cdot 76) + \cdot 1382 \sin 3(\omega t - 63 \cdot 1) + \cdot 0265 \sin 5(\omega t - 69 \cdot 2)].$$

7. It is easy to show that as the secondary (galvanometer) current is inappreciable, the energy (D say) dissipated per cycle in the iron of the ring is given by

$$D = n_1 \int_0^T C \frac{dF}{dt} dt,$$

which becomes (see § 6)

$$D = n_1 cf \int_0^T \gamma d\beta.$$

Hence when the galvanometer readings for β are plotted as ordinates against the corresponding ones for γ as abscissæ for a complete period a closed curve is obtained, whose area when multiplied by $n_1 cf$ gives D the total energy dissipated per cycle in the iron of the ring.

Dividing D by the volume $l\alpha$ of iron we get for quotient the space average throughout the ring of the iron loss per cubic centimetre per cycle. In the sequel this quantity will be represented by I.

I can also be obtained from the harmonic expressions for H and B in § 6.

For as

$$Hl = 4\pi n_1 C, \qquad B\alpha = F,$$
$$D = \frac{l\alpha}{4\pi} \int_0^T H\,dB \; ;$$

hence

$$I = \frac{1}{4\pi} \int H\,dB,$$

and it is easy to show that if

$$H = H_1[\sin \omega t + h_3 \sin 3(\omega t - \phi_3) + h_5 \sin 5(\omega t - \phi_5) + \&c.]$$
$$B = B_1[\sin (\omega t - \theta_1) + b_3 \sin 3(\omega t - \theta_3) + b_5 \sin 5(\omega t - \theta_5) + \&c.],$$

then

$$\frac{1}{4\pi} \int H\,dB = \frac{H_1 B_1}{4} \left\{ \sin \theta_1 + 3h_3 b_3 \sin 3(\theta_3 - \phi_3) + 5h_5 b_5 \sin 5(\theta_5 - \phi_5) + \&c. \right\}.$$

Applying this formula to the expressions for H and B in § 6 we find, for the experiment being discussed, that

$$I = 1346.$$

8. For the two rings used what has been called the static hysteresis (U, say) was determined for various induction densities by Ewing and Klaasen's method. It was found that the Steinmetz coefficients σ, where

$$U = \sigma B_{\text{Max.}}^{1\cdot 6},$$

for them were not constant but varied with the induction, and that the variation was quite as great for the narrow ring as for the broad one.

In fig. 2 the Steinmetz coefficients σ for these two rings are plotted against maximum induction (B_0 say). It will be seen that σ tends to be very small in weak fields, possibly vanishing with B_0, that it increases rapidly to a maximum, then diminishes to a minimum, and then steadily increases. It is possible that the maximum σ point, which is very marked, may give the induction at which some definite physical change due to magnetization begins to take place in the iron, perhaps that at which magnetization begins to produce lengthening.

In the case of magnetization by means of alternating currents, it is fair to assume that the static hysteresis will depend on the maximum value of B, (B_0 say), during the

Fig. 2.
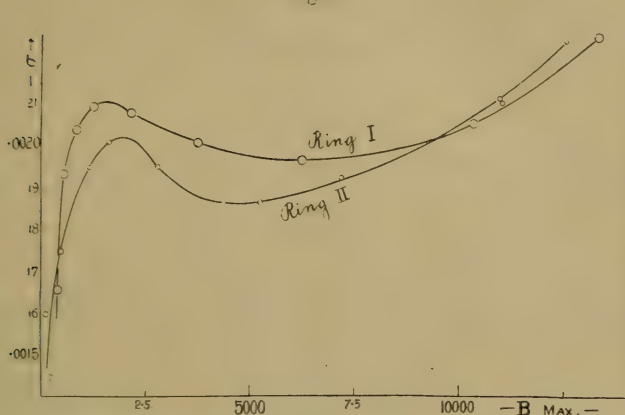
Steinmetz Coefficient σ v. Max. Induction ($U = \sigma B^{1 \cdot 6}$).

alternation. B_0 is obtained from the maximum β reading by multiplying the latter by b (see § 6), and

$$U = \sigma B_0^{1 \cdot 6},$$

in which for σ we take from the curve in fig. 2 for the ring used the value corresponding to B_0.

For the experiment being discussed,

$$\text{Max. } \beta = \beta_0 = 201 \text{ (see § 5)}$$
$$B_0 = 17 \cdot 52 \beta_0 = 3522 \text{ (see § 6)},$$

for which induction for ring I.

$$\sigma = \cdot 002012,$$

hence

$$U = 952.$$

9. No formula is available by means of which the eddy-current loss in an annular lamina can be exactly determined, but a fair approximation to it can be obtained in the following way.

Searle and Bedford * have shown that if X be the space average of the heat dissipated in a thin strip by eddy-currents,

$$\frac{dX}{dt} = \frac{x^2}{\rho} \left(\frac{db}{dt} \right)^2 \left\{ \frac{1}{12} - \cdot 0525 \frac{x}{y} \right\},$$

* Phil. Trans. vol. cxcviii.

where x is the thickness, y the width of the strip, ρ the specific resistance, and b the induction, provided that db/dt has the same value at all points of a cross section.

In ring laminæ when magnetized in the usual way neither b nor db/dt is constant across their section, but an upper limit will be given to the rate of dissipation per cm.³ at any point where the induction is b by the equation

$$\frac{d\mathrm{X}}{dt} = \frac{x^2}{12\rho}\left(\frac{db}{dt}\right)^2,$$

and the rate of dissipation of energy by eddy-currents in the whole ring will be

$$= \frac{x^2}{12\rho} \times \text{vol. of ring} \times \text{space average of } \left(\frac{db}{dt}\right)^2 \text{ throughout the ring.}$$

Now it can be shown that this average will not differ much from $(d\mathrm{B}/dt)^2$ where B has the meaning already assigned to it, namely, the average induction across any section. The latter statement is roughly indicated by the fact that though the amplitude of b and hence of db/bt may vary considerably from the inner to the outer radius of the ring, being greatest at the inner radius as the amplitude of the magnetizing force (which varies inversely as the distance from the centre of the ring) is greatest there, still, since the inner circumference is less than the outer one, there will, in making up the space average of $(db/dt)^2$, be a smaller relative volume of the iron at the high induction than at the low induction. Hence, finally, if E be the average eddy-current loss throughout the ring per cm.³ per cycle, we have, approximately, that

$$\mathrm{E} = \frac{x^2}{12\rho}\int_0^\mathrm{T}\left(\frac{d\mathrm{B}}{dt}\right)^2 dt.$$

If

$$\mathrm{B} = \mathrm{B}_1[\sin \omega t + b_3 \sin 3(\omega t - \theta_3) + b_5 \sin 5(\omega t - \theta_5) + \&c.]$$

then

$$\int_0^\mathrm{T}\left(\frac{d\mathrm{B}}{dt}\right)^2 dt = \frac{2\pi^2}{\mathrm{T}}\mathfrak{B}^2,$$

where

$$\mathfrak{B}^2 = \mathrm{B}_1^2\{1 + 9b_3^2 + 25b_5^2 + \&c.\}$$

and

$$\mathrm{E} = \frac{\pi^2 x^2}{6\rho\mathrm{T}}\mathfrak{B}^2.$$

It will be seen that $\mathfrak{B}$ is a quantity of considerable importance in this theory, and it will be called the *effective induction.*

It is the amplitude of the sinusoidal induction-wave that

would by its variation generate in any circuit looped on it the same virtual E.M.F. as would be produced in the same circuit by the actual induction B ; or otherwise stated,

$$\mathfrak{B} = \sqrt{2} \ \text{R.M.S.} \left(\frac{dB}{dt} \right).$$

When we write the equation for E in the form

$$E = e \frac{\mathfrak{B}^2}{T},$$

e will be a constant for a particular ring and we find, using the values for x and ρ given in § 3, that,

$$\text{for Ring I.} \ e = 2 \cdot 95 \ \times 10^{-7},$$
$$\text{and for Ring II.} \ e = 2 \cdot 373 \times 10^{-7}.$$

In the experiment with Ring I. that is being discussed,

$$\mathfrak{B}^2 = 17 \cdot 2 \times 10^6$$
$$T = \cdot 0307$$
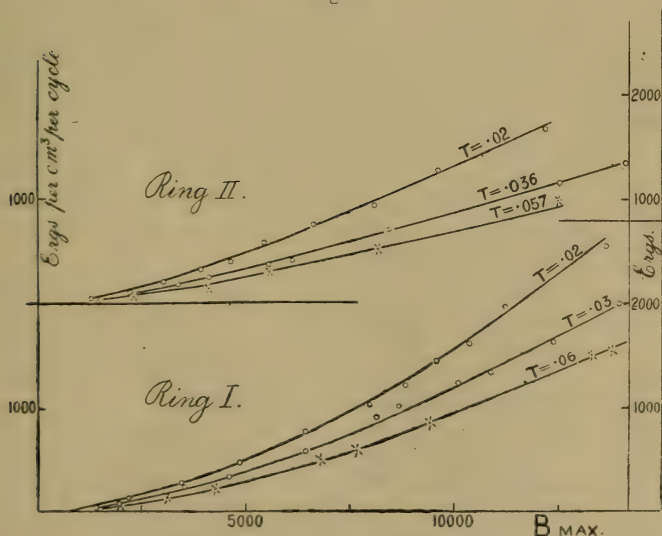
hence
$$E = 165.$$

10. In every case with either ring it was found that the

Fig. 3.

Kinetic Hysteresis v. Max. Induction.

sum of the statical hysteresis (U) and the theoretical eddy-current loss (E) was less than the total iron loss (I) as

determined by the wave-tracer. The difference $I - (U + E)$, which Fleming has called the kinetic hysteresis, was determined for each experiment, and is given in the tables that are to follow; and its values for different induction densities and frequencies for both rings are plotted in fig. 3 (p. 111) against B_0 the maximum induction in each case. The results expressed by these curves seem to thoroughly verify the observations of Steinmetz and of Siemens both as to the existence of such a quantity as kinetic hysteresis and as to the general character of its variation with induction density and period. As this question has been much disputed the verification, by a new method, of the results obtained by previous investigators is not without significance.

11. In Table I. (p. 113) are given the analytical results deduced as indicated in the preceding paragraphs from a series of experiments with Ring I., in which the period was approximately ·019 sec. The wave-forms of the magnetizing currents in experiments 1 to 13 were approximately sinusoidal, and the E.M.F. impressed on the circuit was practically of the sine form in all experiments with the exception of No. 15, Table I.

In fig. 4 (p. 114) the more important characteristics of the induction-waves given in Table I. Nos. 1 to 13, *i. e.* produced by q.p. sine currents, are plotted against the amplitudes B_1 of the first harmonic of these waves. These curves are typical of any series of induction-waves of constant period produced by currents of similar wave-forms. No such regularity, however, would be obtained if the wave-forms of the magnetizing currents were allowed to vary, as will be seen by marking on fig. 4 the points for the characteristics of the induction-wave of Experiment 15, in which the H wave was greatly distorted (made saddle-shaped) by artificially distorting the applied E.M.F. wave.

The characteristics μ_0 and θ, which are the connecting links between the H wave and the B wave it produces, both fall to small values for small values of B_1, θ probably vanishing with B_1. Fig. 4 shows clearly how both rise to maxima and then diminish as B_1 increases.

The curve for b_3 the ratio of the third (B_3) to the first harmonic B_1 of B is striking, apparently issuing from the origin (see series 2, Table II. p. 115, & fig. 5, p. 118), it rises quickly between $B_1 = 0$ and $B_1 = 1000$ (q.p.), from which it continues for larger values of B_1 as a straight line. Hence for all values of B_1 greater than 1000 the amplitude B_3 of the third harmonic of B is of the form

$$\alpha B_1 + \beta B_1^2.$$

Table I.—Ring I.

$T = 2\pi/\omega = \cdot019 \ (\mu.\rho.)$.
$\mu_0 = B_1/H_1$.
$\mathfrak{B} = \sqrt{2} \ . \ \text{R.M.S.} \ \left(\dfrac{dB}{dt}\right)$.

Type forms $\begin{cases} H = \dfrac{\text{M.M.F.}}{\text{length}} = H_1\{\sin \omega t + h_3 \sin 3(\omega t - \phi_3) + h_5 \sin 5(\omega t - \phi_5) + \ldots\}. \\ B = \dfrac{\text{Flux}}{\text{area}} = B_1\{\sin(\omega t - \theta) + b_3 \sin 3(\omega t - \theta - \psi_3) + b_5 \sin 5(\omega t - \theta - \psi_5) + \ldots\}. \end{cases}$

$1 = \dfrac{\text{Iron loss per cycle in ring}}{\text{vol. of iron in ring}}$.

$\iota = \dfrac{\text{Statical hysteresis per cycle in ring}}{\text{vol. of iron in ring}}$.

$E = \begin{Bmatrix} \text{Theoretical eddy-current} \\ \text{loss per cm.}^3 \text{ per cycle.} \end{Bmatrix}$

No.	T.	H_1	$-h_3$	ϕ_3	h_5	ϕ_5	B_1	θ	b_3	ψ_3	b_5	ψ_5	μ_0	Max. B.	$\mathfrak{B}$	1	U.	E.	1−U−E.
1	·01877	0·773	·0463	−0·12			448	25·73	·085	26·76	·021	35·25	580	450	465	·372	31	3·4	28
2	·01887	1·156	·042	1·9			1147	40·65	·099	21·89	·020	30·14	992	1124	1204	·216	158	22·6	354
3	·01849	1·261	·033	4·17			1407	44·55	·096	19·84	·016	26·32	1116	1370	1468	·311	219	34·4	576
4	·01894	1·35	·037	2·67			1689	45·85	·1025	17·32	·017	25·15	1251	1643	1772	·409	293	49	67
5	·01860	1·536	·034	1·14			2291	50·34	·108	16·05	·016	24	1492	2199	2414	·680	463	92·5	1245
6	·0190	1·93	·025	0·0			3708	53·93	·120	12·73	·017	18·85	1921	3482	3956	1·452	936	243	273
7	·01875	2·377	·012	17·4			5360	53·54	·128	8·10	·015	13	2255	4884	5753	2·573	1588	521	464
8	·01892	2·892	0				7100	53·66	·140	7·60	·017	12	2476	6466	7800	4·170	2457	948	765
9	·01878	3·497	·020	57	·005	21·0	9040	49·61	·148	4·82	·021	6·76	2585	8032	9934	6·027	3443	1550	1034
10	·01905	3·897	·014	41·4	·007	14·5	10650	48·18	·157	4·56	·025	6·5	2580	8855	11170	7·260	4115	1935	1210
11	·01868	4·347	·017	50·36	·008	16·0	11010	46·05	·162	3·84	·030	4·92	2534	9604	11670	8·614	4762	2408	1444
12	·01835	4·943	·022	42·85	·008	8·8	12070	45·10	·170	3·28	·033	4·90	2442	10440	13610	10170	5540	3015	1615
13	·01840	5·67	·040	36·53	·014	7·4	13020	40·35	·175	2·65	·034	3·30	2296	11250	14500	11820	6380	3560	1880

End of approximately sinusoidal magnetizing waves.

No.	T.	H_1	$-h_3$	ϕ_3	h_5	ϕ_5	B_1	θ	b_3	ψ_3	b_5	ψ_5	μ_0	Max. B.	$\mathfrak{B}$	1	U.	E.	1−U−E.
14	·01869	9·42	·1635	8·72	−·056	29·78	13800	34·94	·186	4·77	·043	9·18	1677	13670	18542	17500	9530	5427	2543
15	·01844	2·886	−·218	−·45	·042	26·09	6488	40·84	·134	−2·9	·022	−1·5	2248	5780	7037	3451	2059	792	600

Within the limits of experimental error b_5 may be taken as a straight line, so that B_5 is also of the form

$$\gamma B_1 + \delta B_1^2.$$

The curves for ψ_3 and ψ_5 show the gradual change of position as regards phase of the third and fifth harmonics with respect to B_1 the first harmonic of B as B_1 increases.

Fig. 4.
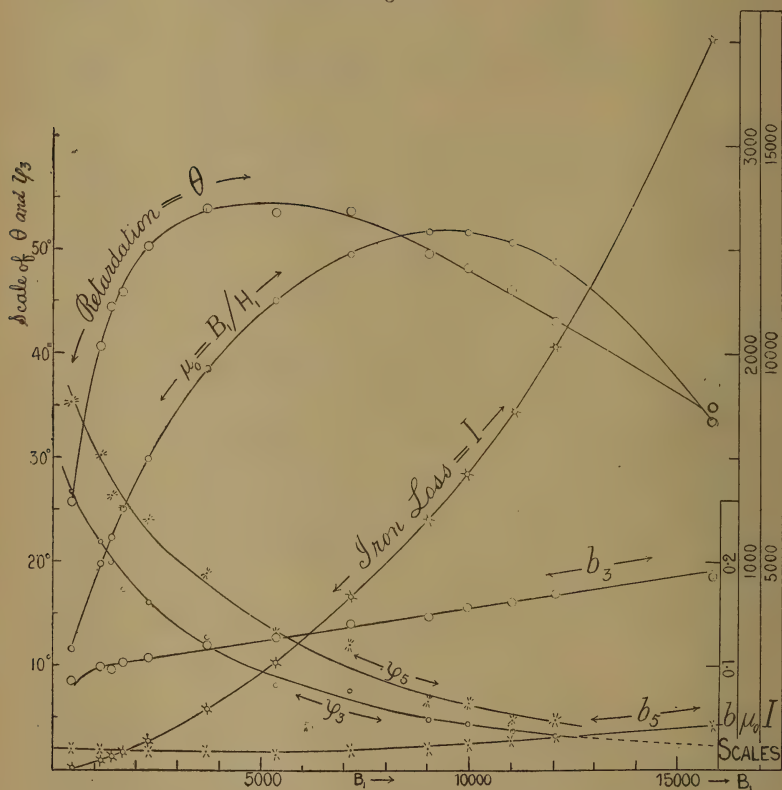
Both ψ_3 and ψ_5 seem to attain small limiting values for very high values of B_1, which means that all the harmonics of B pass through their zero values in the same direction at nearly the same instant when B_1 is very large.

In Tables II. and III. are given the analytical results for two other series of experiments with Ring I. for periods ·03 sec. and ·058 sec. approximately, and in Tables IV., V., and VI. are given the results of three series of experiments with Ring II.

Table II.—Ring I.

No.	T.	H.	$-h_3$.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	h_3.	ψ_3.	b_5.	ψ_5.	μ_0.	Max. B.	𝔅.	L.	U.	E.	I-U-E.
1...	·0302	·4	·054	51	...	...	148	13·9	·053	37·8	·0153	45·3	370	145	150	3·2			
2...	·0299	·577	·048	2	...	...	258	19·4	·070	32·0	·014	40·1	447	259	264	12·2			
3...	·03	·783	·047	4·2	...	...	483	25·11	·081	26·8	·019	38	617	479	500	39·5	353	25	1·7
4...	·0303	·955	·043	2·12	...	...	776	32·41	·103	23·1	·024	32·7	813	768	818	984	832	65	8·7
5...	·0305	1·202	·038	2·46	...	...	1432	41·18	·113	18·5	·024	27·5	1192	1388	1520	282	224	22·3	36
6...	·0303	1·366	·043	4	·005	12·25	2025	47	·1253	16·4	·0284	24·4	1481	1944	2182	505	381	46	78
7...	·0307	1·807	·031	7	·004	23·5	3802	51·76	·1382	11·4	·0265	17·4	2104	3522	4147	1344	952	165	227
8...	·0297	2·114	·021	13·9	·008	25·3	5050	51·7	·147	9·43	·028	14·9	2389	4613	5568	2078	1445	308	325
9...	·0295	2·708	·014	25	·008	11·7	7284	49·13	·155	6·65	·030	10·2	2690	6462	8105	3692	2455	657	580
10...	·0296	3·366	·018	35·6	·009	11·8	9318	46·07	·167	4·77	·035	6·9	2769	8142	10550	5590	3580	1110	900
11...	·0295	4·53	·044	29·1	·012	7·4	11720	39·89	·181	2·34	·042	3·9	2587	10120	13580	8310	5234	1844	1232
							End of approximately sinusoidal magnetizing waves.												
12...	·0295	4·737	·115	13·9	·036	3·0	12400	43·88	·163	6·17	·034	12·0	2618	10900	13964	9308	6026	1950	1332
13...	·0295	6·243	·148	11·6	-·046	34·3	14210	39·28	·178	5·46	·038	10·7	2276	12420	16340	12115	7817	2065	1633
14...	·0301	9·88	·1895	6·5	-·058	26·5	16310	31·1	·196	4·4	·052	8·2	1660	13970	19440	15630	9930	3704	2000

I 2

Table III.—Ring I.

No.	T.	H_1	$-h_3$	ϕ_3	h_5	ϕ_5	B_1	θ	b_3	ψ_3	l_5	ψ_5	μ_0	Max. B.	$\mathfrak{B}$	I.	U.	E.	I-U-E.
1	·0574	·774	·038	1·6	...	...	483	23·62	·089	24·08	·021	34·9	624	477	500	36·8	...	...	6·7
2	·057	·971	·04	0·6	...	...	847	31·34	·123	19·2	·026	28·7	873	825	911	105	·94	4·3	28
3	·0572	1·211	·032	7·4	...	...	1625	39·8	·131	16·5	·031	22·6	1342	1550	1763	311	267	16	57
4	·0576	1·348	·045	7·5	·016	8	2102	42·5	·143	14·2	·038	27·7	1560	1965	2322	470	386	27	133
5	·0578	1·653	·026	9·9	·008	16·3	3398	46·0	·155	11·2	·042	15·5	2055	3118	3814	997	790	74	213
6	·0582	1·94	·024	15·1	·009	15·7	4765	46·35	·165	8·7	·043	12·4	2456	4275	5413	1645	1284	148	213

End of approximately sinusoidal magnetizing waves.

No.	T.	H_1	$-h_3$	ϕ_3	h_5	ϕ_5	B_1	θ	b_3	ψ_3	l_5	ψ_5	μ_0	Max. B.	$\mathfrak{B}$	I.	U.	E.	I-U-E.
7	·0593	2·583	·074	14·6	·027	17·9	7685	48·46	·170	7·6	·043	12·0	2975	6800	8785	3545	2668	384	493
8	·0585	2·91	·082	14·5	·026	2·8	8690	46·84	·174	7·1	·042	11·4	2987	7686	9975	4337	3255	501	581
9	·0602	3·737	·108	13·9	·036	0·5	10780	43·17	·179	5·7	·038	10·7	2885	9440	12450	6255	4620	760	875
10	·0590	6·41	·293	15·6	·061	10·7	14710	37·71	·123	8·6	·018	19·9	2295	13330	15750	11730	8982	1241	1507
11	·0643	9·328	·189	3·45	·074	22·6	16165	28·93	·203	4·0	·053	7·0	1733	13800	19500	13090	9800	1748	1542

Table IV.—Ring II.

No.	T.	H_1	$-h_3$	ϕ_3	h_5	ϕ_5	B_1	θ	b_3	ψ_3	l_5	ψ_5	μ_0	Max. B.	$\mathfrak{B}$	I.	U.	E.	I-U-E.
1	·0210	·582	·034	-1·4	...	...	291	21·2	·073	25·9	·017	35·9	500	291	300	15	69	6	10
2	·0216	·894	·036	1·8	...	...	712	32·4	·082	22·6	·015	34·4	796	703	736	85	193	23	44
3	·0204	1·161	·029	6·2	...	...	1357	41·45	·089	19·2	·016	26	1169	1316	1410	260	453	68	100
4	·0205	1·46	·0306	7·2	...	...	2325	47·2	·101	16	·016	24·5	1592	2230	2430	621	724	132	199
5	·0212	1·737	·03	6·8	...	...	3225	48·97	·116	15·1	·022	22·5	1857	3047	3435	1055	1059	225	320
6	·0209	2·046	·032	17·5	...	...	4173	49·45	·118	12·5	·023	19·3	2040	3916	4450	1604	1383	333	394
7	·0204	2·276	·036	21·4	·008	15	5011	48·75	·1195	11·5	·020	19·25	2202	4657	5350	2110	1775	470	579
8	·0204	2·647	·042	18·6	·008	18·7	5910	47·45	·127	10·6	·022	17·2	2233	5443	6356	2824	2477	722	749
9	·0208	3·18	·052	19·8	·013	13·8	7300	44·6	·136	8·8	·029	15	2295	6638	7956	3948	3575	1163	969
10	·0206	4·234	·068	19·1	·015	5·3	9074	38·78	·151	7·32	·031	12·7	2143	8094	10030	5707	4763	1730	1267
11	·0203	5·65	·076	16·6	·016	0	10900	32·51	·155	6·09	·035	9·19	1930	9612	12165	7760	7716	3318	1676
12	·01994	10·57	·076	11·5	-·0256	25·8	14250	21·4	·184	2·8	·052	4·4	1350	12250	16700	12710			

Table V.— Ring II.

No.	T.	H_1.	$-h_3$.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	h_3.	ψ_3.	h_5.	ψ_5.	μ_0.	Max. B.	$\mathfrak{B}$.	L.	U.	E.	I–U–E.
1	·0352	0·881	·051	4·6	...	...	802	34·17	·10	22·6	·025	30·8	938	788	836	97	78	47	14·3
2	·0358	1·445	·047	7	·005	5·0	2610	45·82	·131	14·7	·038	19·8	1861	2448	2850	670	521	54	95
3	·0360	1·734	·048	11·6	·005	12·5	3636	46·36	·140	13·0	·032	18·8	2097	3372	3990	1120	839	105	176
4	·0359	1·997	·050	16·2	·011	13	4503	45·32	·146	11·1	·032	17·0	2255	4128	4970	1555	1147	163	245
5	·0360	2·573	·065	17·6	·017	10·2	6110	42·07	·149	9·2	·032	14·6	2375	5560	6770	2513	1841	302	370
6	·0358	2·843	·073	19·4	·019	6·8	6770	40·43	·15	8·7	·032	13·3	2381	6133	7500	2948	2172	373	403
7	·0359	4·425	·088	16·7	·023	2·1	9570	32·22	·158	5·6	·037	8·5	2163	8420	10740	5247	3780	763	704
8	·0350	9·97	·210	3·23	–·07	25	14030	24·65	·148	8·5	·034	14·6	1407	12540	15550	10900	8100	1641	1160
9	·0359	14·1	·171	2·57	–·068	17·1	16180	20·96	·176	6·33	·053	11·5	1147	14120	18860	14140	10470	2354	1316

Table VI.—Ring II.

No.	T.	H_1.	$-h_3$.	ϕ_3.	h_5.	ϕ_5.	B_1.	θ.	h_3.	ψ_3.	h_5.	ψ_5.	μ_0.	Max. B.	$\mathfrak{B}$.	L.	U.	E.	I–U–E.
1	·0572	1·13	·0363	0·9	...	...	1525	38·7	·133	15·4	·0355	22·1	1350	1437	1670	265	224	10·5	30·5
2	·0567	1·392	·0268	11·4	·01	1·9	2525	41·54	·143	12·7	·0374	17·7	1814	2330	2795	573	486	32·5	54·5
3	·0573	1·959	·0581	15·3	·02	3·3	4557	41·2	·163	9·6	·0437	14·6	2326	4138	5170	1400	1151	111	138
4	·0572	2·56	·065	13·65	·015	1·87	6206	38·88	·171	8·1	·0465	11·6	2425	5544	7120	2350	1832	210	308
5	·0568	4·185	·090	13·0	–·023	29·1	9340	30·93	·173	5·4	·049	7·7	2232	8168	10770	4550	3543	484	523
6	·0565	10·38	·1583	1·38	–·062	11·8	14370	22	·176	6·1	·05	10·65	1384	12490	16710	10230	8067	1174	990

In fig. 5 some of the characteristics of the induction-waves produced in Ring I. for two different speeds are plotted against B_1 for the purpose of showing the general effect of change of frequency on these characteristics.

Fig. 5.
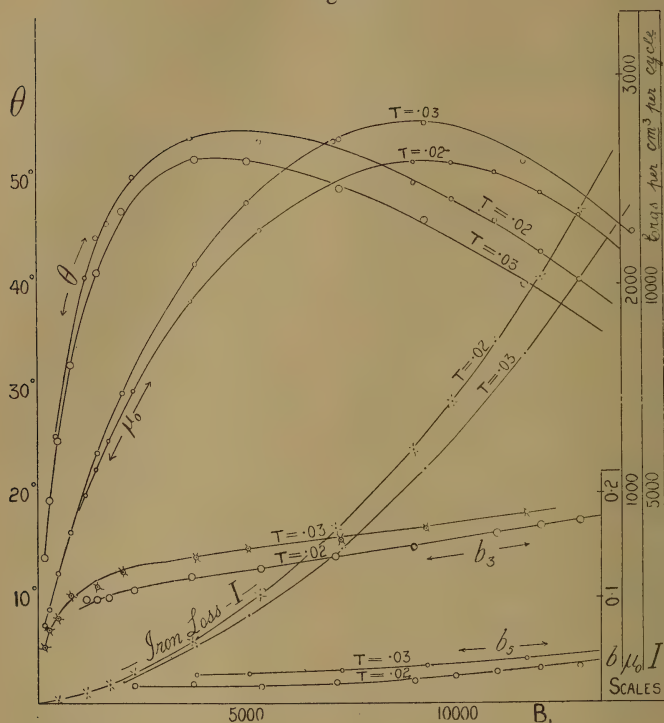
12. For the two rings experimented with I find that I, the total iron loss per cm.^{3} per cycle, is given very approximately in terms of $\mathfrak{B}$, the effective induction, and n the number of periods per second, when $\mathfrak{B}$ lies between 1000 and 12,000, by a simple formula analogous to that of Steinmetz for statical hysteresis.

Thus for Ring I.,

$$I = (\cdot 00186 + \cdot 000026 \; n) \; \mathfrak{B}^{1 \cdot 57},$$

and for Ring II.,

$$I = (\cdot 001684 + \cdot 0000272 \; n) \; \mathfrak{B}^{1 \cdot 57}.$$

That the same power of the effective induction should appear in the formula for each ring is striking, and suggests

the probability that for any given frequency when the wave-form of H is approximately sinusoidal the total iron loss per cm.^{3} per cycle is proportional to

$$\mathfrak{B}^{1\cdot57},$$

or to some power of $\mathfrak{B}$ differing little from $1\cdot57$ independently of the kind of iron or the thickness of the laminæ.

In order to show the degree of accuracy with which the above formulæ give the iron losses, Table VII. has been compiled. In it are given in parallel columns the values of I obtained experimentally by means of the wave-tracer and those determined from the above formula for Ring II. for different values of n and $\mathfrak{B}$.

TABLE VII.

Comparing observed values of I for Ring II. with those given by formula $I = (\cdot001684 + \cdot00002717\ n)\ \mathfrak{B}^{1\cdot57}$.

n.	$\mathfrak{B}$.	I. Obs.	I. Calc.	n.	$\mathfrak{B}$.	I. Obs.	I. Calc.
49·07	1410	260	265	27·77	6770	2513	2517
48·71	2430	621	621	27·90	7500	2948	2962
47·24	3435	1055	1056	27·86	10740	5247	5200
47·91	4450	1604	1596	34·03	3061	793	775
49·02	5350	2110	2153	34·22	3780	1080	1083
49·02	6356	2824	2819	34·66	4770	1569	1568
48·03	7956	3948	3975	34·91	5818	2139	2145
48·54	10050	5707	5760	33·49	7536	3165	3170
49·21	12165	7760	7710	17·62	2795	573	557
27·90	2854	670	649	17·45	5170	1400	1459
27·76	3990	1120	1098	17·48	7120	2350	2409
27·81	4970	1550	1551	17·60	10770	4550	4625

13. In the tables giving the analytical results of the different experiments, it will be seen that as H_1 gets large the wave-form of H differs more and more from the sine form, and that in the slower speeds the range over which it was possible to obtain magnetizing currents of approximately sine wave-form was less. This was due to the reaction of the iron in the ring and may be explained as follows:—

If we assume that the applied E.M.F. was sinusoidal [it was nearly so], a current would flow which would produce a flux wave that contained 1st, 3rd, 5th, &c. harmonics. The 3rd, 5th, &c. harmonics produced would induce in the magnetizing circuit currents of periods $3n$, $5n$, &c., which would

appear as 3rd, 5th, &c., harmonics in the magnetizing current, and would be greater as the impedances of the primary circuit for currents of these different periods were less, if we assume that the amplitudes of the upper harmonics of the flux are fixed.

Now in order to obtain the larger magnetizing currents it was necessary to reduce the impedance, and hence in these cases the higher harmonics produced as above by the reaction of the ring became greater and the magnetizing current more distorted. Also in order to get the slower speeds fewer cells were used to supply current to the rotary converter, thus reducing the applied E.M.F., and in order to get the same currents as before the impedances used had to be proportionally reduced. Hence, in the slower speeds the upper harmonics of the current due to the reaction of the ring would be relatively greater and the distortion of its wave-form more marked.

This phenomenon is an interesting case of transformation and what may be called reflexion of energy. Thus the iron receives energy from the first harmonic (frequency n) of the exciting current, some of which it transforms to vibratory energy of frequencies $3n$, $5n$, &c., and sends back energy to the primary circuit by means of currents of these higher frequencies, where in general this reflected energy will be dissipated as heat.

This reflexion of energy occurred in nearly all the experiments recorded in this paper, and the amount in each case for each harmonic is obtained when calculating the iron loss by the analytical method given in § 7.

Thus the energy D received per cycle by the iron and dissipated as heat in it is given by

$$D = n_1 \int_0^T C \frac{dF}{dt} dt$$

$$= n_1 \int_0^T \left\{ C_1 \frac{dF_1}{dt} + C_3 \frac{dF_3}{dt} + C_5 \frac{dF_5}{dt} + \&c. \right\} dt$$

$$= D_1 + D_3 + D_5 + \&c. \text{ (say)},$$

where C_1, C_3, C_5, F_1, F_3, F_5 are the 1st, 3rd, and 5th harmonics of current and flux respectively.

Now if, for instance,

$$D_3 \text{ or } n_1 \int_0^T C_3 \frac{dF_3}{dt} dt$$

is negative, the iron by means of the third harmonics is

sending back per cycle to the magnetizing circuit energy to the amount $-D_3$, and similarly if D_5 &c. are negative.

We have (see § 7)

$$\frac{D}{v} = I = \frac{H_1 B_1}{4} \{ \sin \theta_1 + 3h_3 b_3 \sin 3(\theta_3 - \phi_3) + 5h_5 b_5 \sin 5(\theta_5 - \phi_5) + \&c. \}$$

$$= I_1 + I_3 + I_5 + \&c. \text{ (say)},$$

where v is the volume of iron in the ring:

hence
$$D_1 = I_1 v = v \frac{H_1 B_1}{4} \sin \theta_1$$

$$D_3 = I_3 v = v \frac{H_1 B_1}{4} \times 3h_3 b_3 \sin 3(\theta_3 - \phi_3)$$

$$D_5 = I_5 v = v \frac{H_1 B_1}{4} \times 5h_5 b_5 \sin 5(\theta_5 - \phi_5).$$

For example, in calculating I by means of the above formula for experiment 14 (Table I.) in which

$$H = \quad 9{\cdot}42 \{ \sin \omega t - {\cdot}1635 \sin 3(\omega t - 8{\cdot}72)$$
$$- {\cdot}0565 \sin 5(\omega t - 29{\cdot}78) + {\cdot}0159 \sin 7(\omega t - 15{\cdot}86) \}$$

$$B = 15860 \{ \sin (\omega t - 34{\cdot}94) + {\cdot}1865 \sin 3(\omega t - 39{\cdot}71)$$
$$+ {\cdot}0434 \sin 5(\omega t - 44{\cdot}12) + {\cdot}0117 \sin 7(\omega t - 47{\cdot}9) \}$$

we find that
$$I = 37350 \{ {\cdot}5727 - {\cdot}0914 - {\cdot}0116 - {\cdot}0009 \}$$
$$= 21390 - 3414 - 434 - 34$$
$$= 17508,$$

or
$$I_1 = 21390, \quad I_3 = -3414,$$
$$I_5 = -434, \quad I_7 = -34,$$

which shows that per cm.^{3} of iron per cycle 21390 ergs entered the ring by means of the first harmonics, and of this there was sent back or reflected to the primary circuit 3414 ergs by means of the third, 434 ergs by means of the fifth, and 34 ergs by means of the seventh harmonic, while the remainder, $= I = 17508$ ergs, was dissipated as heat in the iron of the ring.

These reflected current harmonics will obviously, if of sufficient magnitude, greatly modify the wave-form of C, and they will also by their reaction reduce the amplitudes and change the phases of the corresponding harmonics of F

from which they arise, thus modifying the wave-form of F (the flux).

Now I find that when the upper harmonics of the flux are damped as above, μ_0 or B_1/H_1 becomes greater, and the iron loss I for the same value of $\mathfrak{B}$, or even of B_1, becomes greater. To make up for this greater iron loss as well as to supply the energy that is reflected, more energy must enter the iron by means of the first harmonic of C, and this is effected by a considerable increase in θ_1, the angle of lag of F_1 behind C_1.

If the applied E.M.F. (supposed sinusoidal) and the circuit be so controlled that while F_1 or αB_1 is kept constant the impedance is regularly diminished, the reflected energy due to any of the upper harmonics (C_3, F_3, say) will not go on increasing as the impedance diminishes, but will at first increase to a maximum and then diminish, in the limit vanishing when the impedance becomes zero. This would follow if we assumed that on account of some property of the iron there is always associated with a given primary sinusoidal flux oscillation F_1 or αB_1 of frequency n a series of magneto-motive forces M_3, M_5, &c., of frequencies $3n$, $5n$, &c. given in amplitude and fixed in phase relative to F_1.

F_3 would then, for a given impedance, be the flux produced by the vector resultant of M_3 and $4\pi n_1 C_3$, where C_3 is the reflected current produced in the circuit by variation of F_3.

In the simple case in which the circuit outside the ring is non-inductive, and when magnetic lag is neglected, the relations between M_3, C_3, F_3, and the reflected energy can be shown by means of a vector right-angled triangle of which the hypotenuse is proportional to M_3 and is fixed. One of its sides is proportional to C_3 and the other to F_3, while the area of the triangle is proportional to the reflected energy. The locus of the right-angled vertex is thus a semicircle, and as C_3 increases from zero until the triangle becomes isosceles, the reflected energy increases from zero to a maximum, after which any further increase of C_3 causes a diminution in the reflected energy, which finally vanishes with F_3.

If we took account of magnetic lag and of the inductance of the circuit the vector diagram would be more complicated, but the general result would be similar.

The following results of three experiments with Ring I. will illustrate the preceding.

The ring was rewound with two layers of 207 turns each, one or both of which could be used, thus enabling me to

obtain for the same values of F_1 two series of values, one double the other, for the reaction E.M.F.'s due to variation of the upper harmonics of F.

Experiment 1.

T $=\cdot0295$ sec.   Magnetizing current sent through 207 turns.   Full pressure of secondary battery on to rotary converter, and current in magnetizing circuit cut down by iron-less inductances to keep the reflected currents due to higher harmonics of F as small as possible.

$$H = 4\cdot53\{\sin \omega t - \cdot0443 \sin 3(\omega t - 29\cdot11)$$
$$+ \cdot0124 \sin 5(\omega t - 7\cdot41)\}$$

$$B = 11720\{\sin (\omega t - 39\cdot89) + \cdot1814 \sin 3(\omega t - 42\cdot23)$$
$$+ \cdot042 \sin 5(\omega t - 43\cdot8)\},$$

$\mu_0 = B_1/H_1 = 2587.$ $\mathfrak{B} = 13580.$ Max. B $= 10120.$
$I_1 = 8541,$ $I_3 = -203,$ $I_5 = -1,$
$I = I_1 + I_2 + I_3 = 8310,$
$I/\mathfrak{B}^{1\cdot57} = \cdot0027.$

Experiment 2.

T $= \cdot02957$ sec. $n_1 = 414$ turns.

E.M.F. the same as in Exp. 1, but current cut down by non-inductive resistances.

$$H = 4\cdot4\{\sin \omega t - \cdot1758 \sin 3(\omega t - 22\cdot56)$$
$$+ \cdot039 \sin 5(\omega t - 1488)\}$$

$$B = 11850\{\sin (\omega t - 48\cdot04) + \cdot1246 \sin 3(\omega t - 56\cdot82)$$
$$+ \cdot0164 \sin 5(\omega t - 66\cdot73)\}.$$

$\mu_0 = 2693.$ $\mathfrak{B} = 12700.$ Max. B $= 10840.$
$I_1 = 9692.$ $I_3 = -830.$ $I_5 = -42.$
$I = 8820.$ $I/\mathfrak{B}^{1\cdot57} = \cdot00318.$

Experiment 3.

T $= \cdot0294$ sec. $n_1 = 414$ turns.

Impressed E.M.F. reduced and non-inductive resistances proportionally reduced.

$$H = 4\cdot23\{\sin \omega t - \cdot25 \sin 3(\omega t - 30\cdot25)$$
$$+ \cdot0572 \sin 5(\omega t - 32\cdot24)\}$$

$$B = 11560\{\sin (\omega t - 50\cdot77) + \cdot0718 \sin 3(\omega t - 62\cdot19)$$
$$+ \cdot0096 \sin 5(\omega t - 82\cdot7)\}.$$

$\mu_0 = 2734.$ $\mathfrak{B} = 11840.$ Max. $B = 10900.$

$I_1 = 9465.$ $I_3 = -654.$ $I_5 = -32.$

$I = 8780.$ $I/\mathfrak{B}^{1.57} = .00353.$

Though the applied E.M.F.'s in these experiments were only approximately sinusoidal, they illustrate the main features of the phenomenon being discussed. Thus we have with diminution of the impedance of the circuit while B_1 is kept approximately constant, the increase in the reflected upper harmonics of C and the diminution of the corresponding harmonics of F, the increase of μ_0 and of I and the increase and subsequent decrease of the reflected energies I_3 and I_5.

Attention is drawn to the great variation in the ratio of I to $\mathfrak{B}^{1.57}$ with change of wave-form of H, and therefore of B, thus showing that the formula given in § 12 for I can only apply to a series in which the wave-form of H is nearly sinusoidal.

In order to fully investigate the phenomenon drawn attention to in this paragraph, it would be necessary to determine the impressed E.M.F. wave as well as that of H and of B. This is being done, and will form the subject of a future communication.

Finally, the matter dealt with in this section seems to me to have an important practical significance in the case of transformers. In calculating their losses all that has hitherto been counted as iron loss is the energy that is dissipated as heat in the iron. But now we see that it is possible for the iron to send back a considerable amount of energy to the primary circuit, where it also is dissipated as heat. This reflected energy should be counted against the iron.

Thus Experiment 2 above may, for the sake of illustration, be supposed to refer to a transformer on open secondary. The true dissipation of energy associated with its operation per cm.^{3} of iron per cycle is I_1 or 9692 ergs, of which $I = 8820$ ergs, hitherto called the total iron loss, is dissipated as heat in the iron, and $-(I_3 + I_5) = 872$ ergs is reflected by the iron to the primary circuit, where it also is dissipated as heat.

I desire to thank Mr. K. S. Cross and Mr. E. Machin for valuable assistance in carrying out the experiments discussed in this paper.

IX. *A Method of Determining the Thermal Conductivity of Copper.* By G. F. C. SEARLE, *M.A., Peterhouse, University Lecturer in Experimental Physics*[*].

§ 1. THOUGH the laws of the conduction of heat are of importance in a course of instruction in physics, yet the experimental investigation of those laws is, as a rule, thought to be suitable only for advanced students. But, if a substance of very high thermal conductivity, such as copper, be chosen, many of the difficulties fall into the background and then, with suitable apparatus, an elementary student can easily make an approximate determination of the thermal conductivity in " absolute measure."

The arrangement described in this paper is not well suited for very accurate determinations, since it would be difficult to estimate the proper corrections necessary to allow for the effects of the various imperfections. But in this case, as in many others, if the teacher so designs the apparatus that the corrections are certainly small, he may allow the student to neglect them, and he may be content with knowing that his pupil has gained the leading idea which the apparatus was designed to illustrate.

The method has been in use for some years at the Cavendish Laboratory for determining the thermal conductivity of copper ; the fact that my students have been interested in the experiment furnishes me with the excuse for publishing an account of it.

§ 2. The principle of continuous-flow calorimetry is employed in the method. One end of a stout bar of copper is heated by steam, while the other end is kept cool by a stream of water flowing through a narrow copper tube soldered to that end of the bar, and means are provided for determining the temperatures of two intermediate points on the bar. When in addition to these two temperatures the difference of temperature between the inflowing and outflowing water and also the mass of water flowing through per second are known, the conductivity of the copper is easily calculated.

§ 3. The apparatus is shown in fig. 1. The copper rod XY is about 20 cm. long and about 2·5 cm. in diameter. To the end X is fixed a small copper cylinder, closed at the ends, and steam from a small boiler passes through this cylinder. It is convenient to place the boiler below the level of the cylinder, and to make the steam-pipe large enough to allow the water formed by the condensation of the steam to find

its way back to the boiler. At the top of the cylinder the
surplus steam passes out by a small escape pipe. A copper
tube, 2 mm. in internal diameter, is coiled four times round
the end Y and is soldered to the rod ; this tube serves to
convey the cooling water.

The temperatures of the water before and after passing
through the coiled tube are ascertained by means of the
thermometers D and C, which are fixed into the small brass

Fig. 1.
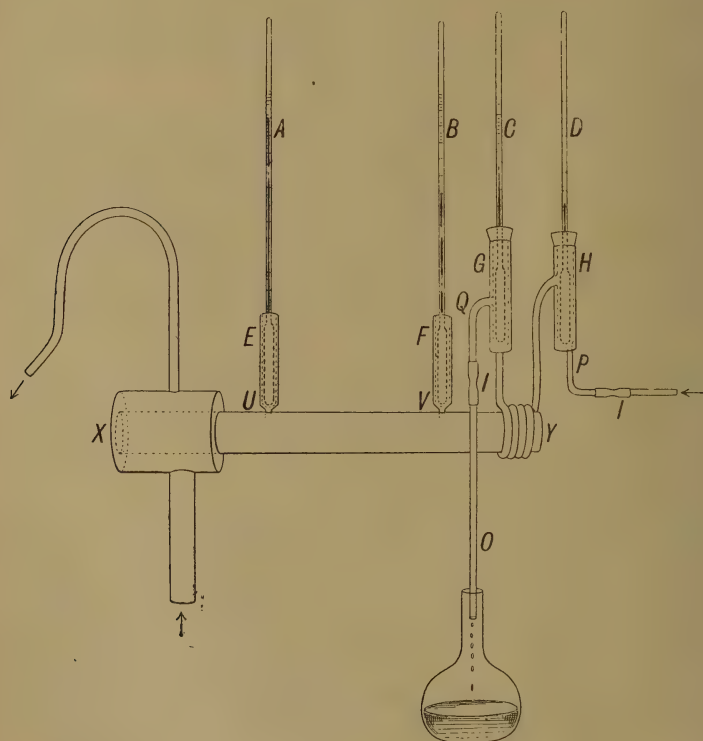
vessels H and G by indiarubber stoppers. Water enters H
by the tube P and leaves G by the tube Q, the necessary
connexions being completed by the indiarubber joints I, I.
A convenient way of obtaining a steady flow of water is to
allow the inflow pipe attached to P to dip like a siphon into
a reservoir in which the water is kept just overflowing by a
stream from a tap. By changing the difference of level
between the surface of the water in the reservoir and the

lower end of the outflow tube O, the current of water can be easily varied. With the apparatus in use in my own class the difference of level is never greater than 1 metre.

The mass of water which flows through per second is readily found by observing the time taken to fill a 250 c.c. flask. The water is, of course, allowed to flow continuously, and the time is counted from the instant when the flask is brought up to receive the water.

If, when the conditions have become steady, the temperature of the water in H be θ_4 and that of the water in G be θ_3, and if M grammes of water flow through in t seconds, then heat is taken away by convection from the end Y at the rate of

$$\frac{M(\theta_3 - \theta_4)}{t}$$

thermal units per second.

§ 4. To determine the temperature-gradient along the part of the rod between the steam-jacket and the cooling-tube, we must know the temperatures at two points on the bar separated by a measured interval. These temperatures are found by means of the thermometers A, B, which fit loosely into holes bored in the stout copper rods E, F nearly to the lower ends of the rods *. The lower ends of these rods are cut away so as to fit into saw-cuts made in the bar at U and V, and the rods are secured to the bar by solder or, preferably, by brazing. If no heat escapes from the copper tubes, the thermometers placed within them will indicate the temperatures of the bars at U and V. The distance between U and V is about 10 cm. in my apparatus. The isothermal surfaces passing through U and V are supposed to be plane. To secure this approximately, the distances of U and V from the steam-jacket and the cooling-tube respectively are not less than the diameter of the bar.

If θ_1 be the temperature at U and θ_2 that at V, when the conditions have become steady, and if the cross-section of the bar be A square cm. and the distance between the centres of the saw-cuts at U and V be l cm., then the rate at which heat flows along the bar is

$$\frac{K(\theta_1 - \theta_2)A}{l}$$

---

* The temperatures at U and V could be determined electrically by measuring the resistances of two coils, each formed of a few turns of silk-covered platinum wire wound round the bar at U and V; or a pair of thermo-junctions, insulated from the bar, might be used.

thermal units per second, where K is the thermal conductivity.

Equating this to the former value, we have

$$K = \frac{Ml}{At} \cdot \frac{\theta_3 - \theta_4}{\theta_1 - \theta_2}.$$

§ 5. It has been assumed in this calculation that all the heat which flows across the section of the bar at U flows also across the section at V, and that the whole of this heat is absorbed by the cooling water. To obtain a practical approximation to this ideal condition, the whole bar with its fittings is very thickly wrapped from end to end with a woollen covering. Wool is also wrapped round the copper tubes E, F and round the vessel G. It is not necessary to wrap up the vessel H, as this vessel is in practice at nearly the atmospheric temperature. For convenience, the apparatus, when wrapped with the wool, is placed in a wooden box provided with feet.

On account of the high conductivity of copper and the large cross-section of the bar, the heat conducted along the bar is very large compared with that which escapes altogether from the bar and with that which passes from one end of the bar to the other by conduction through the woollen covering.

§ 6. To illustrate the working of the apparatus, I give the numerical details of an experiment in which two different rates of flow of water were used *.

$$A = 5{\cdot}267 \text{ cm.}^2 \qquad l = 9{\cdot}95 \text{ cm.}$$
$$M = 250 \text{ grms.}$$

(a) $\theta_1 = 82^\circ{\cdot}0$ C., $\theta_2 = 44^\circ{\cdot}8$ C., $\theta_3 = 23^\circ{\cdot}1$ C., $\theta_4 = 12^\circ{\cdot}5$ C.
$t = 155{\cdot}5$ secs.

Hence $$K = \frac{250 \times 9{\cdot}95}{5{\cdot}267 \times 155{\cdot}5} \cdot \frac{10{\cdot}6}{37{\cdot}2} = {\cdot}865.$$

(b) $\theta_1 = 80^\circ{\cdot}8$ C., $\theta_2 = 40^\circ{\cdot}9$ C., $\theta_3 = 17^\circ{\cdot}3$ C., $\theta_4 = 12^\circ{\cdot}8$ C.
$t = 65{\cdot}6$ secs.

Hence $$K = \frac{250 \times 9{\cdot}95}{5{\cdot}267 \times 65{\cdot}6} \cdot \frac{4{\cdot}5}{39{\cdot}9} = {\cdot}812.$$

In obtaining these results it was assumed that the readings

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* In some copies of the apparatus which have been made, the bar is $1\frac{1}{2}$ inches in diameter and the internal diameter of the cooling-tube is 3·5 mm. The bar is made thicker to diminish the relative importance of the unavoidable errors.

of the four thermometers indicated correctly the temperatures of their bulbs. But this assumption is seldom valid in the case of thermometers such as those used by students and costing about 3s. 6d. each. The error due to this imperfection has the greatest influence in the case of the thermometers C and D, since $\theta_3 - \theta_4$ is much less than $\theta_1 - \theta_2$. An approximate method of obtaining the corrections to be applied to $\theta_1 - \theta_2$ and to $\theta_3 - \theta_4$ is to compare A and B in a bath whose temperature is roughly $\frac{1}{2}(\theta_1 + \theta_2)$ and to compare C and D at the temperature $\frac{1}{2}(\theta_3 + \theta_4)$. When this was done in the above experiment the results became

(a) $\theta_1 - \theta_2 = 36°\cdot2$ C., $\theta_3 - \theta_4 = 10°\cdot9$ C.,   K $= \cdot914$.

(b) $\theta_1 - \theta_2 = 38°\cdot9$ C., $\theta_3 - \theta_4 = 4°\cdot8$ C.,   K $= \cdot888$.

As the thermometers were only read by estimation to $\frac{1}{10}$ degree, the agreement between these two values of K is as close as could be expected.

When sufficient time is available, the student should compare the readings of each of the four thermometers with those of a standard thermometer, whose corrections have been determined at the National Physical Laboratory, by placing them together in a bath of water which is gradually heated. The *whole* of the standard thermometer must be immersed in the bath, but as to each of the others, so much of the stem should be in the air as was the case when it was fixed in the apparatus. In this way the correction due to the " emergent column " is practically avoided. The first-hand knowledge of the limitations of thermometry (with cheap thermometers) so obtained is well worth the time spent upon the comparison.

Cavendish Laboratory,
    Cambridge.

X. *An Interference Apparatus for the Calibration of Extensometers.* By JOHN MORROW, *M.Sc.(Vict.)*, *M.Eng.* (*Liverpool*), *and* ERNEST L. WATKIN, *M.A.* (*Cantab.*), *Lecturers in University College, Bristol* [*].

Introduction.

BEING engaged in a research which requires the use of extensometers of considerable sensitiveness, the authors have found some difficulty in determining the constants of the instruments with sufficient accuracy ; they also

considered it advisable to find out how far the extension readings were reliable.

For these purposes it was decided to calibrate the instruments by interference methods; and accordingly, after some preliminary trials, the apparatus described in this paper was designed. The following points were kept in view as being essential :—(1) That the calibrator should be as far as possible self-contained and easily made ready for use ; (2) that any displacement accidental or otherwise should affect both the interference apparatus and that being calibrated simultaneously and equally ; (3) that it should be unaffected by external disturbances ; and (4) that there should be no error possible in the determination of its own displacement.

In testing strain-meters by comparison with reading-microscopes or comparators there is great difficulty, owing to the unavoidable backlash of the screw, in ascertaining whether the instrument is itself free from a similar defect. The apparatus here described is not open to this objection.

Description of the Apparatus.

In figures 1 & 2, ff is a rigid cast-iron frame into the top of which is screwed a steel tube a. The lower portion forms guides $g\,g$, between which slides a gun-metal sleeve c. The steel tube b is attached to c by a set-screw, and the whole can be raised or lowered by the fine-threaded screw s actuated by the levers $l\,l'$.

The extensometer to be tested is attached by means of its gauge screws to a and b as to an ordinary test-piece. The displacement to be measured is therefore that of b relatively to a, and this is effected as follows :—

To the lower end of a, and at right angles to its axis, is attached a thin piece of optically plane glass p, whilst b carries at its upper extremity nearly in contact with p a lens q of black glass and small curvature. This lens is supported by the levelling table r, by means of which its centre of curvature is placed accurately on the axis of the instrument.

Light of known wave-length passing through a hole in a is directed to this optical system by a piece of plane glass m, set at an angle of 45°; and the interference-rings thus produced are viewed through the microscope k which slides in the upper tube.

When the screw s is turned b is raised or lowered through a small distance, and a ring appears or disappears for every half wave-length of light.

In the eyepiece of the microscope are three cross-wires,
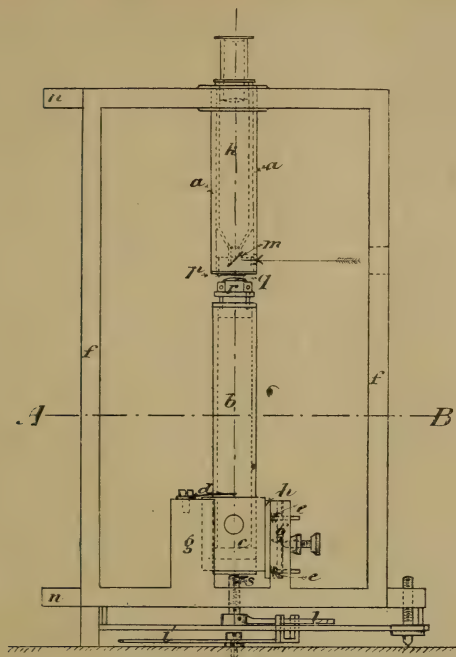
— *Elevation.* —
— *Fig. 1.* —
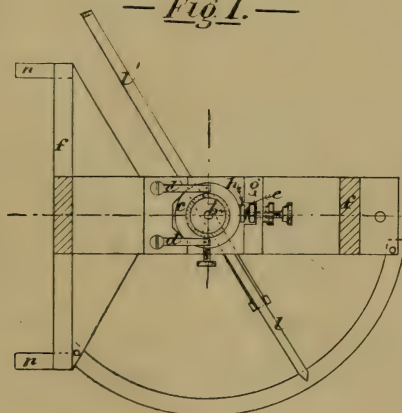
— *Sectional Plan A-B.* —
— *Fig. 2.* —

one at right angles to the other two, both extremities of the diameter of a ring being thus under observation. When in perfect adjustment, the centre of the rings remains midway between the two cross-wires.

The displacement is produced by turning the lever l' which is pivoted independently of the remainder of the apparatus, so that vertical pressure on it is not transmitted to l and b. The sleeve c is kept in contact with two worked surfaces on g by the piece h pressed against it by the springs $e\,e$, and at the same time its contact with the point of s is ensured by the springs $d\,d$.

To attach an extensometer the piece b can be lowered in its sleeve by releasing the set-screw ; the ends of the instrument can then be passed between p and q. Where this is not possible, the springs d are turned outwards so that the tube b can be removed altogether.

When the extensometer is one to be used in a horizontal position, the calibrator can be laid on its side resting on the legs $n\,n$.

The sodium light is brought to a focus at p by a condensing lens. It is obtained by placing a piece of platinum foil in the flame of a Bunsen burner, and allowing a solution of common salt to trickle very slowly down the foil.

Other methods which have been tried do not give a sufficiently intense light for a long period. This one, when properly arranged, burns continuously with little or no attention.

Each new ring corresponds to a distance of $29\cdot46 \times 10^{-6}$ centimetres.

Test of an Extensometer.

The points to be examined in testing an extensometer are— the value of the magnification constant ; the action of the instrument at starting and reversing (in order to discover whether there is any 'lost motion') ; and lastly—within what limits the readings obtained are proportional to the displacement and what correction is to be applied when this is not the case.

The following tests will serve as examples :—

The extensometer was of the differential mirror type arranged so that a displacement of 3×10^{-6} centimetres could easily be detected.

In the first place simultaneous readings of the extensometer and calibrator were taken for successive displacements in the same direction.

The test commenced with the extremities of an outside

diameter of a ring at the cross-wires of the microscope. A displacement exactly equivalent to ten rings was then produced and the movement of the extensometer scale noted. The process being continued, the readings in Table I. were obtained.

TABLE I.

Calibrator No. of Rings.	Extensometer.	
	Reading.	Differences in $\frac{1}{40}$ inch.
0	50·0	
		14·5
10	64·5	
		14·6
20	79·1	
		14·3
30	93·4	
		14·4
40	107·8	
		14·4
50	122·2	
		14·4
60	136·6	
		14·4
70	151·0	

Mean difference for 10 Rings ...... 14·43

The mean difference of 14·43 corresponds to a displacement of $29·46 \times 10^{-5}$ cms. The magnification of the extensometer is therefore 3012.

On another occasion, the instrument was arranged to give a difference of 13·75 units for a displacement of ten rings, and was then tested for accuracy in reversing. Readings were taken for a displacement of five rings followed by a return to the original position. Three such series are given in Table II.

TABLE II.

Calibrator No. of Rings.	Extensometer.					
	Readings	Diff.	Readings	Diff.	Readings	Diff.
0	163·3		119·9		120·0	
		6·8		7·0		6·8
5	170·1		112·9		113·2	
		6·9		6·9		6·9
0	163·2		119·8		120·1	

It will be seen that the instrument proved correct within the limits of accuracy of the observations.

The authors feel that the use of a calibrator of this kind will facilitate accurate work in elasticity and lead to a greater uniformity in the determination of coefficients.

University College, Bristol.
  June, 1904.

XI. *On the Lateral Vibration of Bars.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

HAVING recently, when dealing with the transverse vibrations * of bars, made liberal use of the approximate method developed in Lord Rayleigh's 'Theory of Sound,' I have read with interest Mr. Garrett's paper on this subject in your November number. The use of photography, to which Mr. Garrett's paper is mainly devoted, seems not unlikely to be of material service, at least in cases where no exact mathematical solution is obtainable. It is even conceivable that it might supply the means for a critical investigation as to the limits within which the ordinary Bernoulli-Euler solution for thin rods should be restricted.

The earlier part, however, of Mr. Garrett's paper seems hardly to do justice either to Lord Rayleigh's method † or to the subject of lateral vibrations. If the degree of closeness of Rayleigh's assumed type to the actual type of vibration in a clamped-free rod had required investigation, experiment is not, as Mr. Garrett seems to think, the only or even the most natural resource. The exact solution in this case is known, and if necessary it could have been compared with the simple type assumed by Rayleigh. This question, however, does not really arise. The object of Rayleigh's method is to obtain a close approximation to the *frequency* of the gravest vibration in a simple way, and the justification required by any assumed type of vibration is that the frequency derived from it should be a close approach to the truth. The justification for the method itself is the general theoretical result established earlier in Lord Rayleigh's vol. i. §§ 88, 89, &c., that a close approach is obtainable to the true period of vibration even by comparatively crude hypotheses as to the type of vibration, and that the frequency thus obtained must

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* Phil. Mag. May 1904, p. 504.
† 'Theory of Sound,' vol. i. § 182.

always be in excess of the true frequency. The method is based on recognized dynamical principles, and when different assumed types of vibration give different results for the frequency, there is no room for doubt as to which is the most nearly correct.

The method advanced by Mr. Garrett, on the other hand, is not apparently based on any *dynamical* principle. He simply calculates the moment about the clamped end of what he calls " the forces causing motion " and equates it to the elastic couple at that end, both results being derived from an assumed type of displacement which differs from that of the actual dynamical problem. A mistake of sign really runs through Mr. Garrett's work on pp. 583 & 584. His results, *e.g.* his equation (5), require $\ddot{y}_1/y_1$ to be a *positive* quantity, whereas it really equals $-k^2$, where $k/2\pi$ is the frequency. To get the sign correct, Mr. Garrett ought to have employed the " reversed effective forces " instead of the " forces causing motion."

As to the types of vibration employed by Mr. Garrett in his two illustrations, they are simply those employed by Lord Rayleigh himself in his § 182. Mr. Garrett differs, however, in the results he obtains for the frequency. Thus, in the first case, $k/2\pi$ denoting the frequency, and A a certain function of the dimensions and physical properties of the bar, the results obtained are as follows :—

	Garrett.	Rayleigh.	Exact theory.
$k^2 =$	A(120/11)	A(140/11)	A(136·0/11)

The error in Garrett's result is thus almost exactly four times that in Rayleigh's. The assumed type of vibration in this first case answers to the deflexion of the bar by a statical force at the end. In his next case Mr. Garrett, still following Rayleigh, assumes the displacement to be of the type answering to a statical force applied at some point short of the end. The result (9) he obtains is again different from Rayleigh's. In this instance the best result obtained by Garrett (see his pp. 582 & 589) differs 1·4 per cent. from the true frequency, whereas the best result obtained by Rayleigh is only 0·23 per cent. in error.

Mr. Garrett seems to see some difficulty in Rayleigh's method which I fail to recognize. In such an elementary case as a clamped-free bar it involves nothing more serious than two simple integrals [*e. g.*, in Garrett's first case $x^4(3l-x)^2$ and $(l-x)^2$ taken between 0 and l], operations which may surely be left to the reader of such a work as Rayleigh's.

It should also be noted that Rayleigh's method is equally applicable to all the simpler types of vibration. The procedure is the same in all cases, and like accuracy is obtained. In the case even of lateral vibrations, when the rod is " supported," or is clamped at both ends, Mr. Garrett's method would seem to require modification, and supposing such modification possible, no assurance could be felt in the absence of some guiding general principle as to the degree of accuracy of the results, or as to the direction in which they would differ from the true frequencies. In the two cases actually considered, Mr. Garrett's results for the frequency and those obtained by Lord Rayleigh's method from the same assumed type of vibration differ from the truth in opposite directions. If Garrett's method admits of generalization, and can be proved to supply a minimum as Rayleigh's method supplies a maximum value for the frequency, it may prove of real service in acoustics.

December 3, 1904.   CHARLES CHREE.

XII. *On the Practical Determination of the Mean Spherical Candle-Power of Incandescent and Arc Lamps. By* G. B. DYKE, *B.Sc., Salomons Scholar of the Institution of Electrical Engineers* *.

[Plate III.]

THE following experiments were undertaken at the suggestion of Dr. J. A. Fleming, F.R.S., as a piece of work during the tenure of a Salomons Scholarship, and were carried out at the Pender Laboratory, University College, London, under his direction.

The principal object of the research was to investigate the relation between the mean spherical candle-power of various types of incandescent and arc lamps, and the candle-power measured in a horizontal direction.

It is obvious that some improved method of expressing the efficiency of electric lamps is needed, having regard to the fact that a nominal 16-candle power incandescent lamp may give from 12 to 20-candle power in different directions in a horizontal plane, corresponding to a variation in so-called efficiency of from 5 to 3 watts per candle.

Such a method has been pointed out by Dr. Fleming †,

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* Communicated by the Physical Society : read November 11, 1904.
† " The Photometry of Electric Lamps," by Dr. J. A. Fleming, F.R.S. Journal Instit. Electric Engineers, vol. xxxii. p. 165.

consisting in stating the whole flux of light in lumens per watt, which gives a figure of about 3·5 for glow-lamps.

The expression of the efficiency in this manner involves the determination of the mean spherical candle-power ; and the method described in this paper affords accurate results with no greater expenditure of time than that required for one, or at most two, measurements by existing methods.

Apparatus and Method.

The experiments were carried out with the following objects in view:—

(i.) To obtain a series of curves showing the variations of candle-power in incandescent lamps, in a horizontal plane.

(ii.) To obtain reduction factors by which the mean horizontal candle-power may be calculated from the maximum horizontal candle-power, for different types of filament.

(iii.) To obtain reduction factors for obtaining the mean spherical candle-power from the maximum or mean horizontal candle-power.

Objects (i.) and (ii.) were met by designing a carriage, to run on the existing photometric bench in the Pender Electrical Laboratory, provided with a scale of degrees and means for rotating the lamp.

The carriage may be briefly described as follows :—

A strong cast-iron frame A (Pl. III. fig. 1) carries four flanged wheels B and a cord grip C. In the centre of the frame a bearing for a vertical shaft D is arranged, supported at E by a collar-bearing. This shaft is hollow for more than half its length, and carries a second sliding tube, which can be fixed in any position by the nut F. To the upper end of this is screwed the lamp-holder G.

On the lower portion of the shaft is fixed a bevel-wheel H and a pair of slip-rings J arranged in a horizontal plane on an ebonite base. On these rings press a pair of gauze brushes K K fixed in spring-holders and mounted on insulating supports.

A scale of degrees is arranged at L (shown dotted in plan), the pointer being fixed to the collar-bearing E.

The vertical shaft is driven by a motor M carried on the same carriage. The motor is mounted in a gun-metal ring N by three screws, which, together with the slots at O O, allow of a small movement in any direction.

A pinion on the motor-shaft engages with the bevel-wheel H, the reduction ratio being 4 to 1. The motor voltage is adjusted until the lamp rotates about four times per second.

Two pairs of flexible leads are brought from suitable

plugs, supplying current to motor and lamp respectively, and also a pair for measuring the voltage on the lamp-terminals.   All the electrical measurements were accurately made on a potentiometer.

The slip-rings are connected by flexible leads passed through the hollow shaft to the terminals of the lamp-holder.   A cord grip P relieves the motor and brush supports from the drag of the flexible cord.

The carriage is pulled along the photometric bench by an endless cord passing over pulleys at the ends of the bench and gripped by a cord grip C. The position of the lamp on the bench is read by means of the wire Q.

The advantage of this rotator is that it is self-contained, and hence can be arranged so as to enable a fixed photo-meter-screen to be used, making it much more convenient to work with.

The method employed was to fix the lamp in the holder with the filament so placed that the axis of the block on which the filament was formed coincided with the axis of the photometric bench.

This was called the zero position.

The voltage on the lamp terminals was kept at that required to give the marked candle-power in the zero position.

Readings of candle-power were then taken at frequent intervals round the lamp, special attention being paid to those positions in which the candle-power was changing rapidly.

Finally the lamp was rotated, and a series of readings of the mean horizontal candle-power taken.

The value thus obtained was checked against that given as the mean ordinate of the curve plotted from the step-by-step readings; the results in almost every case agreeing to within three-quarters of 1 per cent.

To obtain rapidly the relation between mean spherical and mean horizontal candle-power, a modification of the method devised by Mr. C. P. Matthews was employed.

This method consists in placing several pairs of mirrors round the circumference of a semicircle, and so arranging them that light emitted from a source placed at the centre of the system shall be incident on a white screen, also situated at the centre, at the same angle with the vertical as that which it made on emission from the source.   Then, if the source is symmetrical about the diameter of the semicircle, and if Lambert's cosine law holds for the screen used, it can be shown that the intensity of illumination on the screen is proportional to the mean spherical candle-power.

A proof of this is given in Mr. Matthew's paper [*]; but an interesting method is to deduce it from the well-known Rousseau diagram, as follows:—

Let n = number of pairs of mirrors used.

I_θ = intensity of illumination in a direction making an angle θ with the horizontal.

$I_{max.}$ = maximum intensity = OB.

M.S.C.P. = mean spherical candle-power.

and   M.H.C.P. = mean horizontal candle-power.

Let OG1 (Pl. III. fig. 2) be the polar intensity curve of a source symmetrical about the vertical AB.

Let CDEF be the Rousseau diagram obtained in the usual way from the polar curve.

Divide the semi-circumference A 1 B into $(n+1)$ parts at the points 6, 5, 4, ... 7.

Project these points horizontally on to CF, and through the projections draw ordinates to the Rousseau curve, shown dotted in the figure.

Bisect the arcs A6, 65, &c., and project the points of bisection in a similar manner, obtaining the full-line ordinates of the curve.

Then, if n is sufficiently large, the whole area CDEF may be considered to be made up of small strips, e. g., $a\,b\,c\,d$, cut off by the full-line ordinates of the curve.

Hence the area CDEF $= \Sigma$ (area $a\,b\,c\,d$)

$$= \Sigma\, \{ad \times \tfrac{1}{2}\,(a\,b + c\,d)\}.$$

Now, with a sufficient degree of approximation,

$$\tfrac{1}{2}\,(a\,b + c\,d) = I_\theta,$$

and

$$a\,d = \frac{\pi}{n+1}\cos\theta\,I_{max.}$$

Hence the area CDEF $= \dfrac{\pi}{n+1}\,I_{max.}\,\Sigma\,(I_\theta\cos\theta).$

But M.S.C.P. $= \dfrac{\text{area CDEF}}{\text{CF}}$

$$= \frac{\text{area CDEF}}{2 \times I_{max.}}.$$

Hence M.S.C.P. $= \dfrac{\pi}{2\,(n+1)}\,\Sigma\,(I_\theta\cos\theta).$

[*] "An Integrating Photometer for Glow-Lamps and Sources of like Intensity," by C. P. Matthews. Trans. Amer. Inst. Elec. Engin. vol. xix. Nov. 1902, p. 1467.

Now if D=total distance in feet by any one path from source to screen,

and K=reflexion coefficient of a pair of mirrors,

and if the cosine relation holds,

then the intensity of illumination of the screen in candle

feet$=\dfrac{K}{D^2} \Sigma (I_\theta \cos \theta)$.

Hence M.S.C.P.$=\dfrac{K}{D^2} \dfrac{\pi}{2\,(n+1)} \times$ illumination of screen in C.F.

The following description briefly outlines the construction of the apparatus designed to obtain the ratio $\dfrac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$.

A wrought-iron band, 4 in. × ¼ in., A (fig. 3), stiffened with a small L section, is bent into a semicircle 6 feet in diameter, and fixed by wood supports with its diameter vertical.

Eleven bosses, B, fitted to the outside of the semicircle, carry the mirror supports C ; these supports slide radially through the bosses and may be fixed in any position by set screws.

Each mirror support carries a pair of brass plates to which mirrors are clipped, each plate being independently adjustable.

The arrangement is shown in Pl. III. fig. 3. The distance from centre to centre of the mirrors of each pair is 8¾ inches.

The lamp to be tested is mounted on the rotator at D, at the centre of the semicircle and in the plane of one set of mirrors.

The photometer, of the Lummer-Brodhun pattern, is placed at E, also at the centre, but in the plane of the other set of mirrors.

Both lamp and photometer are so mounted as to allow of their being set accurately to the central position.

The photometer has a 1-inch disk, and the case is cut away at F, so as to permit light to reach the whole surface of the disk from any angle in the vertical plane containing the disk.

The object being to find the relation between the mean spherical and mean horizontal candle-powers, the side G of the photometer must be illuminated with an intensity proportional to the mean horizontal candle-power.

This is done by arranging the pair of mirrors H to slide in a horizontal plane along a railway J formed of an inverted T section.

A matchwood screen K prevents any light reaching the photometer except by way of the mirrors.

This screen is provided with windows opposite the mirrors fitted with slides, so that each pair of mirrors may, if desired, be used separately : this arrangement is of great service during the setting of the mirrors.

The position of the sliding pair is read on a scale arranged on the box-shaped screen protecting them from stray light. This scale is graduated to read off directly the distance from lamp to photometer.

The above system of screens is of course only suited for use in a dark room, but, with a simple arrangement of curtains, no difficulty should be found in using the apparatus in broad daylight.

An approximate idea of the dimensions of the apparatus may be formed from the scales included in fig. 1 and fig. 3.

It will be seen that the formula given above for the determination of the mean spherical candle-power depends for its validity upon the truth of the cosine relation, and also on the identity of the reflexion coefficients of the various pairs of mirrors.

Taking first the cosine relation, this was investigated in the following manner :—

The mirrors on one side of the upper quadrant of the apparatus having been removed, a lamp-holder was devised to slip on the rods of the mirror supports and to be clamped with a set screw in such a position that a lamp placed in the holder in a radial direction pointed directly head-on to the photometer disk.

A 32 C.P. lamp was used for this purpose.

A 16 C.P. lamp, which had been previously calibrated, was then placed in the rotator and, all the window-slides having been closed, the 32 C.P. lamp was placed on each of the mirror supports in turn and successive balances obtained, and the corresponding scale-readings taken.

From these readings, and a knowledge of the reflexion coefficient of the sliding pair of mirrors, the illumination of the photometer screen was deduced and the curves of fig. 4 obtained : the full-line curve taken from the experimental results, and the dotted curve showing the values which would have been obtained had the cosine relation held.

From these two curves fig. 5 follows immediately, and shows the percentage variation from the cosine relation for the particular Lummer-Brodhun screen under test.

The reflexion coefficients of the mirrors were obtained by the usual method on an ordinary photometric bench, each pair of mirrors being tested together.

To compensate for both these disturbing influences, each pair of mirrors was placed at such a distance from the photometer that the illumination on the screen due to each was the same as it would have been, had the cosine relation been rigidly followed and the reflexion coefficients been all the same as the coefficient for No. 1 pair of mirrors (the mirror pairs being numbered as in fig. 2).

The following table shows the results of these investigations and adjustments.

No. of mirror pair.	Illumination less than cosine. per cent.	Reflexion coefficient. per cent.	Illumination less than No. 1. per cent.	Distance lessened. per cent.	Total distance lamp to photometer. inches.	Distance mirror to photometer inches.
1	0·0	68·5	0·0	0·0	73·75	32·50
2	0·1	68·5	0·1	0·1	73·68	32·47
3	0·9	71·4	−2·0	−1·0	74·49	32·87
4	2·1	68·9	1·7	0·9	73·10	32·18
5	3·6	69·1	3·0	1·5	72·66	31·96
6	6·1	69·5	5·1	2·6	71·84	31·55
7	6·1	66·8	7·8	3·9	70·88	31·06
8	3·6	67·3	4·8	2·4	71·99	31·62
9	2·1	69·2	1·4	0·7	73·24	32·25
10	0·9	69·2	0·2	0·1	73·68	32·47
11	0·1	65·1	3·5	1·8	72·44	31·85
12	0·0	70·5		(sliding pair)		

These adjustments having been made it will be sufficient to insert in the formula

$$K = ·685 \quad \text{and} \quad D = \frac{73·75}{12} = 6·15 \text{ feet.}$$

The formula given above was

$$\text{M.S.C.P.} = \frac{D^2}{K} \cdot \frac{\pi}{2(n+1)} \times \text{illumination of screen in C.F.}$$

But the illumination of the screen

$$= k \frac{\text{M.H.C.P.}}{d^2},$$

where $d =$ total distance in feet from source to screen by way of the sliding pair, and $k =$ reflexion coefficient of sliding pair of mirrors.

Hence $\quad \text{M.S.C.P.} = \frac{k}{K} \left(\frac{D}{d}\right)^2 \frac{\pi}{2(n+1)} \cdot \text{M.H.C.P.},$

or, putting in the values of the constants :

$$\text{M.S.C.P.} = \frac{70 \cdot 5}{68 \cdot 5} \times \left(\frac{6 \cdot 15}{d}\right)^2 \times \frac{\pi}{2(11+1)} \times \text{M.H.C.P.}$$

$$= \frac{5 \cdot 09}{d^2} \times \text{M.H.C.P.}$$

$$\frac{\text{M.S.C.P.}}{\text{M.H.C.P.}} = \frac{5 \cdot 09}{d^2} \text{ where } d \text{ is in feet.}$$

Hence a scale could be easily constructed from which the ratio $\dfrac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$ might be read off directly.

Before passing on to the results of the tests, some interesting figures relating to the absorption of ground glass might be quoted.

In designing the apparatus described above, attempts were made, at the suggestion of Dr. Fleming, to diminish the dimensions of the integrating photometer by fixing ground-glass screens in the windows between the mirrors in order to absorb a portion of the light and so reduce the illumination on the photometer-screen, which with an instrument of small radius would otherwise be uncomfortably brilliant, especially when dealing with arc lamps.

The validity of this device must of course depend upon the constancy of the absorption coefficient of the screen when illuminated by lights of very different spectral composition, and the following experiments were made to investigate this point.

The screen used was a sheet of glass of an average thickness of ·08 inch, ground by sand-blast on both sides.

This screen was set up in a vertical position, 25 inches from the photometer-screen and at right angles to the photometric bench. The source of light under test was set up behind the screen at a fixed distance from it, a photometric balance being obtained by means of a lamp, on the other side of the photometer, whose distance could be adjusted.

The sources used were a 50 C.P. incandescent Focus lamp, a 100 watt Nernst lamp, and a hand-fed continuous-current arc.

However, the results obtained with the Arc lamp can scarcely be compared with the others, as the distance of the lamp from the screen was much greater in this case than in the others, due to the intensity of the light. In the case of the Focus and Nernst lamps the distance from lamp to screen was 25 inches.

Readings were obtained with and without the screen in position, the absorbing power being taken as the ratio of the illumination on the photometer without the screen to the illumination with the screen.

Observations were made on the Nernst lamp from 20 per cent. below its marked volts until the automatic heater came into action, and on the Focus lamp from 2 per cent. above to 10 per cent. below its marked volts.

The results of these experiments are shown in the accompanying curve (Pl. III. fig. 6), in which absorbing power is plotted against watts per candle.

The curve shows the general result that the absorption increases with the watts per candle, whence it follows that the red rays suffer a greater absorption than the blue.

It was at once seen from these results that any such screen could not possibly be used for sources of different spectral character, and hence the somewhat bulky apparatus of fig. 3 was necessitated.

Results.

Incandescent Lamps.

In all, nine types of incandescent lamps were tested; the total number of lamps under observation being twenty-five.

The horizontal candle-power was investigated as indicated above, the average number of positions photometered being 26 for 16 C.P. lamps and 84 for 32 C.P. lamps; the greater number in the case of the 32 C.P. lamps being necessitated by the greater distance between lamp and photometer, and the consequent diminished sharpness in the gradations of illumination of the photometer as the lamp was turned round by hand.

The following is a table of the different types of lamps investigated :—

1. Ediswan Flat-Loop Filament.   50–16–B 1·12.
2. Ediswan Double-Loop Filament.   100–16–A ·64.
3. Edison Double-Loop Anchored Filament.   110–V–16 C.P.–E.
4. Ediswan Triple-Loop Doubly Anchored Filament. 110–16–A.
5. Ediswan Triple-Loop Doubly Anchored Filament. 210–16–B.
6. Royal Ediswan Double-Loop Filament.   110–16–A ·58.
7. Ediswan Double-Loop Filament.   110–8–A ·29.
8. Ediswan Quadruple-Loop Doubly Supported Filament. 110–32–B.
9. Premier Double-Loop Filament.   32–110–D. C.

The curves in fig. 7 represent graphically the result of the observations, the ordinates reading candle-power and the abscissæ degrees of angle in a horizontal plane round the lamp ; the zero angle, as before mentioned, corresponding to that position of the lamp in which the axis of the loops of the filament is parallel with the axis of the photometric bench.

The full-line curves show the variation in candle-power in a horizontal plane, and the line of strokes gives the M.H.C.P.

These curves show in a very evident manner the very large changes in candle-power which may occur in any lamp when photometered in different positions.

The sharp peaks and depressions which occur in some of the lamps are particularly noticeable ; and with a view to investigating the appearance of the filament when observed in these directions, photographs of the lamp when incandescent were taken in a few of these positions.

These show that, as would be expected, the sudden depressions in the curves are due to parts of the filament becoming hidden, either behind another part of the filament, or, in some of the multiloop lamps, behind the anchoring support.

The high peaks, on the other hand, are, as Mr. Russell * pointed out some little time ago, due to the back of the bulb acting as a mirror, the peaks of the curve corresponding to the positions in which part of the filament happens to coincide with the principal focus of the part of the bulb directly behind it. The photographs, two of which are reproduced in Pl. III. fig. 8, show this very well indeed. Whilst all of them show a large number of images of the filament in the glass, the sharpness of these images proves that they are at approximately the same distance from the camera as the filament ; but in the cases where a peak occurs, and only in such cases, the photograph shows a more or less pear-shaped and blurred bright region ; indicating the presence of an image which is not at the same distance.

With regard to the relations between the mean spherical, mean horizontal, and usually measured candle-power, the results of the observations on the types of incandescent lamps tested are shown in the following tables, in which the following abbreviations have been used :—

* " Mean Horizontal and Mean Spherical Candle-power," by Alex. Russell, M.A., Journ. Inst. Elect. Eng. vol. xxxii. p. 635.

CP. = mean of C.P.'s in positions in which axis of loops of filament is parallel to photometric bench.

M.H.C.P. = mean horizontal candle-power.

M.S.C.P. = mean spherical candle-power.

Lamp No.	C.P.	M.H.C.P.	M.S.C.P.	Reduction Factors.		
				$\dfrac{\text{M.H.C.P.}}{\text{C.P.}}$	$\dfrac{\text{M.S.C.P.}}{\text{C.P.}}$	$\dfrac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$
1.........	16·6	16·1	12·5	·97	·75	·78
2.........	16·6	16·0	12·8	·97	·77	·80
3.........	16·4	16·2	13·0	·99	·79	·80
4.........	15·9	14·8	12·5	·93	·79	·85
5_A	16·0	14·9	11·7	·94	·73	·78
6_A	16·0	15·1	12·2	·94	·76	·81
7_A	8·0	7·5	6·2	·93	·77	·83
8_A	30·5	29·3	24·5	·96	·80	·84
9_A	31·8	29·9	24·2	·94	·76	·81

It is obvious, however, that these reduction factors are of very little use unless they remain sensibly constant for different lamps of the same type of filament.

The following table shows the reduction factors for seven lamps of type 5 :—

Lamp No.	Reduction Factors.		
	$\dfrac{\text{M.H.C.P.}}{\text{C.P.}}$	$\dfrac{\text{M.S.C.P.}}{\text{C.P.}}$	$\dfrac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$
5_A	·94	·73	·78
5_B	·92	·71	·78
5_D	·92	·72	·78
5_E	·96	·74	·78
5_F	·93	·72	·78
5_G	·95	·74	·78
5_H	1·00	·79	·79

This table shows clearly that only one reduction factor is of practical utility, namely, the ratio $\dfrac{\text{M.S.C.P.}}{\text{M.H.C.P.}}$.

This one factor, however, is remarkably constant and appears to be very little altered by a distorted filament; thus, although in 5_H the filament was so much bent as to

almost touch the bulb in one place, the factor is only altered by 1 per cent.

The unsatisfactory nature of the present method of photometering incandescent lamps is seen in the second reduction factor $\dfrac{\text{M.S.C.P.}}{\text{C.P.}}$.

This shows that the mean intensity of the light emitted is only about 75 per cent. of the nominal or marked intensity, and also that the mean and marked intensities bear no definite relation to one another, even in the case of lamps of the same type of filament.

A more satisfactory method would be to determine the mean horizontal candle-power of each lamp by using a rotator similar to that described in the paper, and then to obtain the M.S.C.P., employing a factor previously determined for the particular type of filament by means of an integrating photometer as explained.

The marking on lamps thus photometered would then be proportional to the amount of light emitted, and hence would be a figure definitely determining their utility as a means of illumination.

In this manner the suggestion made by Dr. Fleming in his paper on the "Photometry of Electric Lamps" * could be practically carried out with but little more trouble than the present unscientific method of marking incandescent lamps.

Arc Lamps.

The Arc lamp being symmetrical about the vertical will not require revolution in the integrator, but the readings will give directly the ratio of the mean spherical to the horizontal candle-power.

Preliminary experiments only have been made with arc lamps, but these have been sufficient to show that arc lamps may be dealt with in the integrator with but little more trouble than incandescent lamps.

* Journ. Inst. Elec. Eng. vol. xxxii. p. 165.

XIII. *On Radioactivity of Mineral Springs.* *By* G. A. Blanc, *Dr. in Physics of the University of Rome* *.

THE researches made by Sella, J. J. Thomson, Himstedt, and other physicists have shown that ordinary tap-water generally contains a radioactive gas or emanation that can be easily obtained by boiling the water or by drawing air through it.

Mineral waters, and the deposits which they form, have hitherto generally shown considerable activity, the subject having been studied by Elster and Geitel, Strutt, Lord Blythswood and Allen, Curie and Laborde, and others.

I shall give here the results of an investigation concerning the radioactive properties of the materials deposited by a certain number of mineral and thermal waters belonging to the Alpine region, which I have had the opportunity of examining during the summer.

To measure the activity of the various sediments I adopted the apparatus shown in fig. 1, which has been used by Sella in similar researches. It consists of a glass vessel

Fig. 1.
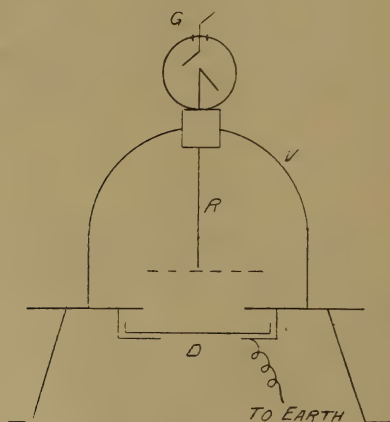
V with an opening at the top through which a metal rod R is fixed by means of a sulphur insulator. To the lower end of this rod a wire-gauze disk can be fixed horizontally, while the upper part carries an aluminium leaf and is covered by an electroscope case. The vessel stands on a metallic plate provided with an opening in its centre, under which a dish D may be placed containing the substance to be tested. The charge is given to the apparatus by a movable contact C of the usual type.

* Communicated by Lord Kelvin.

In this case the deviations of the aluminium leaf were read by means of a microscope containing a micrometer eyepiece, the measurements being made by observing the time taken by the aluminium leaf to fall from a determined scale-division to another ; this method has the advantage of insuring a constant sensibility of the apparatus throughout all the observations.

The readings were reduced to the potential fall in volts per hour, the effects due to the natural ionization of the air of the room and to the imperfections of insulation being each time subtracted.

Ten grams of each material, reduced into a fine powder and uniformly spread on the dish, so as to form a very thin layer, were successively tested in the apparatus, with the results that follow (see Table, p. 150).

The effect due to an equal weight of uranium nitrate is given as a term of comparison.

As one can see, the activity of the different sediments deposited by the waters diminishes greatly with the increasing distance between the spring and the spot at which they are formed. A similar observation had already been made by Elster and Geitel with the deposits of Baden.

I have noted a curious fact in connexion with this at the so-called " Source d'Alun " of Aix-les Bains ; the waters there show at their surface a viscous substance, partly organic, called " Barégine," which has a considerably greater activity than any other material that I could collect in this particular spring. The theory of the origin of this substance, as given by naturalists, states that it is formed at the spot where the waters emerge from the soil.

I had noted that by keeping some of the sediments for a certain time in the testing-vessel the conductivity of the air inside steadily increased until it reached a maximum. This being evidently due to the formation of a radioactive emana-tion, I undertook to determine its rate of decay.

The apparatus which I used for this purpose is represented in fig. 2 (p. 151).

A certain amount of the sediment to be tested was intro-duced into a flask F and left there for some days, the flask being heated from time to time. Then the two stopcocks S_1 and S_2 were opened and air was drawn by means of a water-pump from F through a wash-bottle W containing sulphuric acid, a tube P containing phosphoric pentoxide, and a bulb B tightly packed with glass-wool, into the testing-vessel V, after which the stopcocks S_3 and S_4 were carefully closed.

Name of the Spring.	Nature of Water.	Temp. C°.	Nature of Substance Tested.	Potential fall in volts per hour.
Echaillon (Maurienne)	Saline.	34°	Deposit from serbatory, coloured red by iron.	312·0
„ „	„	„	Deposit from well, yellow in colour.	122·9
Salins-Moutiers, Grande Source......	„	35°	Reddish sediment from the spring.	128·8
„ „ „	„	„	Red sediment from canal-walls.	72·0
Salins-Moutiers, Petite Source	„	37°	Brown-red sediment from canal-walls.	56·0
Aix-les-Bains, Source d'Alum	Sulphurous-Sodic.	46°	Barégine.	19·4
„ „	„	„	Hard grey sediment from canal-walls.	11·2
Aix-les-Bains, Source de Soufre ...	„	45°	Greyish mud from the spring.	4·9
Allevard (Isère)	Sulphurous.	16°	Grey sediment, very rich in sulphur.	4·0
Challes (near Chambéry)	„	10°	Grey mud „ „	3·3
La Boisse (Chambéry)...............	Ferruginous.	„	Deposit coloured red by iron.	2·6
La Bauche (Ain)	„	12°	Mud coloured red by iron.	0·7
			Uranium nitrate.	5752·0

By examining in this way some of the most active materials, I could see that each produced a certain amount of emanation, the activity of which decayed with time at rates not very dissimilar from that of radium emanation.

Fig. 2.
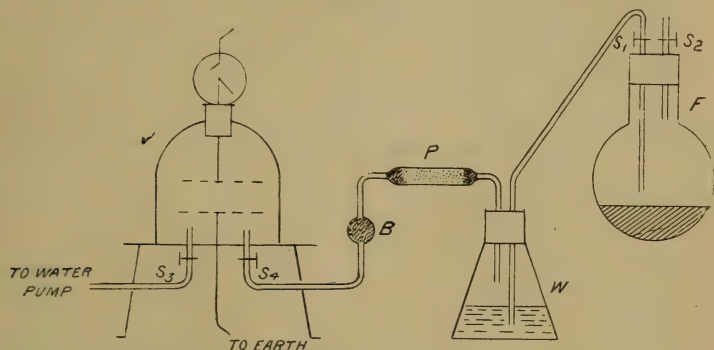
The sediment of the Source d'Alun of Aix-les-Bains produced considerable effects, the conductivity in the testing-vessel rising to several hundred times the normal value. After the introduction of the emanation into the testing-vessel the conductivity increased steadily until it reached a maximum, and then began to decay according to an exponential law, falling to half its value in 3·2 days.

The activity of the emanation obtained from this sediment has, as one can see, a rate of decay somewhat more rapid than should be expected for radium emanation, but the result is in close agreement with the one obtained by Ebert and Ewers with emanation contained in air removed from the soil *. It also differs only slightly from the value given by Adams for the rate of decay of emanation obtained from Cambridge tap-water †.

But a fact that struck me in the course of these measurements was that some of the most active deposits, as for instance those of Salins-Moutiers, when tested in the manner which I have now described, seemed to produce a very small amount of emanation as compared with the large amount obtained from other sediments which appeared to be much less active when examined by the method described in the first part of this note.

* *Phys. Zeit.* iv. p. 162 (1902).
† Phil. Mag. Nov. 1903.

It seemed to me therefore evident that the greatest part of the activity shown by a certain number of sediments must be due to the presence of a radioactive substance without any emanating power, or maybe whose emanation lost its activity very rapidly with time. This hypothesis could also explain how the waters of Salins-Moutiers, when examined by Curie and Laborde, had proved to contain very little emanation*, while the sediments were among the most active which I had found.

To settle this I proceeded in the following way :—Having left for a few minutes some 50 grams of sediment from the Grande Source of Salins-Moutiers in the dish of the apparatus shown in fig. 1, I managed to slide rapidly a metal screen over the dish, so as to close the opening under the vessel and remove the air inside it from the direct action of the sediment. (I had previously ascertained that the metal plate used as a screen was sufficient to cut away any direct radiation from the active substance.) In this way I was able to see that the conductivity of the air in the testing-vessel did not drop instantly when the screen was introduced, but that it gradually decayed with time.

By noting the rate at which the aluminium leaf fell I was able to determine with sufficient approximation the law according to which the emanation lost its activity. Repeated measures showed that the conductivity is reduced to half its value in about one minute.

Similar results were obtained by examining in the same manner the various sediments of the " Petite Source " of Salins-Moutiers and of the Echaillon spring.

The results obtained with the sediments of Echaillon are shown graphically in fig. 3. These results seemed to point to the conclusion that a great part of the activity shown by the materials now mentioned was due to the presence of a radioactive constituent similar to, if not identical with, thorium.

I thought therefore that it would be interesting to see whether an inactive body could be made temporarily radio-active by exposing it to the emanation, and in this case verify whether the rate of decay of the excited activity was the same as in the case of thorium. After keeping for a couple of hours a disk of tinfoil in a glass vessel containing a certain amount of sediment from any one of the three springs mentioned above, it showed a notable activity. Considerably stronger

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* *Comptes Rendus*, cxxxviii. p. 1150.

effects were obtained by keeping the tinfoil disk charged at a
potential of about −600 volts, by means of a Zamboni pile.
The activity decayed with time, falling to half its value in

Fig. 3.
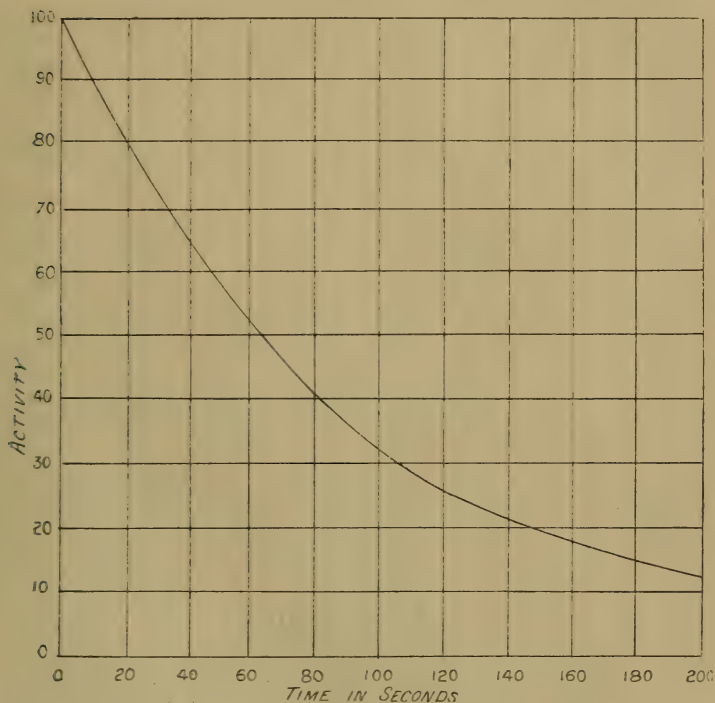
about eleven hours. The resulting curve is shown in fig. 4
(p. 154). The close agreement that exists between these
measurements and the values obtained by other observers for
thorium, seems to establish clearly the existence of this metal
in the sediments of a certain number of thermal springs.

I could detect no similar effects with the other deposits,
which, as I have mentioned above, give rise to an emanation
that loses half of its activity in 3·2 days. The excited activity
obtained from this emanation decayed at a rate very like the
one of radium-excited activity, *i. e.* falling to half its value
in about half-an-hour.

Fig. 4.
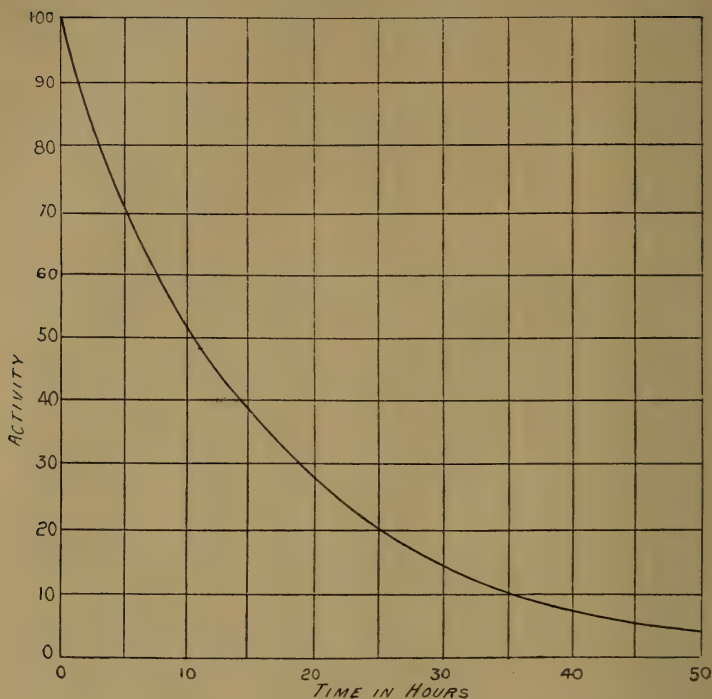
Conclusion.

The results given above show that besides the activity and emanation probably due to traces of radium which several physicists have observed in the sediments of a great number of mineral and thermal springs, some deposits contain a radioactive constituent, in all probability thorium, the emanation of which loses half of its activity in about one minute, and is capable of producing excited activity that decays with time, falling to half its value in about eleven hours.

I may add that attempts are now being made with the object of separating by chemical means the radioactive constituents and of determining as exactly as possible the amount of each present in the different sediments.

In conclusion I desire to express my heartiest thanks to Lord Kelvin, who was good enough to take some interest in the present research, and did me the honour of witnessing some of my experiments.

XIV. *The Measurement of Small Differences of Phase.* By W. E. SUMPNER, *D.Sc.**

WHILE investigating recently the behaviour of watt-meters, transformers, and similar apparatus, the writer has found it necessary to measure small phase-differences occurring in the working of alternating-current plant. The methods hitherto available for such measurements are not at all satisfactory when the angles to be determined are small. Phase-meters have been constructed for commercial purposes, but the angular deflexion of the pointer of these instruments is, as a rule, smaller than the phase-difference to be measured, so that when the latter is as small, or smaller, than one degree, such instruments, even if perfectly accurate, are quite unsuited for the purpose. All other known methods necessitate the simultaneous reading of three instruments. The best-known method needs a wattmeter, a voltmeter, and an ammeter. Among wattmeters we may include all instruments of the double current, or double voltage, type, whether dynamometers, current-balances, or electrometers. In all these cases the value of $\cos \theta$, where θ is the angle of phase to be determined, is measured by dividing the wattmeter reading by the product of the readings of the other two instruments. The percentage error made in measuring $\cos \theta$ is thus greater than the corresponding errors made in reading the separate instruments. The method only gives fair results when the angle θ is large, and instruments of suitable range are available. When θ is small the method is hopeless, for since $\cos 1°$ is ·99985, and $\cos 0°·5$ is ·99996, it is clear that no such method involving the measurement of three deflexions can be anything like accurate enough for determining values of θ less than one degree.

All other known methods are based upon the three-voltmeter method of measuring power factor, or phase-difference. This method has been much criticised at different times, but its limitations were all fully pointed out in the original paper (see Proc. Roy. Soc. vol. xlix. March 1891) in which Professor Ayrton and the present writer first drew attention to it. It is a method for measuring power, or power factor, which cannot compare in ease or simplicity with the wattmeter method, when a suitable wattmeter is available, and the accuracy of the instrument is not in doubt. But wattmeters and other alternating-current meters, owing to the absence of iron-cored magnetic circuits, are not nearly so sensitive

as, and do not possess anything like the range of, the corresponding direct-current instruments.   A wattmeter of the right range is not always available, and in such cases the three-voltmeter method, or some modification of it, has often proved a convenient substitute.

The errors arising in practice in measuring phase by the three-voltmeter method are serious for low-power factor circuits, that is for values of θ approaching 90 degrees, but are not so important when high-power factors have to be measured.   The perfection of the method in theory, and its limitations in practice, are exactly comparable with the determination of the angles of a triangle from measurements of its three sides.   If these sides are measured accurately the angles can be correctly calculated in all cases, but for given percentage errors made in estimating the sides, the resulting error made in calculating the angle will largely depend on the shape of the triangle.   Thus, in fig. 1, the

Fig. 1.
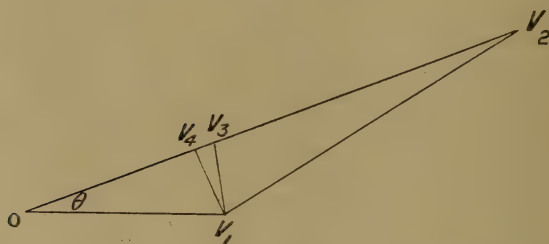
angle θ will be determined much less accurately from measurements of the sides of the triangle OV_1V_2, where OV_1 and OV_2 are very different in magnitude, than from equally accurate measurements of the sides of OV_1V_3, in which OV_1 and OV_3 are supposed to be nearly of equal length.   The measurement will be most accurate, for given percentage errors in the sides, if the length of the perpendicular V_1V_4 on the line OV_2 can be measured, since then the value of $\sin \theta$ will be known as accurately as the ratio of V_1V_4 to OV_1 can be determined.   If the sides of the figure represent voltages in different phases, the measurement of the ratio in question can in this case be determined as accurately as two voltmeter readings can be read and compared.

If we denote by v_1 and v_2 the lengths of the sides of the triangle forming the angle θ, and if the third side of the triangle opposite θ is represented by v, it can easily be shown that

$$\cos \theta = 1 - \tfrac{1}{2}\phi^2, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where
$$\phi^2 = \frac{v^2 - d^2}{v_1 v_2}, \text{ and } d = v_1 - v_2.$$

This is true whatever the value of θ, but for small values θ becomes identical with ϕ, and we have in circular measure

$$\theta = \sqrt{\frac{(v-d)(v+d)}{v_1 v_2}} = \frac{v}{\sqrt{v_1 v_2}} \sqrt{1 - \frac{d^2}{v^2}} \quad . \quad . \quad (2)$$

Some years ago the writer tested the phase-difference between the primary and secondary voltages of a small equal-ratio transformer by joining a low-reading hot-wire voltmeter to two terminals, one on each coil, and by connecting the other two terminals with a wire. Some numbers taken from an old note-book and referring to a test with the secondary on open circuit give $v_1 = 72$; $v_2 = 73\cdot8$; $v = 1\cdot9$. Here $d = 1\cdot8$, and the value of θ works out to be $0\cdot0084$ radian, or $0\cdot48$ degree, for which $\cos \theta$ is $0\cdot99996$. But recent tests on the same transformer by a more accurate method have proved that this estimate of θ is no less than 24 times too great, that its real value is only $0\cdot00037$ radian, or $0\cdot021$ degree, and that consequently the value of $\cos \theta$ only differs from unity by 7 parts in a hundred million.

The explanation of the error made in the earlier test is simply the difficulty of determining the difference d with sufficient accuracy when it is about the same magnitude as v and small compared with v_1 or v_2. If, assuming v_1 correct, v_2 had been read $73\cdot9$ instead of $73\cdot8$, the angle θ would have worked out to be zero, so that an error of only about one-tenth per cent. in reading one of the voltmeters completely accounts for the difference between the two results. This was not noticed at the time, and as the test was not repeated, the error was not discovered till long afterwards.

If in fig. 1 the voltage OV_2 is adjustable without alteration of phase, the voltmeter measuring the side opposite θ can be made to give a minimum reading by altering OV_2. The minimum reading will be $V_1 V_4$ perpendicular to OV_4, and we can determine $\sin \theta = v/v_1$ as accurately as we can read the voltmeters.

Thus the phase relation of the voltages of two coupled alternators can be determined by a two-voltmeter method in this way, by simply adjusting the excitation of one of them, this one being preferably run on open circuit. But in any case, by shunting the larger of the two voltages forming the angle θ with a non-inductive resistance, it is possible, by tapping this resistance at various points, to get a minimum reading of the voltmeter v, the ratio of which to the unshunted

voltage will give the value of $\sin \theta$. This method is particularly suited to determine small angles of phase-difference, since voltmeters having different ranges may be used, and $\sin \theta$ or θ can be measured as accurately as the two readings can be determined. The method is not always applicable, since the conductors absorbing the voltages to be compared may be in series. In such cases the following bridge method can be used. Suppose that in fig. 2 A B represents

Fig. 2.
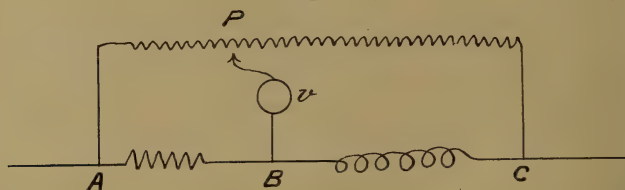
a non-inductive resistance, and B C an inductive resistance in series with it ; the phase-difference ϕ between the current flowing through B C, and the voltage across B C, can be determined by connecting the points A and C to the terminals of a suitable non-inductive resistance A P C, and finding upon it a point P such that when this point is electrically connected with B through a voltmeter the reading v of this instrument is a minimum. The vector figure representing the voltages is shown in fig. 3. In this figure B P

Fig. 3.
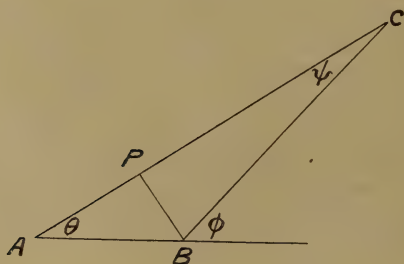
represents v and is perpendicular to A C. If v_1 and v_2 are the voltages A B and B C respectively, the phase-angles θ and ψ by which these voltages differ from the voltage A C are such that

$$\sin \theta = \frac{v}{v_1}, \quad \sin \psi = \frac{v}{v_2}, \quad \phi = \theta + \psi.$$

We also have $v_1 v_2 \sin \phi = v v_3$ where v_3 is the voltage A C.

The former equations illustrate a method of determining ϕ by observing v, v_1, and v_2, while the latter equation indicates a method of determining $\sin \phi$ from these measurements together with an observation of v_3. It is to be noted that the conductor A P C need not necessarily be noninductive. It should be of the same inductive character all through, and such that it is possible to adjust the point P connected with the voltmeter. The point P will then move along the vector A C (see fig. 3) as the adjustment is varied. Moreover there is no disadvantage in choosing a small noninductive resistance A B such that the voltage v_1 is small compared with v_2, provided always that suitable voltmeters are available for making the requisite measurements. One of the disadvantages of the three-voltmeter method for the measurement of power factor, is that the noninductive resistance A B should absorb about the same voltage as the conductor B C, and hence a greater amount of power than the load under test. If the bridge method just described is used, the load B C may absorb several hundred volts, while v_1 and v may each be less than a volt. Where A B is negligible compared with B C, it of course follows that ψ is small compared with θ, and that θ and ϕ can in general be considered the same. Now in order to determine one of the angles by the above method, all that is really needed is to have two voltages, one like A B in phase with the current, and the other in phase with either B C or A C. One of these voltages should be fixed in magnitude, and the other adjustable. Such voltages can be obtained with all requisite accuracy for most purposes by the aid of current and voltage transformers. The former must have its primary in the main circuit, and its secondary closed through a non-inductive resistance, the voltage on which will be in phase with A B. The latter must be a constant potential transformer with its primary across A C or B C, and its secondary closed through a noninductive resistance, the voltage on which will be in phase with that on the primary. The secondary voltage need not be greater than corresponds with the length A P in the figure. The two secondary circuits must be connected together at one point to determine a common potential represented by the point A in fig. 3. Such an arrangement makes the method adaptable to loads absorbing high potentials, or taking heavy currents. The phase-differences introduced by such transformers, if of good design, can in practice be reduced to about a tenth of a degree, and hence, for values of θ such as are usual, no appreciable error will be introduced into the measurement by them.

The writer has tested this two-voltmeter method in all the

various forms above indicated, and found it satisfactory. It is not so simple or so accurate as the wattmeter method when an instrument of the right range is available, and when the value of θ to be determined is large. But its great adaptability to loads of all kinds makes it a useful laboratory method. Under ordinary circumstances, a hot-wire voltmeter reading up to 1 volt can be used for taking the two measurements v and v_1, while the voltage v_2 on the load can be taken by the instrument necessarily used on the circuit. It is then only necessary to find two noninductive resistances, one such as to absorb about one volt when traversed by the load current, and the other suitable for the voltage v_2 or v_3, and having a portion of it (taking about a volt) of such a structure that a voltmeter can be connected to it at various points.

It is, however, when the phase-difference to be determined is small that the method is most useful and most accurate, while the wattmeter method altogether fails. The vector diagram, shown in fig. 3, applies whatever the frequency or wave form of the current, and even when the current is unidirectional but varying in strength. In a test in which a hand-regulated direct-current arc was put in series with a noninductive resistance, it was found, with $v_1 = 70$ volts and $v_2 = 40$ volts, that the minimum value of v was less than $0\cdot 1$ volt. A lower voltage could not be measured with any certainty with the hot-wire voltmeter used. It follows from this test, that the power factor of the direct-current arc cannot differ from unity by more than 5 parts in a million, and the phase-difference ϕ between the voltage and the current works out to be about $0\cdot 003$ radian. A similar test, made on the small equal-ratio transformer already referred to, with the primary subjected to over 80 volts, gave as minimum voltage v, a value estimated to be only $0\cdot 03$ volt. This would correspond with a phase-difference of only $0\cdot 00037$ radian, or $0\cdot 021$ degree, and a value of $\cos \phi$ differing from unity by only seven parts in a hundred million.

Voltmeters for measuring minute alternate voltages are not procurable. For small phase-angles v will be small compared with either v_1 or v_2. If the apparatus to be tested is such that the application of high voltages to it would either do it injury or alter its working conditions, the method becomes impracticable when the angles to be measured are small.

In order to overcome the difficulty arising from the absence of sensitive alternate-current voltmeters, the writer has adopted the drastic method of commutation. By such means it is possible to make use of sensitive direct-current instruments

for the measurement of alternating voltages. Suppose the alternator used for the tests is made to drive a synchronous motor constructed with at least four poles and having on its rotating spindle a commutator with as many metallic sectors as there are current reversals in the time of a revolution. Let half of the sectors (the first, third, &c.) be in metallic connexion with each other, and the other sectors be insulated. A pair of brushes suitably spaced on the commutator will then be in metallic connexion during half the period, and insulated from each other for the other half. But the instant at which the brushes are connected will not be the same as that at which the current reversal takes place, except for special positions of the brushes. If the brushes are adjusted to this position, and the alternating voltage to be measured has its terminals connected to the brushes with a direct-current voltmeter in circuit, this instrument will measure half the mean value of the ordinate of the positive portion of the alternating wave of potential, while the true value of the alternating voltage will be the square root of mean square of this ordinate. The sectors are supposed to be all equal to each other, and the width of the gap between successive sectors should be small compared with the circumferential width of the sectors, though for wave-forms approximately sinuous the gap width can be an appreciable fraction of the sector width without causing any appreciable effect on the readings. Theory shows that if the gap width is a small fraction ϵ of the sector width, the fractional error of the reading is represented by $\pi^2 \epsilon^2/8$ for sine currents. It follows that if ϵ is one per cent. the error is only 1/80 per cent., while ϵ has to be 9 per cent. before the error amounts to one per cent. The sensitiveness of the arrangement might have been doubled if the commutator had been constructed so as to reverse the voltage at every half period by connecting alternate sectors to two slip-rings. But further sensitiveness was not required, and hence only the simplest form of commutator was tried.

For each wave-form there is a definite value f for the ratio of the square root of mean square of the ordinates to the arithmetical mean. For sine waves this ratio is

$$f = \frac{\pi}{2\sqrt{2}} = 1 \cdot 11.$$

For such waves the reading of the direct-current voltmeter would have to be multiplied by $2 \cdot 22$ to get the reading of an alternate-current instrument arranged so as to measure the voltage directly. For other wave-forms the multiplier would be $2f$, where f may be called the form factor of the

alternating voltage. By using a commutator in this way, the most delicate Thomson reflecting-galvanometer can be utilized for measuring alternating voltages of the same magnitude as the smallest direct-current voltages to which such an instrument responds.

The method would in practice prove very tedious owing to the need for carefully adjusting the brushes before taking a measurement, since the adjustment required will depend on the phase of the voltage under test. But if the commutator be rotated nearly, but not quite, synchronously with the speed corresponding with the frequency, the galvanometer or voltmeter will show "beats," the interval between which lengthens as the speed of synchronism is approached. Under such conditions, the phase of the brush contact will automatically adjust itself whatever the phase of the voltage under test ; and the maximum reading of the direct-current instrument will measure half the arithmetical mean of the ordinates of the curve representing the alternating voltage.

Any method of driving in which the speed can be delicately adjusted can be used for running such a commutator. A shunt motor with adjustable resistance either in series or in the shunt circuit would of course do. But for alternating currents of usual frequencies, a particularly convenient piece of apparatus is available in the induction motor. The speed of the rotor of such a motor, when running light, differs from the synchronous speed by a minute amount called the "slip," the amount of which in many cases is less than one-tenth per cent. For currents of 50 cycles per second such a value of the slip would correspond with beats at intervals of 20 seconds, or with a 10 second interval between the maximum positive and the maximum negative readings of the direct-current instrument.

A commutator of very simple construction, as above described, when attached to the spindle of an induction motor, and used with a direct-current instrument, forms in practice a satisfactory means of measuring minute alternate-current voltages. The writer has found it very serviceable in connexion with the measurement of alternating magnetic fields. The reliability of the apparatus was thoroughly tested in various ways, as, for instance, by putting in parallel (1) a standard voltmeter suitable for either direct or alternate voltages, and (2) a direct-current voltmeter in series with the commutator. The ratio of the two readings in volts of the two instruments was found to be exactly 2 on direct-current circuits with the commutator running ; but for the alternating voltages produced by a small rotary transformer run from the direct-current side, the ratio was found

to be 2·28 when the instruments were shunted to a portion of a non-inductive resistance connected with two slip-rings of the rotary. These ratios were the same whatever instruments were used, and for different resistances put in series with the direct-current instrument, provided due allowance was made for these resistances when interpreting in volts the reading of the latter instrument. But although the ratio of the reading of the alternate-current voltmeter to the maximum reading of the direct-current instrument was always found to be 2·28 when only noninductive resistances were included in the circuit connecting two slip-rings of the rotary, this number was found to vary with the inductiveness of the load current taken from these slip-rings. Thus the parallel arrangement of instrument just described, when shunted to a suitable noninductive resistance traversed by the current passing through an iron-cored choking-coil consisting of one of the coils of an old hedgehog transformer, yielded a ratio of 2·36 for the readings of the two voltmeters when the current was 10·6 amperes, and the frequency 43 cycles per second. A similar experiment on a hand-regulated alternating-current arc for a current of 11 amperes of the same frequency, yielded a ratio of 2·55. In each case about a dozen different observations were taken, and the individual values of the ratio found in each set of tests agreed with each other as accurately as it was possible to read the instruments observed. Since the ratio between the two readings is $2f$ where f is the form factor of the voltage, or the ratio of the square root of mean square of the voltage to its arithmetical mean value, it follows that the form factor of the slip-ring voltage is 1·14, that of the current through the choking-coil is 1·18, and that of the alternating-current arc is 1·275, the theoretical value for sine-currents being 1·11. The wave-forms of the alternating current through an arc, or through a large choking-coil with a strongly magnetized iron core, are quite exceptional. For ordinary currents produced by the rotary, it was found sufficiently accurate to take a constant of 2·3 for direct-current voltmeters used with the commutator.

It now became possible to measure the voltages represented by the lines in figs. 1 and 3, even when some of these voltages were small compared with a millivolt. Moreover, as the angles to be measured are determined from the ratios of voltages, and not from their actual values, it was possible, by measuring each voltage by the commutator method, to eliminate the constant of the instrument, assuming only that the form factor is the same for all the voltages measured.

Thus the inductiveness of a pair of lead plates in slightly acidulated water was tested by the method shown in fig. 3.

A current of 8·2 amperes was passed through a noninductive resistance A B in series with the lead plates B C. The voltage B C on the plates was 18·7 ; that of A B was $0 \cdot 6 \kappa$, while BP the minimum voltage v was $0 \cdot 004 \kappa$, where κ is the constant previously referred to. It follows that $\sin \theta$ is 0·0067 and $\cos \theta$ is 0·999977. A number of similar measurements were made with very satisfactory results.

A small current-transformer for oscillograph purposes was tested for phase-difference between primary and secondary currents. This transformer was designed to reduce currents in the ratio of 20 to 1, and for use on a 6000 volt circuit, for a primary current not exceeding 10 amperes, and to supply a secondary current to any closed circuit not absorbing more than 2 volts. Noninductive resistances were put in series with the primary and secondary windings, and were electrically connected so as to form a point represented by O in fig. 1. The former for a current of 8 amperes absorbed 0·66 volt, represented by OV_1. The secondary resistance could be tapped at various points, and the minimum voltage v represented by $V_1 V_4$ produced a reading of 0·50 millivolt. Allowing for the constant 2·3, we have ·00115 volt as the value of v. It follows that $\sin \theta$ is ·00175, and θ is 0·10 degree for a primary current of 8 amperes. For a primary current of 10 amperes θ was found to be 0·088 degree, and for a current of 7 amperes 0·114 degree. Such values of θ yield values of $\cos \theta$ differing from unity by about one part in a million, and are such that the wave-forms of primary and secondary currents must correspond with almost absolute accuracy.

Several small instrument transformers as supplied commercially for wattmeters were tested under their normal working conditions, the noninductive resistance put in the secondaries being very small compared with the total secondary resistance, and only absorbing a few millivolts. In one case it was found that the phase-difference between the primary and secondary currents of a current-transformer was 2·32 degrees, and in another as much as 4·2 degrees.

A number of larger transformers for power purposes were tested for phase-differences between primary and secondary voltages. In several cases the phase-difference on open circuit was found to be of the order of a tenth of a degree, and on full load of the order of one degree. On an inductive load the phase-difference found was not so large as a rule as for the same secondary current through a noninductive circuit. The curve shown in fig. 4 represents the result of some tests on two transformers of 3 kilowatt capacity intended to work between voltages of 100 and 1000, and with currents of 100 cycles per second. One transformer was

Fig. 4.
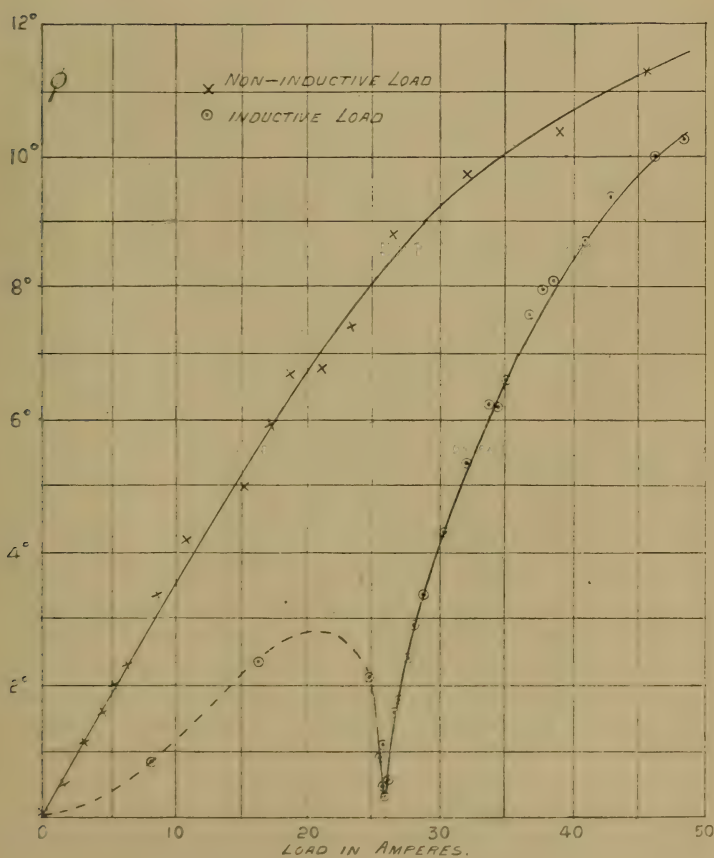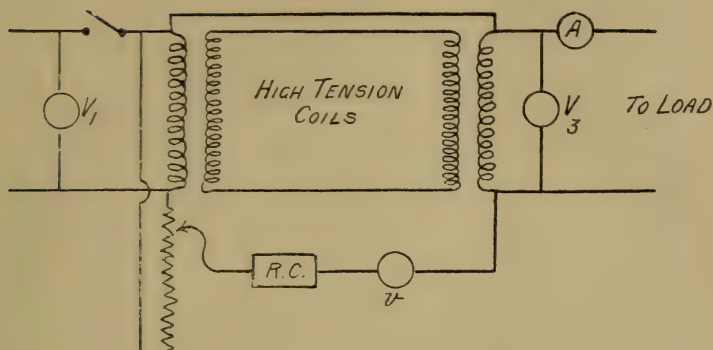
used to step up the volts from V_1 to V_2, the other to step down from V_2 to V_3. The arrangement of the circuits is shown in the diagram at the top of fig. 4. A noninductive resistance was put across V_1, and the phase-difference between V_1 and V_3 was tested by the two-voltmeter method already described. The phase-difference found is for the double transformation and is approximately twice that for each transformer. The load for the first curve was composed of lamps, and is noninductive ; that for the second curve consisted of a hedgehog transformer, the secondary of which was, for the first test, open-circuited, but afterwards closed through a number of lamps in parallel. The primary took a current of nearly 25 amperes at a power factor of 0·09 when the secondary was open-circuited ; but as the lamp-load on the latter was increased the primary current and power factor each rose. The voltages V_1 and V_3 used, instead of being about 100 at 100 cycles, for which the transformers were designed, were about 50 volts at about 50 cycles, so that the magnetic fluxes in the cores were about the same as if the normal voltages and frequencies had been used. The full-load current is 30 amperes for each of the low-voltage coils. The phase-difference between V_1 and V_3 for the two transformers with the second on open circuit, is only 0·109 degree or 0·054 degree per transformer. For a noninductive load it increases regularly with the current. For the full-load current of 30 amperes it is 9·4 degrees, and for 50 amperes it is 11·5 degrees. For the inductive load there was a remarkable, and sharply defined, minimum of 0·36 degree for a current of 25·5 amperes at low-power factor. This minimum was carefully tested. It occurred for a small load on the secondary which could be varied very gradually, the corresponding change in the primary current being only just measurable. For larger loads on the secondary, and therefore for power factors approaching unity as the current increased, the phase-difference curve is seen to approach that corresponding with a noninductive load. The portion of the curve for small inductive currents was not tested owing to the absence of a suitable load, but two special tests were made for lower currents at a power factor of about 0·1, as shown on the dotted portion of the curve, which of course must have the same starting-point for zero current as the curve for noninductive load. In all probability the dotted part of the curve denotes a "leading" condition of current, and the rest of it a "lagging" state of current.

In conclusion the writer desires to express his thanks to Mr. David Owen for much valuable assistance in carrying out several of the measurements above referred to.

XV. *On Clausius' Theorem for Irreversible Cycles, and on the Increase of Entropy* [*].

To the Editors of the Philosophical Magazine.

GENTLEMEN,   Berlin, Nov. 2, 1904.

I BEG leave to make a few observations on Prof. W. McF. Orr's paper[†] of the above title, in which he criticises among other things my treatment of Thermodynamics [‡], for otherwise I fear that it may perhaps give rise to one or two misconceptions.

1. I will not quarrel with Prof. Orr as to whether he is right in saying (p. 509) that I use the words "reversible" and "irreversible" in an unusual sense, since these words are seldom expressly defined. Yet, I really must show by an example that the form of the definition I use is practical.

Clausius, as is well known, founded his proof of the Second Law of Thermodynamics on the simple proposition that *heat cannot of itself pass from a colder to a hotter body.* Here it is not only stated, as Clausius repeatedly and expressly pointed out, that heat does not pass directly from a colder to a hotter body, but that heat can *in no way whatsoever* be conveyed from a colder to a hotter body without leaving behind some lasting change (*i. e.* without compensation).

If I now say, *the process of heat conduction is irreversible,* this proposition, according to my definition of irreversibility, means exactly the same as Clausius' fundamental proposition. Whether this proposition is in reality true cannot be directly settled and requires a special investigation ; but if once we assume its truth, then the whole import of the Second Law of Thermodynamics can be deduced from it. If, on the other hand, we understand by the irreversibility of a process, only that it cannot be directly reversed, then the proposition that *the passage of heat from a higher to a lower temperature is irreversible* is, to be sure, self-evident, but it is of no value for the derivation of the Second Law of Thermodynamics, for we are not in a position to draw therefrom any conclusions regarding other processes.

2. Prof. Orr says (p. 511) " Planck gives one Definition of Reversibility, but uses another." As I searched for a proof of this assertion, the only thing I could find was the following statement :—" Under ordinary circumstances, however, no body can expand without producing a change of

density in some other body." The truth of this statement, and the conclusions therefrom, I must directly contest. There is nothing to hinder us from supposing that the gas-holder together with the weights, which are acting against the gas pressure, are placed in a vacuum. If, then, the gas pressure lifts the weights, the gas expands without producing a change of density in some other body. That it is impossible to obtain an absolute vacuum, that absolutely unchangeable weights do not in reality exist, and that there may be other difficulties opposing the realization of this process, does not of course affect the validity of the proof. I am sorry to see from this misconception, that the note, which Dr. Ogg on my advice added to that particular part of his translation of my Thermodynamics, has not been sufficient to make the point clear.

3. If a process, *e. g.* the flow of a gas into a vacuum, takes place so violently that one can no longer define temperature and density, then the usual definition of Entropy is inapplicable. Of course, on this point I completely agree with Orr and Bertrand. My observations with regard to the entropy of a gas (p. 512) which is not in a state of equilibrium, refer only to the case where we may speak of a temperature and a density of the gas in all its parts. But, as is well known, Herr L. Boltzmann has shown from the point of view of the Kinetic Theory of Gases, that, even in the case of violent motion, a definition of Entropy, which includes the usual one as a special case, may be deduced from the Theory of Probabilities. The suppositions of the Kinetic Theory of Gases, however, were purposely excluded from my Treatise on Thermodynamics.

4. Prof. Orr gives (p. 518) " A shorter proof of the principle of Increase of Entropy, if the substance of Planck's definition of Irreversibility be adopted." Closer examination shows that this proof starts with the assumption expressed in Lord Kelvin's version of the Second Law, and that, therefore, the essential import of the thing to be proved is already assumed to be true. I cannot, therefore, accept this proof as an improvement on my own.

With these remarks I shall conclude. I should not, however, have again given my views on these questions at such length, had it not been that I wish to express my pleasure at having found in Prof. Orr's paper much that was stimulating and interesting to me.

I am, Yours faithfully,
Dr. MAX PLANCK.

XVI. *Note on the Tangential Stress due to Light incident obliquely on an Absorbing Surface.* By J. H. POYNTING, *Sc.D., F.R.S., Mason Professor of Physics, Birmingham* *.

THE existence of pressure on a surface due to the incidence of a normal beam of light, first deduced as a consequence of the electromagnetic theory by Maxwell, has been fully confirmed by the experiments of Lebedew, and quite independently by the exact work of Nichols and Hull. These experiments show that the pressure exists and that it is equal to the energy per c.c. or to the energy-density in the incident beam.

In so far as it produces this pressure we may regard the beam as a stream of momentum, the direction of the momentum being along the line of propagation, and the amount of momentum passing per second through unit area cross-section of the beam being equal to the density of the energy in it. Let E denote this energy-density. If the beam is inclined at θ to the normal to a surface on which it falls, the momentum stream on to unit area of the surface is $E \cos \theta$ per second, and this is the force which the beam will exert in its own direction. If the beam is entirely absorbed, the result is a pressure $E \cos^2 \theta$ along the normal and a tangential stress in the plane of incidence $E \sin \theta \cos \theta = \frac{1}{2} E \sin 2\theta$. If μ of the incident beam is reflected, the normal pressure is $(1 + \mu) E \cos^2 \theta$, and the tangential stress is $\frac{1-\mu}{2} E \sin 2\theta$ †.

When there is absorption the tangential stress has a maximum value at $45°$ if μ is constant. When there is no absorption the tangential stress disappears.

The tangential stress is much more easily detected than the normal pressure. For the action of the gas surrounding the surface is normal to it and is with difficulty disentangled from the normal light-pressure. But the gas action is at right angles to the tangential stress, and it is merely necessary to arrange a surface free to move in its own plane to eliminate the action of the normal forces and to reveal the tangential stress.

With the assistance of my colleague Dr. Guy Barlow, to whom I am much indebted for help in the work, I have made the following experiment to show the existence of the stress.

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* Read at Section A, British Association, Cambridge, August 1904. Communicated by the Author.

† These expressions are given in "Radiation in the Solar System," Phil. Trans. A 202. p. 539.

Two circular glass disks, each 2·75 sq. cm. area, were fixed at the ends of a horizontal light glass rod 5 cm. long, the disks being perpendicular to the rod and fixed to it at their highest points. One of the disks was lampblacked, and the other silvered. The rod was placed in a light wire cradle and suspended by a fine quartz fibre about 25 cm. long in a brass case with glazed sides. On the cradle was a mirror by which deflexions could be observed with a telescope on a millimetre-scale 1·8 metres distant. The moment of inertia of the system was 2·35 gm. cm.^{2} and the time of vibration was 146 seconds. A deflexion of 1 scale-division therefore corresponded to a tangential force on a disk of about one two-millionth of a dyne—more exactly ·483 × 10^{-6} dyne.

The air was pumped from the case till the pressure was less than 1 cm. of mercury. At this pressure the irregularity of the disturbances due to the residual gas is very greatly reduced. A parallel beam of light from a Nernst lamp was then directed so as to be incident obliquely on the lamp-blacked disk. From the arrangement of the disks it is obvious that a uniformly distributed normal force would have no moment tending to twist the system, while a tangential force would have a moment and would twist it. In all cases the disk moved away from the source of light. The deflexion was a maximum when the incidence was not very far from 45°, and fell off on each side of the maximum value.

As there are various sources of error not yet removed, we have not made a complete series of measurements but have only made sure that the effect is of the order to be expected from the theory, by finding the deflexion for an angle of 45°.

The beam from the Nernst lamp when incident at 45° turned the rod through 16·5 scale-divisions. Assuming total absorption, the tangential force should be $\frac{1}{2}$E sin 2θ × area of disk = $\frac{1}{2}$E × 2·75.

Equating to the value of the force given by the deflexion, viz., 0·483 × 10^{-6} × 16·5, we have E = 5·8 × 10^{-6}.

The same beam was then directed on to a small lamp-blacked silver disk of known heat capacity, through a glass plate of thickness equal to that of the side of the case. The initial rise of temperature per second was measured by a thermojunction of constantan wire soldered to the disk. The energy-density of the stream was thus found to be E = 6·5 × 10^{-6}.

The agreement of the two values is quite as close as could be expected in so rough a determination.

When the beam was directed on to the silver disk at the

other end of the torsion-rod, the deflexion was much less, as was to be expected.

We have also made some qualitative experiments with a blackened glass cylinder—a ring cut from a test-tube—suspended by a quartz fibre with its axis vertical. When a beam fell on this in any direction not along a diameter, there was always a twist in the direction corresponding to the tangential stress.

XVII. *The Union of Hydrogen and Oxygen at Low Pressures through the Passage of Electricity.* By Rev. P. J. KIRKBY, *Fellow of New College, Oxford* [*].

IN the Phil. Mag. for Feb. 1904 I described some experiments which exhibited the effect of passing electricity with a luminous discharge through hydrogen and oxygen, mixed in equivalent proportions and at low pressures. The experiments showed that a partial union of the gases into water-vapour is the result; and that if the pressure and the potential-difference of the plates during the discharge remain steady, the quantity of water-vapour formed is proportional to the quantity of electricity which has passed—in other words, to the product of the current and the time. They showed, moreover, that if Δp mm. is the fall of pressure associated with the passage of ΔQ coulombs at the pressure p mm.; and if X is the voltage-difference of the plates during the discharge, then, within the limits of experimental errors,

$$\frac{1}{p} \frac{\Delta p}{\Delta Q} \text{ is a function of } \frac{X}{p}.$$

The experiments described in the following paper were undertaken with the view of deciding, first, how changes in the distance between the parallel electrodes, between which the discharge took place, would affect these results; and, secondly, whether the same effects would be observed if non-oxidizable plates were used.

While the general arrangement of the apparatus was substantially unchanged, there were several alterations in detail. A more satisfactory method of preparing hydrogen and oxygen was adopted. The gases were prepared simultaneously by the electrolysis of pure barium hydrate contained in a small sealed glass vessel. By this method the gases are generated free from hydrocarbons and, according to Mr. H. B. Baker, free also from ozone and peroxide of hydrogen (Trans. Chem. Soc. xxxviii. 1902). The mixed gases were

dried in a vessel containing pentoxide of phosphorus, and were then admitted to a glass bell-jar which contained the parallel plates between which the current was to be passed.

The bell-jar was connected by glass tubing to an ordinary manometer and to two McLeod gauges which combined to measure pressures below 40 mm. It was also connected to a large mercury-pump. But the tube leading to the latter could be closed by a stopcock ; for it was obviously important to confine the hydrogen and oxygen within as small a space as possible, with a view to determine the more accurately the falls of pressure resulting from their partial union.

The arrangement within the bell-jar is shown in fig. 1, which represents a section of it by a plane through its axis.

Fig. 1.
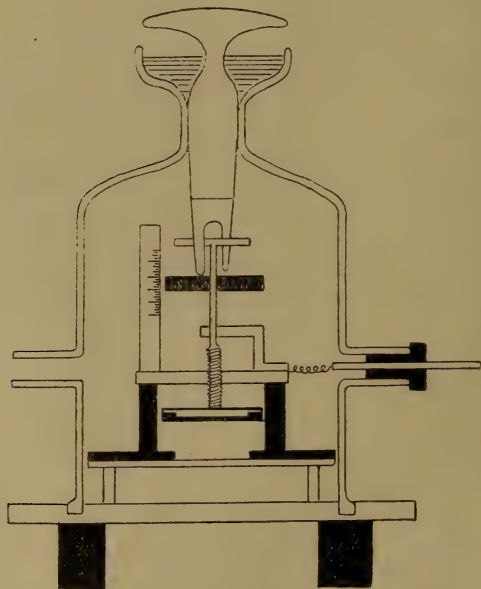
The dark parts represent ebonite pieces : the light, metallic pieces, except the stopper which was glass. The figure shows how by turning the stopper the screw, whose pitch was 1 mm., could be turned and the upper plate raised or lowered thereby, and how the distance between the plates could be most accurately determined by the vertical scale and by the graduated drum of ebonite whose rim almost touched the scale. The possibility of a leak between the stopper and the neck of the jar was obviated by pouring mercury into the cup above the neck, as shown in the figure.

It will be seen from the figure that the lower electrode was in metallic connexion with the brass base of the apparatus upon which the bell-jar rested ; and that the upper electrode was in metallic connexion with the brass rod which passed through the ebonite plug fixed in one of the side tubes of the bell-jar. Thus the electrodes could be easily connected by wires to the terminals of the battery which supplied the current.

The bell-jar was fastened down to the brass base by means of elastic glue, and the necessary joints in the bell-jar, where it was impossible to fuse the glass, were made in the same way. Every other joint in the glass tubing along which the gases had to pass was made by fusing together the glass. All the stopcocks and drying vessels had mercury cups which prevented air from passing in. By these means the apparatus was made satisfactorily air-tight. In fact, joints which are carefully made with elastic glue, even between large perimeters in contact, prove often to be surprisingly airtight, and last for months.

Phosphorus pentoxide was placed in vessels (not shown in the figure) within the bell-jar, and also in a large tube connecting the bell-jar to the McLeod gauges. This absorbed the water-vapour as soon as it was formed.

The bell-jar was about 10 cms. in diameter, and its height up to its shoulders was about 12 cms. The lower plate was protected by an ebonite guard-ring throughout these experiments, and when silver electrodes were substituted the upper plate was similarly surrounded.

To prepare for a series of experiments, the bell-jar was alternately exhausted by the mercury-pump and replenished by small quantities of freshly prepared and dried hydrogen and oxygen. In this way the proportion of the old to the newly introduced gas was reduced to a very small percentage. The stopcock cutting off the bell-jar, the manometer, and the McLeod gauges from the rest of the apparatus was then closed, the total volume thus cut off being nearly 1130 c.c.

The method of these experiments was throughout the same as the method described in my former paper. The parallel plates AB, CD of the apparatus were connected to the poles of a battery of small lead cells, ranging up to 1080 volts, through a high resistance R, and through a sensitive ammeter V (fig. 2, p. 174).

Thus with a suitable voltage and resistance a current attended by a glow was passed for an observed time through the mixed gases and was measured by the ammeter. It was, in the case of all the observations recorded below, steady

enough to reduce the errors of observation to within a very few per cent. In this way the quantity of electricity passed

Fig. 2.
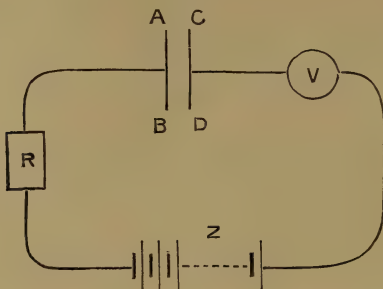
through was known, while the pressures before and after its passage were determined by one of the McLeod gauges. If these are denoted by p_0, p_1, the mean pressure p during the experiment is given by

$$p = \tfrac{1}{2}(p_0 + p_1),$$

while the fall of pressure Δp is given by

$$\Delta p = p_0 - p_1.$$

Sufficient electricity was, as a rule, passed through the gas to make Δp large enough not to involve errors of more than 2 or 3 per cent. At pressures lower than those recorded below this was not possible without sacrificing the accurate determination of the quantity in coulombs; for the current ceases at these low pressures to be steady for more than a very short time.

In some of the earlier experiments a voltmeter of resistance 510 ohms, occasionally halved by a shunt of 510 ohms, was used as the ammeter. For the rest a high-resistance voltmeter of 38,000 ohms was employed. The current was, therefore, considerably varied.

If X is the potential-difference of the parallel plates during the passage of the steady current C, Z the voltage of the battery, and R the whole ohmic resistance of the circuit distinct from the gas, then obviously

$$X = Z - CR,$$

the battery having no appreciable internal resistance.

The first set of experiments were made with two parallel zinc electrodes AB, CD, the circle of metal surrounded by the guard-ring being 3 cm. in diameter. These are recorded in Tables I., II., III., IV., in which, to recapitulate,

D = the distance in cms. between the parallel plates ;

p = the mean pressure in mms. of mercury during the experiment ;

Δp = the fall of pressure in mms. ;

C = the steady current in milliamperes ;

ΔQ = the quantity of electricity in coulombs passed through the hydrogen and oxygen ;

X = the difference of potential in volts between the parallel plates during the passage of the current.

The capacity of the part of the apparatus where the fall of pressure took place was 1130 c.c. approximately.

TABLE I.   D = ·25 cm.

p.	Δp.	C.	ΔQ.	$\dfrac{\Delta p}{\Delta Q}$.	X.	$\dfrac{X}{p \cdot D}$	$\dfrac{\Delta p}{p \cdot \Delta Q \cdot D}$
9·3	2·75	3·8	1·6	1·73	392	169	·74
9·0	2·26	8·6	1·29	1·75	397	176	·78
6·9	2·2	3·8	1·37	1·61	392	229	·94
3·27	·51	2·5	·303	1·68	386	472	2·06
2·76	·51	4·95	·302	1·69	397	576	2·45

TABLE II.   D = ·5 cm.

p.	Δp.	C.	ΔQ.	$\dfrac{\Delta p}{\Delta Q}$.	X.	$\dfrac{X}{p \cdot D}$	$\dfrac{\Delta p}{p \cdot \Delta Q \cdot D}$
8·8	1·62	3·5	·85	1·91	405	93	·14
7·1	1·71	3·6	·86	2·0	402	113	·56
5·6	1·29	3·6	·645	2·0	402	144	·72
3·8	·55	2·5	·31	1·8	386	202	·94
3·37	·424	8·2	·247	1·71	416	246	1·02
3·0	·67	8·1	·4	1·67	406	276	1·12
2·67	·27	3·9	·157	1·72	383	286	1·28
2·16	·72	16·4	·43	1·69	408	378	1·56

TABLE III.   D = 1 cm.

p.	Δp.	C.	ΔQ.	$\dfrac{\Delta p}{\Delta Q}$.	X.	$\dfrac{X}{p \cdot D}$	$\dfrac{\Delta p}{p \cdot \Delta Q \cdot D}$
4·54	·92	4·7	·42	2·2	415	91	·48
4·48	·525	8·0	·24	2·17	435	97	·485
3·56	·59	5·0	·3	1·97	412	116	·55
2·85	·62	8·1	·326	1·9	425	149	·67
2·32	·38	3·9	·232	1·64	393	169	·71
2·23	·56	8·1	·325	1·73	403	181	·78
1·67	·25	15·3	·14	1·77	426	256	1·06
1·18	·178	7·6	·11	1·62	426	362	1·37

TABLE IV.   D = 1·5 cm.

p.	Δp.	C.	ΔQ.	$\dfrac{\Delta p}{\Delta Q}$.	X.	$\dfrac{X}{p \cdot D}$	$\dfrac{\Delta p}{p \cdot \Delta Q \cdot D}$
3·92	·565	8·0	·24	2·36	435	75	·4
3·45	·484	7·3	·218	2·22	430	83	·426
3·03	·432	5·0	·2	2·16	413	90	·47
2·11	·87	8·1	·485	1·8	429	135	·57
1·44	·345	8·1	·19	1·8	415	192	·83
1·43	·231	13·9	·134	1·72	448	209	·8

Before proceeding to discuss the tables in detail, I may point out one remarkable result which they establish, viz., the substantial constancy of $\Delta p / \Delta Q$

Now it is easy to show that $4 \dfrac{\Delta p}{\Delta Q}$ is very nearly the number of molecules of water-vapour which result from the formation and passage to the electrodes of each pair of ions. For if N is the number of molecules in a cubic centimetre of a gas at 15° C. and 760 mm., then in an apparatus of S c.c. capacity, a fall of pressure Δp results from the disappearance of

$$\frac{\mathrm{N} \times \mathrm{S} \times \Delta p}{760} \text{ molecules.}$$

Again, since one coulomb contains 3×10^9 electrostatic units, ΔQ coulombs is carried to an electrode by

$$\Delta Q \times \frac{3 \times 10^9}{e}$$

ions, if e is the electrostatic charge on each.

Therefore, as each pair of ions passes to the electrodes,

$$\frac{\mathrm{N}e \times \mathrm{S} \times \Delta p}{760 \times 3 \times 10^9 \times \Delta Q} \text{ molecules of the gases disappear.}$$

$\mathrm{N}e = 1\cdot 22 \times 10^{10}$ (Townsend, Phil. Trans. 1899),

and

$\mathrm{S} = 1130$ c.c., nearly.

Therefore the fraction last written is almost precisely $6 \dfrac{\Delta p}{\Delta Q}$. Thus the formation of $4 \dfrac{\Delta p}{\Delta Q}$ molecules of water-vapour is associated with the passage to the electrodes of each pair of ions.

Now the tables show that, in spite of great variations in the pressure and in the distance between the electrodes, the value of $\dfrac{\Delta p}{\Delta Q}$ is nevertheless always about 2. It varies between $1\cdot6$ and $2\cdot4$; but it is more nearly constant than these limits indicate.

It follows then, that with parallel plane electrodes, whatever their distance apart (at least, so long as it falls within the range defined by the tables), and for all such small pressures as appear in the tables, from 6 to 9 times as many molecules of water-vapour are formed as pairs of ions. This result is independent of the capacity of the apparatus, and, as will be seen presently, of the nature of the electrodes.

To return to the tables, if the values of $\dfrac{X}{p}$, $\dfrac{\Delta p}{p \Delta Q}$ are taken as coordinates, each table determines a series of points lying close to a curve which in each case is nearly rectilinear.

Further, if the values of $\dfrac{X}{p D}$, $\dfrac{\Delta p}{p \Delta Q D}$, given in the last two columns of the tables, are taken as abscissas and ordinates respectively, the observations can be represented by points lying near the same straight line. (See fig. 3.)

This shows that, within the limits of experimental error, $\dfrac{\Delta p}{p \Delta Q D}$ is a function, linear or very nearly linear, of $\dfrac{X}{p D}$.

In order to ascertain what part the electrodes played in these effects, the zinc plates hitherto used were removed from the bell-jar, and were replaced by commercially pure silver plates. Pure silver is non-oxidizable; and therefore if any appreciable fraction of the combination of the gases was due to the substance of the electrodes or to oxidation at one or both of them, different results would be obtained when the experiments were repeated with silver plates.

These were specially made by Messrs. Johnson & Matthey. The proportion of pure silver in their composition was 998 : 999. The upper plate was 3 cm. in diameter, and was surrounded by an ebonite guard-ring flush with the metal. The lower one had a diameter of 8·2 cm.; but it too was supplied with an ebonite guard-ring fastened on it, and leaving a circular aperture of 3 cm. diameter. Thus the discharge was confined to pass between two circular areas, each of 3 cm. diameter, and unsymmetrical discharges were prevented.

Tables V., VI., VII. (p. 178) give the results of the experiments made in the same manner as before, the silver plates being at the respective distances ·4, ·8, 1 cm.

As in the previous tables, so also in these, the values of $\dfrac{\Delta p}{\Delta Q}$ lie within a very restricted range of variation. The limits may here be taken to be 1·6 and 2. The substantial constancy of $4\dfrac{\Delta p}{\Delta Q}$ is thus, as already said, independent of the electrodes.

As before, if $\dfrac{X}{p D}$, $\dfrac{\Delta p}{p.\Delta Q.D}$ are taken as coordinates, each of these experiments contributes a point near to the same line which was determined by the observations made with the zinc electrodes.

TABLE V.   D = ·4 cm.

p.	Δp.	C.	ΔQ.	$\dfrac{\Delta p}{\Delta Q}$.	X.	$\dfrac{X}{p \cdot D}$	$\dfrac{\Delta p}{p \cdot \Delta Q \cdot D}$.
7·0	2·9	3·26	1·63	1·78	410	148	·64
4·65	1·7	3·45	1·04	1·64	396	212	·88
3·5	·58	3·6	·36	1·6	384	275	1·15
3·0	·53	3·3	·33	1·61	406	340	1·35
2·9	·58	3·54	·354	1·63	389	332	1·4
2·45	·52	3·2	·32	1·61	414	422	1·65
2·36	·56	3·53	·353	1·59	390	412	1·68
1·95	·484	2·9	·29	1·67	438	560	2·15
1·65	·126	2·55	·077	1·64	464	702	2·5

TABLE VI.   D = ·8 cm.

7·6	2·5	4·1	1·22	2·05	423	70	·34
5·3	1·86	3·1	·94	1·98	412	97	·47
4·9	2·8	4·2	1·47	1·9	408	104	·49
3·32	·37	3·5	·21	1·78	392	148	·68
3·3	·38	4·5	·224	1·71	378	144	·65
2·74	·75	4·6	·456	1·63	372	170	·74
2·7	·87	3·5	·522	1·67	392	181	·77
2·08	·356	3·6	·215	1·66	384	230	1·0
1·73	·356	3·5	·211	1·69	388	280	1·22
1·39	·314	3·4	·204	1·54	398	357	1·29
1·37	·175	3·1	·092	1·91	424	387	1·74
1·17	·206	2·9	·115	1·79	438	468	1·91
1·11	·251	3·2	·193	1·83	412	464	2·06
·98	·235	2·7	·132	1·78	452	578	2·27

TABLE VII.   D = 1 cm.

4·22	·91	3·2	·48	1·9	414	98	·45
3·46	·59	3·2	·32	1·86	416	120	·56
3·0	·35	3·24	·194	1·79	412	137	·6
2·5	·61	3·4	·34	1·81	402	159	·72

Fig. 3 contains the points corresponding to the observations recorded in all the tables. The number attached to each point indicates the table which supplied it.

I repeated a few of the observations with hydrogen and oxygen prepared by the electrolysis of a solution of caustic potash, to decide whether the gases generated in this way, associated as they are with various impurities, would yet follow the same laws of partial union. The results are given in Table VIII. They show no perceptible departure from

Fig. 3.

the results of similar experiments made with the hydrogen and oxygen obtained by the electrolysis of barium hydrate.

TABLE VIII.   D=1 cm.

p.	Δp.	C.	ΔQ.	$\dfrac{\Delta p}{\Delta Q}$.	X.	$\dfrac{X}{p\mathrm{D}}$.	$\dfrac{\Delta p}{p \cdot \Delta Q \cdot \mathrm{D}}$.
4·0	·6	3·13	·313	1·92	420	106	·48
3·4	·58	3·15	·315	1·84	418	124	·55
2·8	·58	3·3	·33	1·75	408	145	·62
2·22	·6	3·36	·336	1·77	402	181	·8
1·63	·59	3·2	·32	1·85	414	254	1·13
1·2	·26	3·0	·151	1·73	428	356	1·45

It follows that the equation

$$\frac{\Delta p}{p\,\mathrm{D}\Delta Q} = f\left(\frac{X}{p\mathrm{D}}\right)$$

is approximately true for the range of pressures dealt with, and that the functional relation is independent of the electrodes.

Moreover, the curve determined by all these observations does not differ sensibly from the right line

$$\frac{\Delta p}{p\mathrm{D}\Delta Q} = \frac{1}{10} + \frac{3X}{800\,p\mathrm{D}}.$$

Thus the results of all the experiments recorded in the tables can be approximately represented by

$$\frac{\Delta p}{\Delta Q} = \frac{p\mathrm{D}}{10} + \frac{3X}{800}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

With regard to X the potential-difference of the plates, it never seems to undergo any great variation. The limits of X for all the experiments recorded above lie between 370 and 500 volts. The variation of X in the tables scarcely exceeds 15 per cent. of its mean. It might, therefore, be doubted whether it is necessary to introduce X at all, in order to coordinate these experiments. The tables show that, in the great majority of cases, X differs by only about 5 per cent. from 400 volts. It might, therefore, be replaced by that constant value without dislocating, to any great extent, more than a very few of the points which have been plotted in fig. 3.

Nevertheless, the necessity of introducing X was shown

quite clearly by the experiments described in my first paper
on this subject. And though the variation in X is far less
prominent in the tables of this paper, the presence of X in
the abscissa, X/p, brings the present observations into satis-
factory agreement, and reduces their discrepancies within the
limits of accountable errors.

The errors of observation may be considered, in most cases,
to fall within about 5 per cent. It should be mentioned that
the experiments in Tables I.–IV. were made with a normal
laboratory temperature. The observations recorded in the
last four tables were taken last midsummer, the temperature
of the laboratory being from 22° to 26° C. As no correction
has been made for temperature, an error of 3 per cent. might
thus arise among the abscissas X/pD.

The errors involved in ΔQ should not, as a rule, exceed
2 per cent. The errors involved in Δp are about the same.
The latter are due partly to errors associated with a McLeod
gauge ; partly to the incomplete absorption of the water-
vapour by the phosphorus pentoxide. It was verified, how-
ever, that this absorption was practically complete and im-
mediate, so that Δp could be determined by observing the
fall of pressure as soon as the current was stopped.

But though the errors in the values of Δp are in most
cases small, yet they increase as the pressure is reduced.
For the current when the pressure is about 1 mm. tends to
run down considerably. And so, if X is to remain sensibly
constant during the experiment, the current can only be
permitted to run for a short time. Thus ΔQ, and therefore
Δp, is very small for the lowest pressures.

Now the few points in fig. 3 which fall abnormally wide
of the mean locus, represent experiments made at a mean
pressure of less than 1·5 mm. Their failure to approach the
mean locus as closely as the rest must, therefore, be attributed
to errors of observation. It does not naturally or necessarily
suggest that for these low pressures the line of fig. 3 is less
accurately than for the higher pressures the true locus of
points whose ordinates are $\dfrac{X}{p\mathrm{D}}$, $\dfrac{\Delta p}{p\mathrm{D}\Delta\mathrm{Q}}$. This view is con-
firmed by the closeness with which other points, corresponding
to the lowest pressures in the tables, approach the line.

The equation

$$\frac{\Delta p}{\Delta \mathrm{Q}} = \frac{p\mathrm{D}}{10} + \frac{3\mathrm{X}}{800}$$

may, then, presumably be considered as nearly true for the
lowest pressures of the tables as it is for the rest.

From this equation it is easy to ascertain what errors arise when the fall of pressure is considerable.

For if $-\dfrac{dp}{dQ}$ is written instead of $\dfrac{\Delta p}{\Delta Q}$, it integrates at once and gives

$$\frac{QD}{10} = \log \frac{p_0 + \dfrac{3X}{80D}}{p_1 + \dfrac{3X}{80D}},$$

where, X being constant, the passage of Q coulombs of electricity through the hydrogen and oxygen lowers the pressure from p_0 to p_1—all the water-vapour being absorbed.

If we put $p + \dfrac{\Delta p}{2}$ for p_0 and $p - \dfrac{\Delta p}{2}$ for p_1 we get

$$\frac{QD}{10} = \frac{\Delta p}{p + \dfrac{3X}{80D}} \left\{ 1 + \frac{1}{12} \frac{\Delta p^2}{\left(p + \dfrac{3X}{80}\right)^2} \right\}$$

nearly. Now the second term inside the bracket is negligible compared with unity for practically all values of Δp, X being always about 400 or more. Therefore equation (1) holds good for all values of Δp, ΔQ, so long as X is constant. It may in fact be written

$$\frac{p_0 - p_1}{Q} = \frac{(p_0 + p_1)D}{20} + \frac{3X}{800}.$$

Again, it was observed during the discharge that the glow between the plates at the higher pressures consisted of a small luminous cylinder of very slight diameter compared with the breadth of the electrodes, while at the lower pressures the discharge filled the whole space between the surface of the electrodes left exposed by the guard-rings. Therefore the last equation must be independent of the size of the parallel plates.

Now the capacity of the apparatus was 1130 c.c. nearly, hence the equation

$$\frac{p_0 - p_1}{Q} = \frac{1130}{S} \left\{ \frac{(p_0 + p_1)D}{20} + \frac{3X}{800} \right\}$$

should apply to discharges between parallel conductors in a vessel of S c.c. capacity.

As for the limits below which p_0 must lie if this equation is to hold, that has not been determined. One experiment in which p_0 was 39 mm., p_1 34, and X 500, was in close

agreement with it. Hence, probably, p_0 may be at least as much as 40 mm., although this cannot be said with certainty.

The lower limit for the mean pressures was about 1 mm. At pressures less than this the results were nugatory. For the current C rapidly decreased, and the sparking was no longer confined between the electrodes, but appeared anywhere within the bell-jar.

It should be added, however, that after about three-quarters of the mixture has been made to combine into water, the union of the rest does not appear to conform to the same laws. This seems to be the case with silver electrodes. With zinc electrodes this irregularity appears to occur sooner, viz., when about two-thirds of the mixture has gone.

Returning now to the equation

$$\frac{\Delta p}{\Delta Q} = \frac{p\mathrm{D}}{10} + \frac{3\mathrm{X}}{800},$$

let us denote by W the number of molecules of water-vapour whose formation is connected with the passage to the electrodes of each pair of ions. Then, since it has been shown above that

$$W = 4\frac{\Delta p}{\Delta Q},$$

it follows that

$$W = 4\left(\frac{p\mathrm{D}}{10} + \frac{3\mathrm{X}}{800}\right),$$

an equation which is independent of the capacity of the apparatus.

Now either $\frac{W}{2}$ or $\frac{3W}{2}$ molecules of a mixture of hydrogen and oxygen must be dissociated into atoms before W molecules of water can be formed; the first if it is not necessary to dissociate the hydrogen, the second if it is necessary.

This necessary dissociation may be assumed to be brought about by the ions colliding with the molecules of the gas as they move to the electrodes. The first term of the right side of the equation

$$W = 4\left(\frac{p\mathrm{D}}{10} + \frac{3\mathrm{X}}{800}\right)$$

then becomes intelligible. For pD is proportional to the quantity of gas between the electrodes, and therefore to the number of collisions made, on the average, by each negative ion, if each is supposed practically to cross the whole distance between the electrodes. But this is probably what happens.

For the ions which constitute the current, as they move to

the electrodes, break up into new ions those molecules of the gas which they strike under certain conditions. To do this, the velocity of the ion at impact must exceed a certain magnitude ; and, possibly also, the position in the molecule of the point of impact must satisfy some condition.

Now this reproduction of ions by collision is probably confined, at least approximately, to the region of the cathode fall, close to the cathode, where the electric force is intense enough to secure to a sufficient number even of positive ions an adequate velocity of impact (see Townsend, Phil. Mag. Dec. 1904, p. 751).

Probably, then, nearly all the negative ions which are formed by collision cross approximately the whole distance D between the plates, and this must be accurately true of any other negative ions in the current, for they are discharged from the cathode.

The second term of our last equation, which is, numerically, of much the most importance, is proportional to the electrical work done in separating the pair of ions and in bringing them to the electrodes. This work is accounted for by the formation of new ions and by the energy imparted to those molecules with which the pair collide, for the energy imparted by impact to the electrodes is probably a negligible fraction of it.

Possibly, then, the second term chiefly represents the activity of dissociation within the region of the cathode fall, and due, partly or wholly, to the positive ions.

In any case, however, if we assume that the atomic dissociation which must precede the formation of water is effected by the motion of ions, it is evidently easier for a colliding ion to dissociate a molecule into atoms than to decompose it into ions ; for, as we have seen, from 6 to 9 times as many molecules of water as pairs of ions were always formed.

The curve here obtained does not agree with the curve given in my former paper (Phil. Mag. Feb. 1904). It should be mentioned that the pressures there given are all 3·5 per cent. too small, owing to an error in standardizing the McLeod gauge. The distance between the electrodes was ·97 cm. Therefore, to compare that curve with the one here given, the abscissas and ordinates of the former curve must be increased by 6·5 per cent. In addition to this, since the present apparatus had just twice the capacity of the former one, the ordinates of the former curve, thus slightly corrected, must be halved.

A comparison made in this manner shows a marked

difference in the two curves. The original curve by this adjustment of its coordinates is projected into a curve which touches, or nearly touches, the line of fig. 3. After that, the former falls below, and diverges widely from, the latter, until the ordinates of the former are less than two-thirds of the corresponding ordinates of fig. 3. To illustrate this difference, it may be pointed out that the values of W (the number of molecules of water formed for each pair of ions), as determined by the original tables ranged from 9·7 to 4·1. The tables of this paper, although the conditions have been greatly varied, reduce these limits and confine W between 9·4 and 6·1.

I attribute this discrepancy to the nature of one of the electrodes used in the first investigation. A large circular hole had been turned out of the lower plate to permit the passage of the ultra-violet light, as I explained; and the cavity was covered by a coarse wire network of copper, which was certainly far from being a plane surface. The silver electrodes employed in the present research were carefully turned plane surfaces. It is true that the lower of the zinc electrodes had a series of narrow slits let into it for experiments with ultra-violet light, but the results given above prove that these slits, as was to be expected, produced no perceptible disturbance in the observations.

The discrepancy between the curves cannot be due to the difference in the mode of generating the gases in the two cases. For though caustic potash was used in the first investigation as the electrolyte, yet, as explained above, Table VIII. shows that it is indifferent whether barium hydrate or caustic potash is used for the purpose of these experiments.

I must express my thanks to Professor Townsend for his valuable suggestions and criticisms. I also wish to thank Mr. H. B. Baker for kindly supplying me with very pure barium hydrate and for some valuable advice.

XVIII. *Notices respecting New Books.*

The Collected Mathematical Papers of JAMES JOSEPH SYLVESTER. Vol. I. *Edited by* H. F. BAKER. Cambridge University Press. 1904.

THIS is the beginning of the long expected and patiently waited for publication of the collected works of Sylvester. It is a finely printed volume of 650 pages, and covers the first sixteen years of his mathematical activity (1837 to 1853). The papers are arranged chronologically, and are in this respect very instructive as

to the manner in which ideas originated and grew in the fertile mind of the author. His early investigations in the methods of elimination and in the theories of determinants and invariants, and his important contributions to the theory of equations and the properties of the Sturmian Functions, fall within this period ; but in addition to the longer memoirs on these subjects there are a number of shorter papers, sometimes amounting to mere notes, which touch with a master's hand upon many other departments of mathematics pure and applied. His poetic fancy and enthusiasm combined at times to produce quaint turns of thought and expression, as when, in acknowledging a hint from Cayley, he speaks of him as one "who habitually discourses pearls and rubies" ; or when, in referring to an asymmetric proof by Salmon he remarks that " Symmetry, like the grace of an eastern robe, has not unfrequently to be purchased at the expense of some sacrifice of freedom and rapidity of action" ; or when he begins his great memoir on the Syzygetic Relations of Two Algebraic Functions with the quotation from Milton's *Comus*, "How charming is divine philosophy, etc." The Editor has done his work with great care, and has added a note presenting another view of Sylvester's main theorems on determinants.

Lehrbuch Der Physik. By O. D. Chwolson. Band II. *Translated into German by* H. Pflaum. Friedrich Vieweg und Sohn. Braunschweig.

We have already had occasion to notice the high merits of the First Volume of this treatise. In the Second Volume we have eleven chapters on Sound comprised within 140 pages ; and then the remaining 880 pages are devoted to Radiant Energy. One feature is the complete disuse of the customary phrases "strahlende Wärme," " Wärmestrahlen," " Wärmestrahlung," &c. The order in which the various departments of this great section are taken up differs considerably from what is usual. This is rendered possible in part by the understanding that the student has already been through a course of elementary physics. Thus, Kirchhoff's Principle is taken up very early, practically at the beginning. Then follow chapters on the speed of propagation of radiant energy, on reflexion, refraction, dispersion, and transformation of one type into another. Photometry, optical instruments, the human eye and colour theories are taken up in succession ; and after a discussion of the usual atmospheric phenomena connected with light, the remaining chapters are devoted to interference, diffraction, polarization, double refraction, all of which are treated of in considerable detail and with a wealth of illustration. In an interesting preface Professor Chwohlson says that his aim has been to write a treatise, not for the expert or specialist in any particular line, but for the student, so that he (the student) " may find what he needs and need what he finds."

Electricity in Agriculture and Horticulture. By Prof. S. LEMSTRÖM.
London : "The Electrician" Printing and Publishing Company,
Ltd. 1904. Pp. iv + 72.

THE interesting and important experiments which form the
subject-matter of this work seem to have definitely established the
fact that, under suitable conditions, plant growth is greatly stimu-
lated by the feeble electric current which may be made to pass
from the soil through the plant to a wire net stretched above it,
a definite difference of potential being maintained between the net
and the earth by means of an electrical influence-machine. A more
favourable result is obtained with the current flowing downwards
than upwards. According to Prof. Lemström, a downward current
introduces into the plant-tissues carbonic acid and nitrogen com-
pounds, while an upward one quickens the circulation of the sap.
In hot and bright weather, the electrical treatment should be sus-
pended, as it has at such times been found damaging to most
vegetables. The percentage increase in the crop due to the
electrical treatment is at least 45 per cent. for land of average
fertility. In view of these results, which seem fairly well established
by numerous experiments, the use of an electrical stimulus certainly
comes within the range of an economic possibility, especially as the
plant required for the purpose is very simple and cheap, and the
power necessary to keep it running is absurdly small. To those
practically interested in agriculture or horticulture, we strongly
recommend the careful study of Prof. Lemström's book, where they
will also find full details regarding the cost of the necessary plant
and directions for using it.

Dr. J. Fricks Physikalische Technik. Von Dr. OTTO LEHMANN.
Erster Band. Erste Abteilung : Braunschweig : F. Vieweg und
Sohn. 1904. Pp. xxiv + 630.

IT is some fifty years since there appeared in Germany a book
on physical workshop practice by Dr. J. Frick, the main object of
which was to counteract the pernicious effects of the chalk-and-
blackboard style of teaching Physics so prevalent in his day in
Germany as elsewhere. The book achieved an immediate and re-
markable success, and ran through several editions during the
author's lifetime. No higher tribute could be paid to its enduring
educational value than the fact that an enlarged modern edition of
it now appears, the laborious task of editing and bringing it up
to date having been undertaken by Prof. Lehmann.

The present volume, which is Part I. of Volume I. of the complete
work, is divided into five chapters, and may be said to deal with
preliminaries. In the first two chapters we have a very
complete discussion of the architectural features of a Physical
"Institute," and the design of the lecture-theatre. The various
fittings connected with this latter, including blackboards, water,
gas, steam, electric circuits and switchboards, projection apparatus,
heating and ventilation, &c. are dealt with in considerable detail,

and many useful hints are given regarding the choice of the most suitable type of apparatus. Chapter III. is devoted to the Preparation Room and the Small Lecture Room, and deals with such matters as the setting up of apparatus for use in the lecture-theatre, the construction of various stands, supports and clamps, gas-burners, oil and spirit lamps and electric heating appliances, joints in glass and metal tubes and pipes, and in electric cables, rheostats, driving belts and cords, and the care of apparatus generally. Chapter IV., which is a comparatively short one, deals with instrument cupboards, inventories of apparatus, the use of the slide-rule and planimeter, and the preparation of lantern-slides. Chapter V. fills nearly one-half of the book, and by the majority of readers will probably be regarded as the most important, since it deals with the workshop and its appliances. The various methods of working in wood, metal and glass, lacquering, soldering, cementing, and kindred processes, all receive due attention. Not the least striking feature of this chapter is the extraordinary wealth of illustration, not only the tools themselves but also the methods of handling them being very carefully and clearly illustrated ; this chapter alone, in fact, contains no fewer than 1023 illustrations, the quality of which leaves nothing to be desired.

We have to congratulate Dr. Lehmann on the successful way in which he has accomplished this part of his arduous task, and have little doubt that his efforts will be gratefully appreciated by all connected with the teaching of physics. The somewhat long preface, in which he defends the educational value of Physics, and criticises various suggestions regarding the way in which the subject should be taught, will also be found of considerable interest. A portrait of the original author of the book—Dr. J. Frick—forms a frontispiece to the volume.

We are sorry to note that the book is printed in Gothic type—so trying to most readers, but particularly annoying to foreigners ; and we cannot but consider this—in a scientific work, at any rate—as a most regrettable mistake.

Il Radio. Di Augusto Righi. Bologna : N. Zanichelli. 1904. (Attualità Scientifiche, No. 6.) Pp. 68.

No simpler or more interesting account of the properties of radium could be wished for than that which Prof. Righi gives us in this small booklet of 68 pages, which forms No. 6 of the series known as *Attualità Scientifiche.*

Il Selenio. Di Lavoro Amaduzzi. Bologna : N. Zanichelli. 1904. (Attualità Scientifiche, No. 7.) Pp. viii + 142.

The remarkable properties of selenium have given rise to a very large number of researches, and have also served to stimulate the mind of the inventor, who saw great practical possibilities in them. If we except photo-telephony, the dreams of the inventor still

remain unrealized. But the large amount of experimental work done in connection with selenium has greatly added to our knowledge of this interesting element. As many of the researches have up to now remained in a form not readily accessible, the author of the present monograph on the subject, which forms the latest addition to the *Attualità Scientifiche*, has performed a very useful service by bringing together the results of the various investigations, and supplying a comprehensive bibliography relating to the subject.

In Chapter I. we are introduced to the principal physical properties of selenium, its density, specific heat, behaviour with regard to temperature, coefficient of expansion, &c. Chapter II. deals with the variation of its resistivity, Chapter III. with the effect of light on its contact-E.M.F., Chapter IV. with selenium cells, Chapter V. with the causes of the photo-electric behaviour of selenium, while the closing Chapter VI. contains an account of the more important practical applications of the element.

Die Riechstoffe. Von Dr. GEORG COHN. Braunschweig: F. Vieweg und Sohn. 1904. Pp. vi+220.

THE rapid development of organic chemistry has necessitated a considerable amount of specialization, and the present work deals with one of the specialized branches of the subject—that of aromatic compounds. Although the use of aromatic substances is as old as history itself, the scientific investigation of these interesting bodies is of very recent date, and may, in fact, be almost regarded as still in its infancy. The monograph under review contains a systematic account of the present state of our knowledge regarding aromatic compounds. Their chemical, physical, and physiological properties are duly considered, and special attention is given to the methods of isolating and preparing them. A very full bibliography and list of German patents relating to the subject are given in the introductory portion of the book.

Materialien der Stereochemie in Form von Jahresberichten. Bearbeitet von C. A. BISCHOFF. Erster Band, 1894–1898 ; Zweiter Band, 1899–1902. Braunschweig: F. Vieweg und Sohn. 1904. Vol. I. pp. cxxxvi+840 ; Vol. II. pp. 1138.

THIS monumental work should prove of exceptional value to the many workers in the domain of stereo-chemistry, and may be regarded as a sequel to the *Handbuch der Stereochemie* by C. A. Bischoff and P. Walden. It covers the period 1894–1902, and contains an account of the advances made in stereo-chemistry during that period, in the form of abstracts of all the more important memoirs published either in periodicals or in book form. Such abstracts are accompanied by critical remarks, and full references are given in each case in the form of footnotes.

For the sake of convenience, the matter dealt with is arranged under the following heads :—

    I. General stereo-chemistry.
    II. Optical isomerism.
    III. Geometrical isomerism of optically inactive bodies.
    IV. Connexion between space-relations and chemical reactions.

The work is divided into nine sections, corresponding to the nine years which it covers, and in each section the above method of sub-division is followed. The chapter on Optical Isomerism in the 1894 section is contributed by Professor E. Wedekind, and the corresponding chapter for 1896 by Professor P. Walden.

Even a cursory examination of the two volumes cannot fail to impress one with the immense amount of work which has been done in this branch of science, and with the absolute necessity for a compilation of this kind. By undertaking this laborious task, the compiler has performed a signal service to science ; and his efforts will be appreciated not only by the professional chemist but by workers in other branches of science who have occasion to consider the many problems involved in stereo-chemistry.

We cannot close this brief notice without making special mention of the very comprehensive index, which is arranged in a manner well calculated to save time to the busy worker who has occasion to make use of the *Jahresberichte der Stereochemie.*

Kritische Studien über die Vorgänge der Autoxydation. Von C. ENGLER *und* J. WEISSBERG. Braunschweig : F. Viewig und Sohn. 1904. Pp. xii + 204.

THE exact mechanism of the various slow oxidation processes has formed the subject of numerous controversies, and even at the present time there are differences of opinion on the exact nature of some of these processes. The authors of the book before us have done good service by bringing together all that is known regarding this subject, and arranging it in a systematic manner, in accordance with the theoretical views developed by them. The historical introduction forms very interesting reading, and full credit is given to the important work of Schönbein, to whose memory the book is dedicated.

Étude sur les Résonances dans les Réseaux de Distribution par Courants Alternatifs. Par G. CHEVRIER. Édité par L'Éclairage Électrique. Paris, 1904. Pp. 76.

ALTHOUGH not containing anything strikingly original, this pamphlet may be recommended to engineers in charge of alternating-current systems of distribution as giving an exceptionally clear account of the phenomena of electrical resonance, and of the conditions favouring their production.

Den Elektriska Gnistan. En Undersökning af de Fysikaliska Villkoren för dess Slocknande. Akademisk Afhandling af JOHN KOCH. Upsala : Wretman. 1904. Pp. 80.

THIS pamphlet gives an account of researches carried out by the

author at the University of Upsala on the spark-discharge, as affected by the material of the spark-electrodes, the resistance included in the circuit, &c. The results obtained are clearly exhibited in the form of tables and curves.

La Télégraphie sans Fil. L'Œuvre de Marconi. Traduit du Scientific American de New-York. Bruxelles: Ramlot Frères et Sœurs. Pp. 64.

THIS is a simply written account of the rise and progress of wireless telegraphy, and is a French version of a series of articles in the ' Scientific American.' It is provided with numerous footnotes in which are briefly explained, for the benefit of the non-technical reader, the meanings of the various technical terms employed in the pamphlet.

Gravitation als Folge einer Umwandlung der Bewegungsform des Aethers im Inneren der wägbaren Materie. Von Dr. HIBER. München: H. Lukaschik, 1903. Pp. 44.

THE author attempts to account for gravitation by assuming the æther to consist of discrete, extremely minute, and somewhat closely-packed particles moving about in a manner similar to that of the molecules of a gas, colliding with the ultimate particles which build up a material atom, giving up some of their energy to them, and receiving the energy back in a different form, with the result that the kinetic energy of the æther particles in the imme-diate neighbourhood of an ultimate material particle becomes reduced, and the material particle becomes surrounded by a sort of atmosphere within which the æther pressure is less than at a considerable distance from the particle. By reason of this effect, two particles at no great distance from each other will be forced towards each other, and this constitutes gravitation. This idea is elaborated by the author, and applied to the explanation of various facts. The pamphlet is entirely non-mathematical, and forms easy reading.

An Introduction to the Study of Spectrum Analysis. By W. MARSHALL WATTS, *D.Sc., F.I.C.* With coloured plate and 135 illustrations in the text. London: Longmans, Green & Co. 1904. Pp. viii + 325.

THE subject of spectrum analysis has always possessed a great fascination not only for serious students of physical science, but also for the general reader, whose interest has, no doubt, been considerably stimulated by the popular articles which have appeared from time to time on this subject in leading periodicals. The book before us should, on this account, find a wide circle of readers, for while it contains a good deal that can only be followed by readers having the advantage of a scientific training, yet there are other portions which are capable of being read with interest and intelli-gent appreciation by any person having a good general education. After an introductory chapter containing an account of the

elementary principles of spectrum analysis and the construction of the spectroscope, we are introduced to flame-, spark-, and absorption-spectra. The diffraction spectrum and its application to wavelength measurements are next considered, and then follows a chapter on the dark lines produced by absorption. The spectra of the stars and nebulæ, the physics of the sun, new stars, double stars, and comets form the subject-matter of the next few chapters. The concave grating and the photography of the spectrum, the relationship between the different lines of a spectrum, and between lines of the spectra of allied elements, band-spectra and spectra of compounds then come in for discussion. The concluding chapters are essentially modern, and deal with the spectroheliograph, the electromagnetic theory of light, the Zeeman effect, and the Michelson echelon diffraction-grating. A very complete catalogue of spectra is given at the end of the book.

We have noticed a misprint on p. 5, line 7 from top, where " $(\mu - i)a$ " should read " $(\mu - 1)a$."

Electro-Chemistry. Part I. *General Theory.* By R. A. LEHFELDT, *D.Sc.* Including a Chapter on the Relation of Chemical Constitution to Conductivity, by T. S. MOORE, B.A., B.Sc. London: Longmans, Green & Co. 1904. Pp. x + 268.

THIS admirable text-book is deserving of the highest praise, and will, we confidently believe, do more towards the advancement of a sound knowledge of electro-chemistry among students of the sister sciences of physics and chemistry than any publication previously issued. Our only fear in reviewing the book is lest our admiration of it should tempt us to the use of expressions which may appear extravagant in their lavish praise. Yet we know of very few books indeed in which such extreme simplicity and charm of exposition are associated with language so severely and scrupulously scientific, so entirely free from that looseness of expression which is by many writers mistaken for simplicity of style. The book is thoroughly modern, in the best sense of the term, and the writer has the rare gift of gauging very accurately the mental attitude of his reader, and, anticipating his difficulties, of dealing with precisely those points which are likely to puzzle him most. The bold and explicit manner in which many knotty points are handled immediately impresses the reader and arouses his interest.

The work is divided into three chapters, the first of which deals with the mechanism of conduction in electrolytes. The second Chapter, contributed by Mr. Moore, will be found mainly of interest to students of chemistry. The concluding Chapter contains a very careful exposition of the theory of electromotive-force.

We have noticed very few errors, and the revision seems to have been done with great care. On p. 26, line 7 from top, we notice " Eletrolysis " for " Electrolysis ; " and the illustration, fig. 10, does not seem to bear any relation to the description of it given on p. 38.

Fig 7.

zene.

Potential between Standard Electrode and Cathode.

Copper Cathode,
$i = 1\cdot85$.

Platinum Cathode,
$i = 1\cdot85$.

40   50   60   70   80   90   100
Centivolts
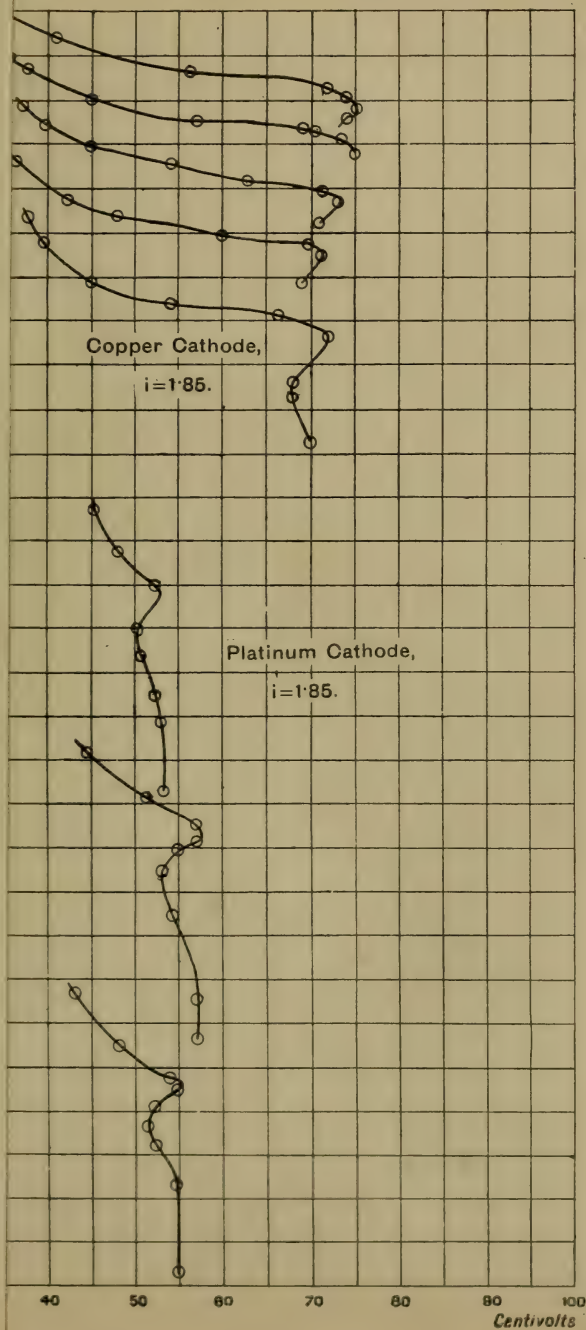
Fig. 5.

Aqueous Solutions of Silver Nitrate and Copper Sulphate.

Abscissae: Difference of Potential between Standard Electrode and Cathode.

Ordinates: Time.

I. ◯ Ag. NO₃.

i=1·757.

10 minutes.

Cu. SO₄, i=2·109. → to about 1·5 volt.

I. repeat.

Cu. SO₄ i=1·757.

Cu. SO₄ i=1·506

40 30 20 10 0 10 20 30 40 50 60 70 80 90 100
Centivolts
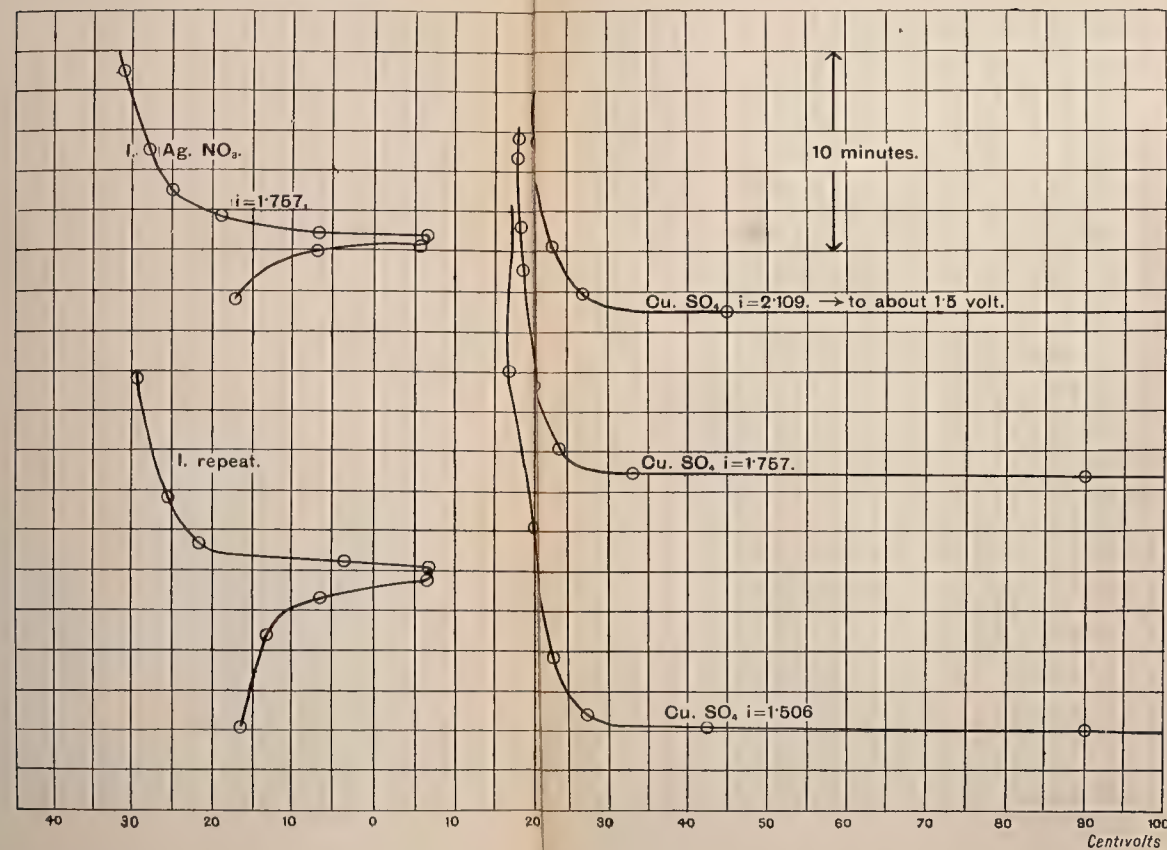
Fig. 6.

Alcoholic Solution of Cuprous Chloride.

Abscissae: Difference of Potential between Standard Electrode and Cathode.

Ordinates: Time.

t=o for Curve I.

t=o for Curve I_A.

I. i=1·054.

10 minutes.

I_A.

t=o for Curve I_B.

I, repeat.

I_B.

I. repeat.

t=o for Curve II.

II. i=1·35.

t=o for Curve II_A.

II_A.

II. repeat.

0 10 20 30 40 50 60 70 80 90 100
Centivolts

Fig 7.

Acid Solution of Nitrobenzene.

Abscissae: Difference of Potential between Standard Electrode and Cathode.

Ordinates: Time.

Copper Cathode,

i=1·85.

10 minutes.

Platinum Cathode,

i=1·85.

0 10 20 30 40 50 60 70 80 90 100
Centivolts

Fig. 11.

Nitrobenzene.

...f Potential between Standard Electrode and Cathode.

...ber Cathode, i=1·85.

...athode, i=1·329.

...pper Cathode, i=1·184.

...peat.

Platinum Cathode, i=1·85.

Platinum Cathode, i=1·329.

Platinum Cathode, i=1·184.

110   120   130   140   150   160

Fig. 8.

Acid Solution of Nitrobenzene.

Abscissae: Difference of Potential between Standard Electrode and Cathode.

Ordinates: Time.

10 minutes

Platinised Platinum

Cathode, i=1·85.

Fig. 9.

Acid Solution of Nitrobenzene.

Abscissae: Difference of Potential between Standard Electrode and Cathode.

Ordinates: Time.

10 minutes.

Copper Cathode,

i=1·329.
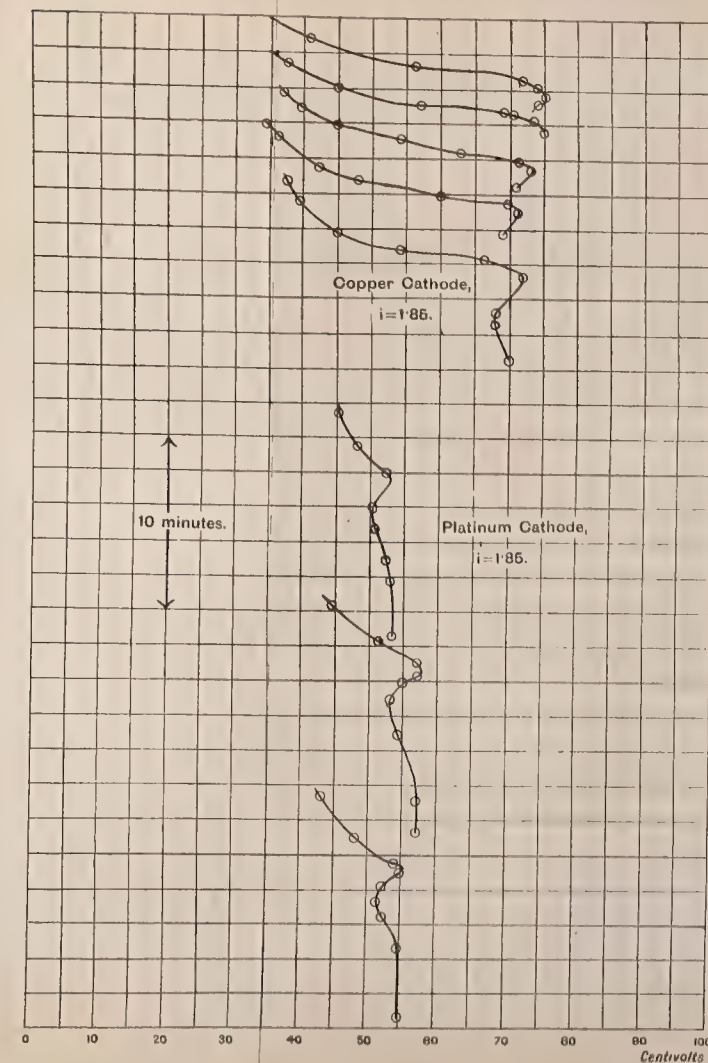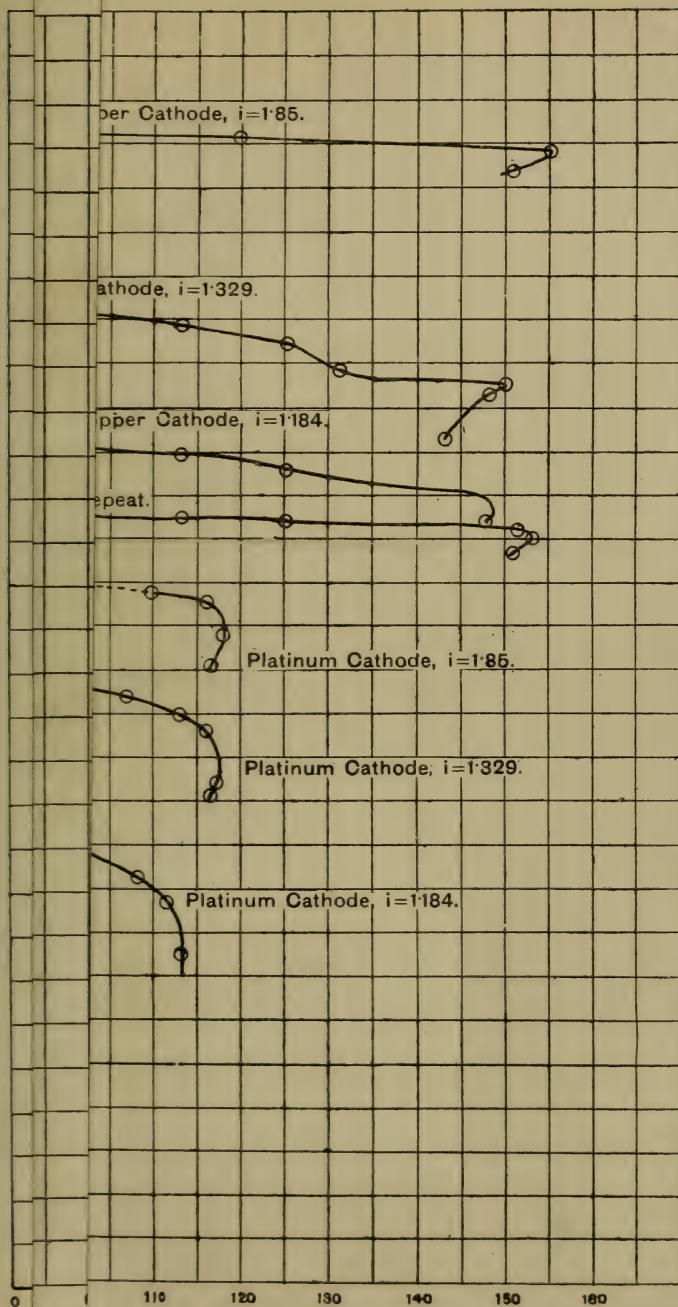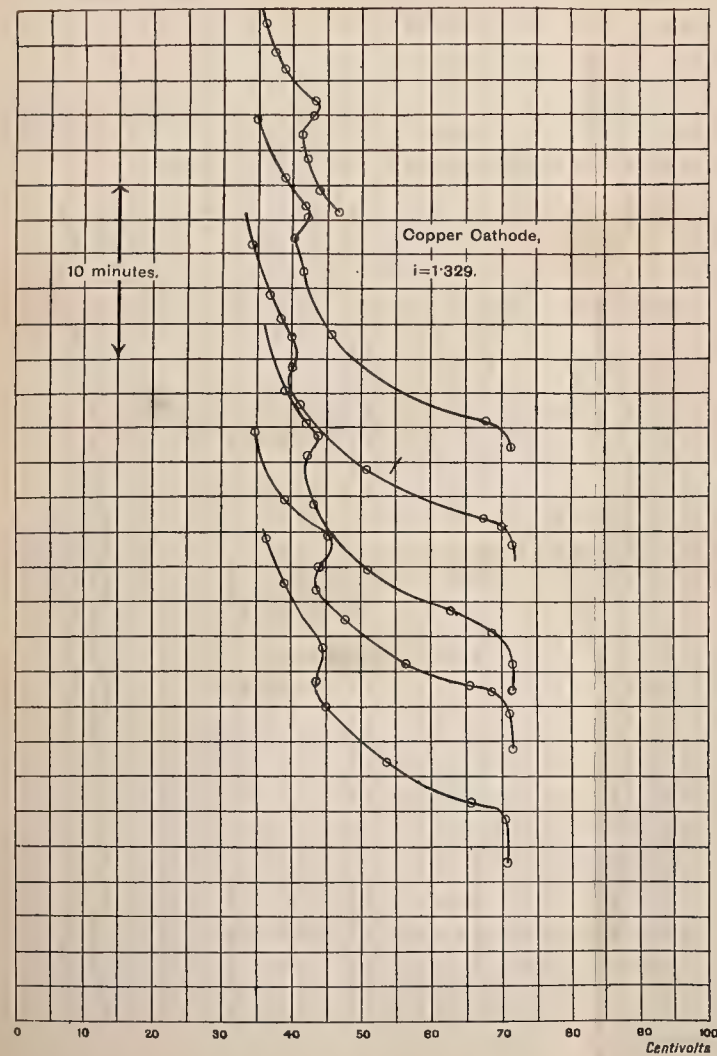
Fig. 10.

Curves to show change in Platinum Cathode brought about by use.

Abscissae: Difference of Potential between Standard Electrode and Cathode.

Ordinates: Time.

10 minutes.

I. repeat.

I.

i=1·329.
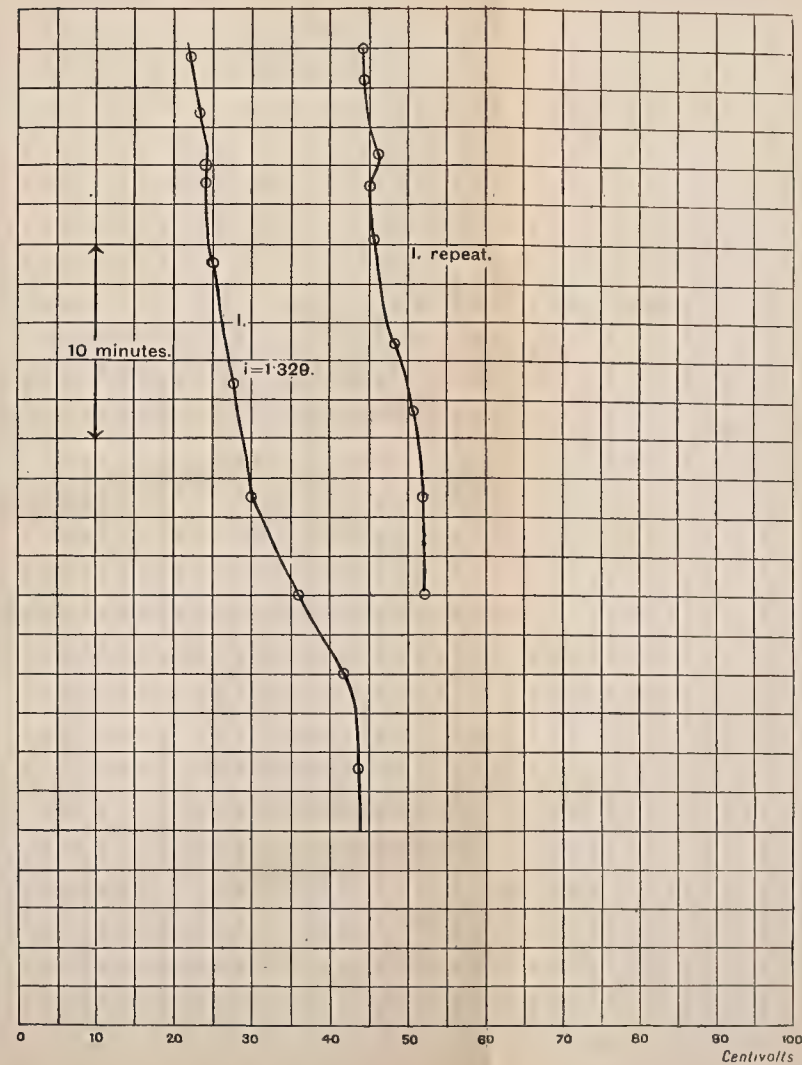
Fig. 11.

Alkaline Solutions of Nitrobenzene.

Abscissae: Difference of Potential between Standard Electrode and Cathode.

Ordinates: Time.

I. Copper Cathode, i=1·85.

II. Copper Cathode, i=1·329.

III. Copper Cathode, i=1·184.

III. repeat.

Platinum Cathode, i=1·85.

Platinum Cathode, i=1·329.

Platinum Cathode, i=1·184.

10 minutes.
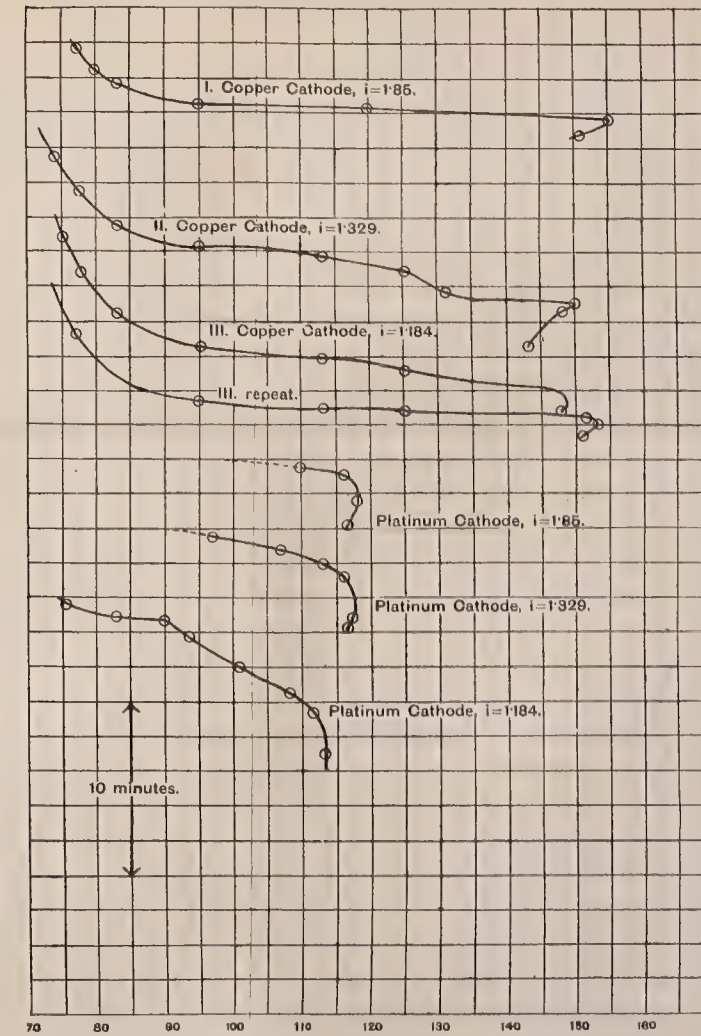
FIG. I.
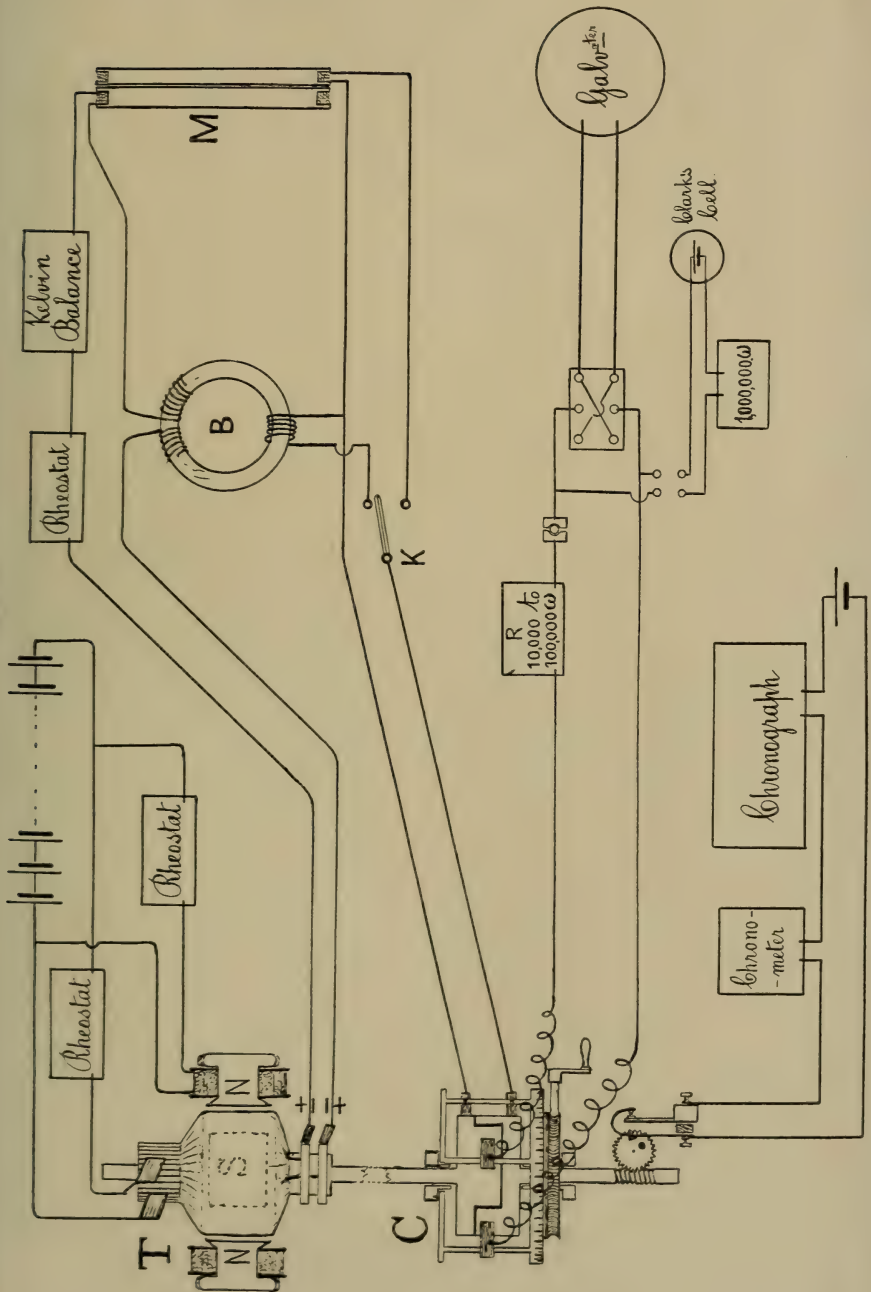
Fig. 1.
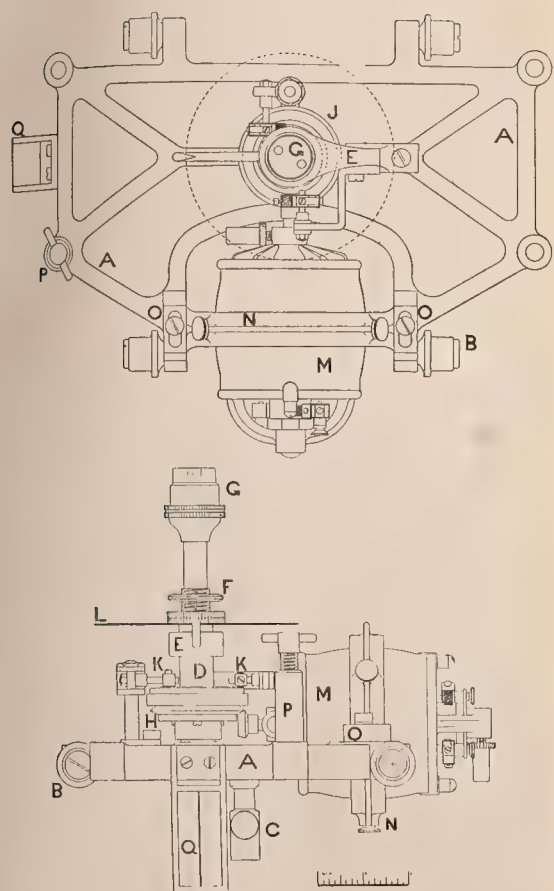
Fig. 2.
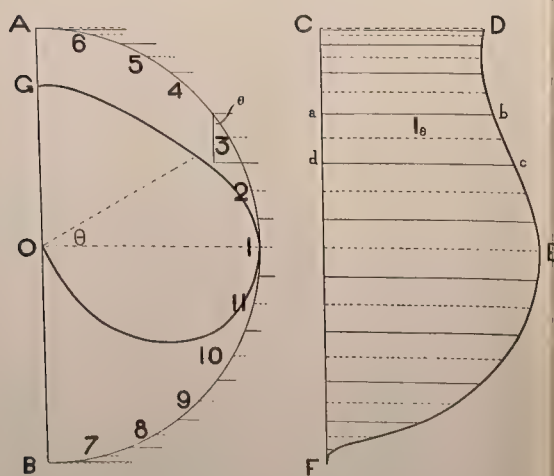
Fig. 3.
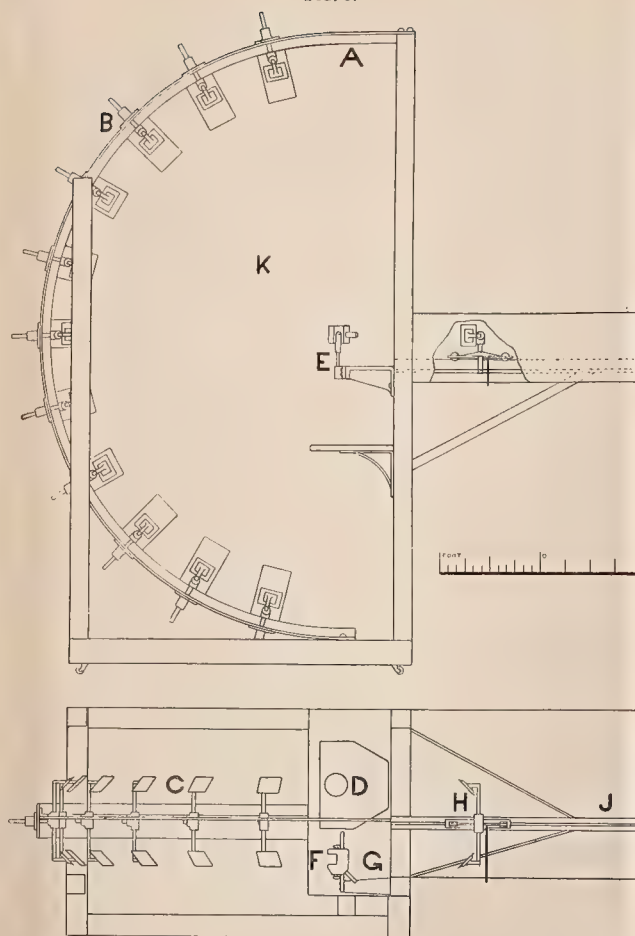
Fig. 4.
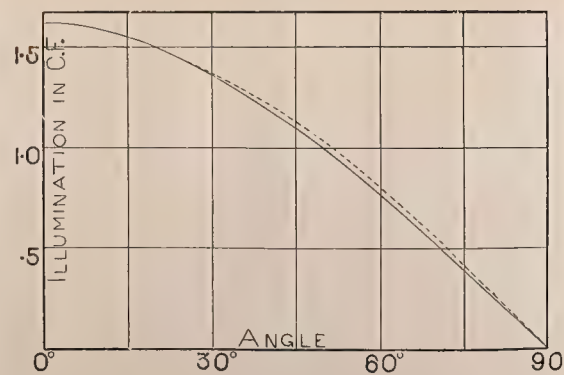
Fig. 5.
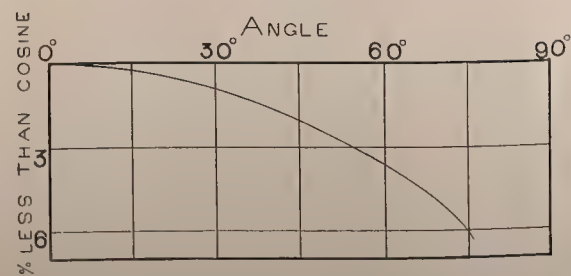
Fig. 6.
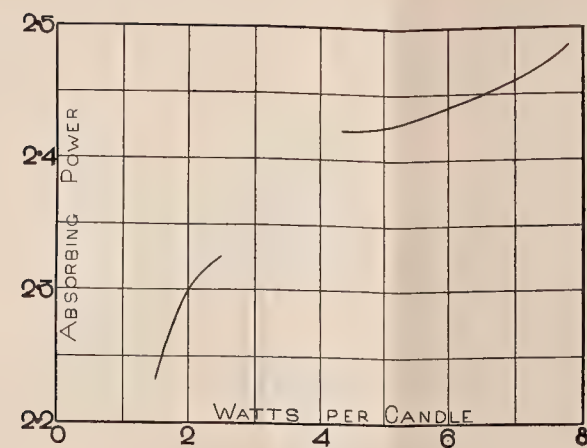
Fig. 7.
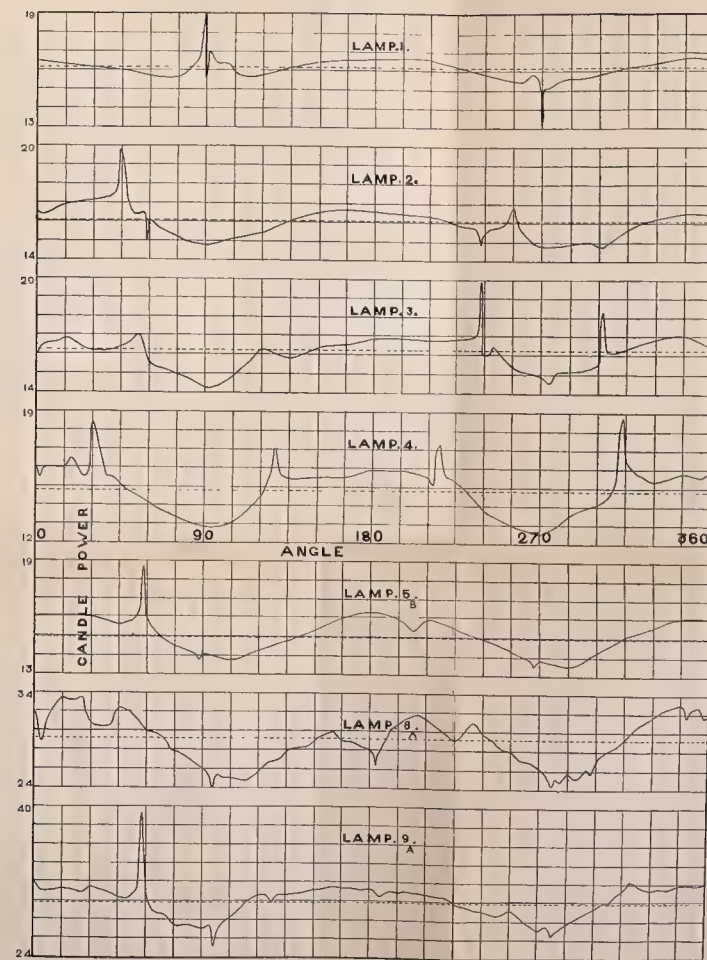
Fig. 8.
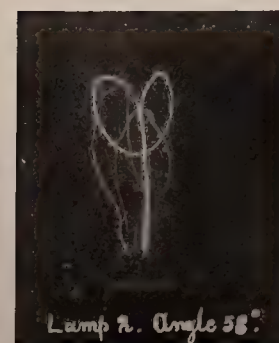
Fig. 8a.
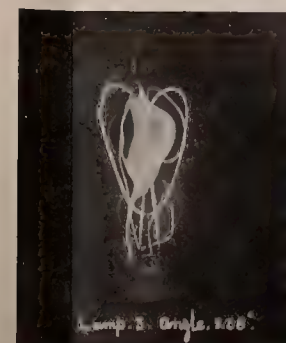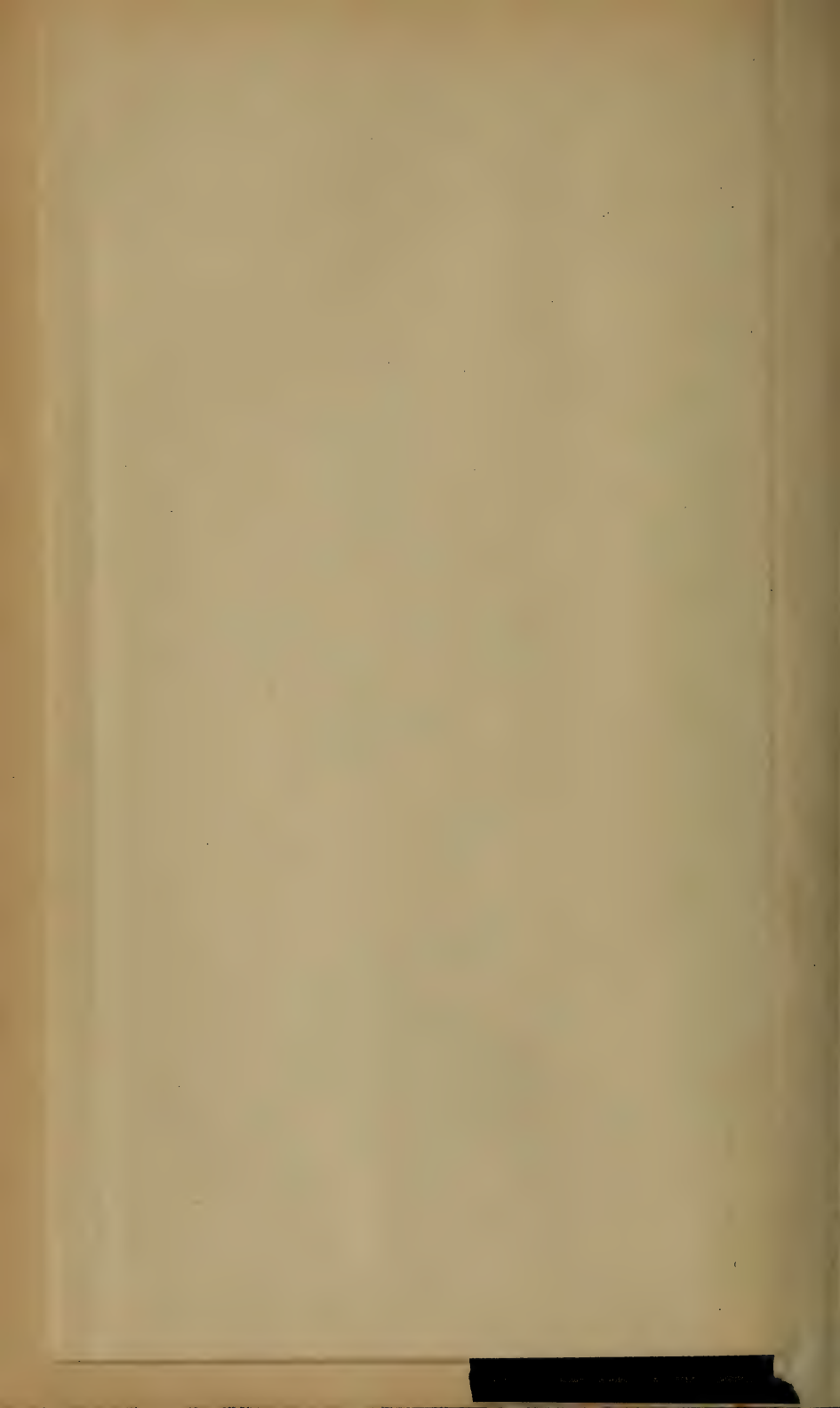
Lamp 2. Angle 58°.

Lamp 3. Angle 226°.

THE

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PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[SIXTH SERIES.]

FEBRUARY 1905.

XIX. *The Cathode Fall from Hot Carbon.* By Professor
J. A. CUNNINGHAM, *B.A.* (*R.U.I. & Cambs.*); *A.R.C.Sc.I.,
Presidency College, Calcutta; formerly* 1851 *Exhibition
Research Scholar, Cavendish Laboratory, Cambridge* *.

THE experiments described in the present paper were
undertaken in continuation of the work already com-
municated to this Magazine (Dec. 1902, p. 684). For that
work a platinum-wire grating through which a heating
current could be passed was used as electrode. But it was
found that platinum became soft, and finally fused just at a
critically interesting stage in the transformations of the
discharge. It was therefore natural to turn to carbon as
the most infusible material. This required the exclusion of
oxygen from the gas employed. It entailed, of course, also
considerably greater difficulty in the making of the apparatus.

The discharge-tube consisted of a stout glass tube 60 centi-
metres long and 25 cms. internal diameter, kept in a horizontal
position. Near the middle of this there were attached two side-
tubes which served to hold the fixed electrodes (A and K) at a
distance of 10·1 cms. apart along the axis of the discharge-tube
(fig. 1). A third narrow tube made connexion with the Töpler
pump, McLeod gauge, and a three-way tap, which served for
the introduction of fresh supplies of gas and to cut off connexion
with the mercury of the pump when not in use, so as to
minimise the diffusion of mercury vapour into the discharge-
tube. Round the cathode the tube was expanded to a
diameter of 3·8 cms. internally. The central portion of the

* Communicated by Prof. J. J. Thomson.

discharge-tube with side-tubes and movable electrodes is
drawn to scale in fig. 1.   The potential at any point along

Fig. 1.
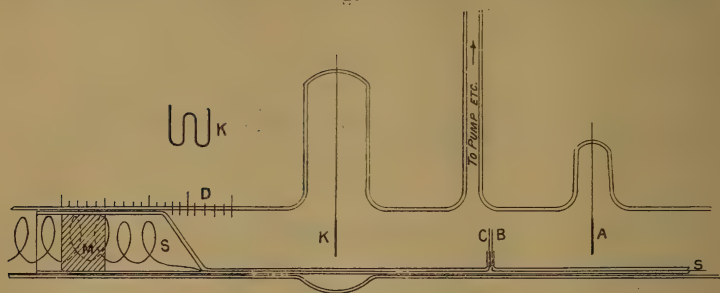
the axis of the discharge was obtained by means of two thin
platinum-wire electrodes (B and C) 1 mm. apart, which were
carried by a sliding-piece consisting of two cylindrical portions
2·4 cms. in external diameter, joined together by a narrow
glass tube 22 cms. long, which lay along the bottom of the
discharge-tube.   This was regarded as the arrangement least
likely to produce a disturbance of the discharge as it was slid
along from place to place.   This anticipation was fully justified
during the course of the experiments.   Not the slightest dis-
turbance of the discharge was noticeable except with very
small currents, when the discharge was often found to wander
about on the cathode even when the movable electrodes were at
rest.   All such results have been rejected.

   The platinum wires from the electrodes B and C passed
along the centre of the narrow connecting tube of the movable
portion (S S), and were welded into copper wires which were
twisted into spirals to allow of free movement, and were again
fused into platinum wires passing through the ends of the
discharge-tube.   The whole structure S S carrying the elec-
trodes B and C was moved along to any desired place by
means of a horseshoe electromagnet acting on the iron ring
M sprung into one of the expanded ends.   A millimetre-
scale was attached to the side of the discharge-tube at D,
and served to fix the position of the intermediate electrodes.
The use of the two moving electrodes was found distinctly
advantageous.   The one (C) nearest to the cathode was con-
nected to one terminal of a Kelvin multicellular voltmeter
whose other terminal was connected with the cathode.
Similarly the potential-difference between the anode A and
the electrode B nearest it was obtained by means of an
Ayrton & Mather electrostatic voltmeter reading from 100

to 300 volts. At the same time a Braun electroscope was kept connected across the fixed electrodes as a check on the total potential-difference.

The discharge was maintained by 500 small secondary cells (Messrs. W. G. Rye & Co.'s large capacity type). A variable resistance consisting of cadmium electrodes dipping into a solution of cadmium iodide in amyl alcohol in a U-tube was included in the circuit; and by its means the current could be regulated with great precision with the aid of a fine adjustment. A telephone was also in series to make sure of a steady discharge. The current was measured by shunting an Ayrton & Mather D'Arsonval galvanometer across a variable and known resistance in the main circuit. In the following tables and curves the deflexions obtained with the different shunts have all been reduced to the equivalent deflexions of the one particular shunt (17 cms. of platinoid wire) which came in for the largest share of use. With it a deflexion of 28·75 cms. corresponded to a current of 10^{-3} ampere, which is therefore the factor for converting all the deflexions given below into amperes.

The anode A consisted of a short carbon filament bent into the form of a simple hoop. The cathode was made of a carbon filament of approximately circular section 1 mm. in diameter. Its total length was 6·5 cms., bent into the shape and dimensions represented in fig. 1, K. From these dimensions its total surface may be estimated at about 2·05 sq. cms. It was ascertained, by making the movable electrodes pass through the middle hoop, that the electric field was symmetrical on the two faces of the cathode.

The heating of this electrode was accomplished by means of a current supplied from a battery of portable storage-cells which were placed on a platform supported by paraffin blocks, so that the whole system was completely insulated. The current could be varied from nothing up to 6 amperes, and accurately adjusted to keep the temperature constant by means of a very convenient form of rheostat similar to that described by Mr. Searle in the Phil. Mag. July, 1903. It was noticed that if a certain current raised the filament to any particular high temperature before or after the discharge, then, while the discharge was passing, it was necessary to increase the current through the electrode in order to maintain the same temperature, *i. e.* to prevent its resistance increasing. This seems to indicate that the surrounding gas on becoming ionized, and so a conductor of electricity, becomes at the same time a better conductor or " convector " of heat, and so helps to cool down the glowing electrode. Or, in other

words, some of the energy supplied to the cathode as a
heating current is consumed in aiding the cathode to dis-
charge negative electricity. Mr. O. W. Richardson [*] has
shown that this is a very considerable quantity, increasing
rapidly with increasing temperature. This view is also sup-
ported by the results shown in fig. 4, where it is evident
that it requires far less electrical potential to extract a large
supply of corpuscles out of the cathode at a high than at a
low temperature. I hope to put this matter to the test of
further experiments specially arranged for that purpose.

It was noticed that at a low temperature, on the other hand,
the starting of the discharge through the gas at once heated
up the cathode to a temperature estimated at from 100° to
250°—the resistance of the carbon diminishing to as low as
0·92 of its value when cold—being markedly warmer with
increased current. It was, of course, in this way that Hittorf [†]
heated up an iridium electrode to a bright yellow heat. He
also (*loc. cit.* p. 128) showed that there was enough heat
developed in different parts of the discharge to heat up
mercury thermometers placed in its path, and this heating
increased as the cathode was approached. E. Wiedemann [‡]
estimated the temperature of the gas molecules carrying a
current at from about 70° to about 86,000°, according to
the diameter of the discharge-tube, being inversely pro-
portional to the cross-sectional area for a constant current.
The temperature at different points along the discharge
was very carefully measured by R. W. Wood [§], who
also found the temperature developed proportional to the
current, *ceteris paribus.* For a current of ·001 ampere at a
gas-pressure of 1·5 mm. it was about 32° in the positive
column. And in all cases he too found the temperature
increasing very rapidly towards the cathode ; so that it may
be estimated from his curves at anywhere between 100° and
300° at the surface of the cathode. The distribution is
exactly what was to be expected from the curves of potential
gradient, to which the heat developed should evidently be
proportional, since the same current flows across every section
of the discharge-tube.

The temperature of the hot electrode was determined by
including it as one arm of a Wheatstone's bridge, and so
measuring its resistance. The second arm of the Wheatstone's
bridge which had to carry the heating current consisted of a

[*] Phil. Trans. A, vol. cci. p. 500 (1903).
[†] Wied. *Ann.* vol. xxi. p. 90 (1884).
[‡] *Ibid.* vol. vi. p. 298 (1879).
[§] *Ibid.* vol. lix. p. 238 (1896).

coil of thick wire (resistance = 1·245 ohms) immersed in an oil-bath of large capacity. This served as the standard of comparison for the resistance of the filament. To get the resistance when cold, the current from one Leclanché cell was used by momentarily depressing a key. It was found that when working at high temperatures the resistance of the cold electrode steadily increased (from 7·91 to 8·75 ohms during five weeks' continuous work—ten to twelve hours a day). Instead of the actual resistance as measured, I have therefore throughout the present paper calculated its ratio (R on the curves) to the resistance of the filament measured immediately before and after. This has the further convenience of being directly referable to the temperature determinations of Le Chatelier[*], obtained by a photometric comparison and generally accepted as the most satisfactory determination of the glowing carbon filaments that has been made[†].

Before starting electrical measurements the discharge-tube was pumped out to $\frac{1}{100}$ of a millimetre, and both filaments cautiously heated up to redness, pumping being continued all the time. The apparatus was then slowly filled with nitrogen (prepared by the action of potassium nitrite on ammonium chloride), and again pumped down to the lowest possible degree accompanied by the heating of the filaments to a still higher temperature. This process was repeated four times. A pocket-spectroscope showed the nitrogen lines only. It was originally intended to use also hydrogen and other gases, but the apparatus cracked at the high temperature reached in the last experiments with nitrogen.

All workers with carbon at low pressures have had trouble on account of its occlusion of gases. Prof. J. J. Thomson[‡] found that so long as a carbon filament continued to give off gas, it gave a positive leak with almost greater facility than a negative leak. This irregularity was not got rid of until the occluded gas was expelled, after which it gave a negative leak only, even at comparatively low temperatures. By virtue of the high potentials used in my experiments, this difficulty did not present itself in the same form, and the following observations may even throw some light on the manner of those irregularities.

With a carbon cathode, it was found that an enormous quantity of nitrogen was absorbed during the passage of the

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[*] *Journal de Physique*, 3me série, vol. i. p. 203.

[†] *Cf.* E. Blaker. Phys. Rev. vol. xii. p. 345 (1901), whose summary I have had to make use of, as the *Journal de Physique* is not available in Calcutta.

[‡] Phil. Mag. vol. xlviii. p. 547 (1899).

discharge so long as the cathode was below a bright yellow
heat, and apparently most rapidly at a dull red. When, how-
ever, the filament was heated to whiteness, the gas began
slowly to come out, only to go back again as soon as the
temperature fell. The pressure remained practically quite
constant over a range of about 1250° to 1360° with currents
varying from 1 to 2 milliamperes. At lower temperatures
fresh gas had to be admitted three times; and at higher
temperatures it had to be pumped out before equilibrium was
established and a fairly steady pressure maintained. If, on
the other hand, the discharge was stopped, the gas began to
be evolved at all temperatures above redness. If now this
filament, after being used for some time as cathode, was made
the anode, the rate of giving off gas was greatly increased at
all temperatures down to coldness. And it was further re-
markable that the power at my disposal (1050 volts) was able
to send a discharge through this evolved gas at a pressure
three times as high as it could when the gas was freshly
prepared. It would therefore appear that it was in the
process of helping out the corpuscles [*] that the positive
atoms of the gas got entangled in the carbon cathode, and
that a certain number of them even retained their ionic
character on getting released.

A large number of measurements were made with cold
electrodes and small current-densities, and the results obtained
were in substantial agreement with those of the numerous
workers (especially Hittorf, Warburg, Graham, H. A. Wilson,
and C. A. Skinner) who have investigated the various phe-
nomena associated with the discharge in a vacuum-tube.
Such results need not be related in detail, but they served to
show that neither the carbon electrodes nor any other part
of my apparatus differed in any essential point from those used
by other experimenters.

One point of difference, on the other hand, I might perhaps
mention, viz., that in my experiments with a striated positive
column I invariably found that a sudden rise of potential
took place to the cathode side of each bright band, succeeded
towards the anode by a region of very small potential-gradient
until the next stria was reached, and accompanied by another
sudden step up of potential. The same sort of thing is to be
noticed in one of Wilson's [†] curves for a striated positive
column in nitrogen; but in general, both his and Graham's [‡]

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[*] *Cf.* H. A. Wilson, Phil. Trans. A, vol. ccii. p. 243, and J. J. Thomson,
' Conduction of Electricity through Gases,' p. 480 (Cambridge, 1903).
[†] Phil. Mag. June 1900, p. 511.
[‡] Wied. *Ann.* vol. lxiv. p. 69 (1898).

curves show a much more gradual maximum intensity towards
the middle of the bright band succeeded by a slightly lower
minimum in the dark space. This difference is probably due
to the composite nature of the striæ in air and other mixed
gases, and is accompanied by a difference in the appearance
of the striæ. In those here alluded to, the striæ presented a
sharp edge of maximum brightness towards the cathode, *just
in front* of which bright part the sudden rise of potential
occurred. The brightness then shaded off gradually towards
the anode, and only became quite dark at a short distance in
front of the next stria.

From the indications of my previous work, I was encouraged
to pay special attention to the influence of increased current
on the distribution of potential along the discharge. The
first method of observation was to place the intermediate
electrodes at any particular point, and then, keeping the
temperature of the cathode accurately constant, to vary the
current through as wide a range as it was possible to obtain
a steady discharge with, and simultaneously to record the
differences of potential. By doing this for a series of points

Fig. 2.

Temperature $\doteqdot 1140°$. Pressure $= 0.55$ mm.
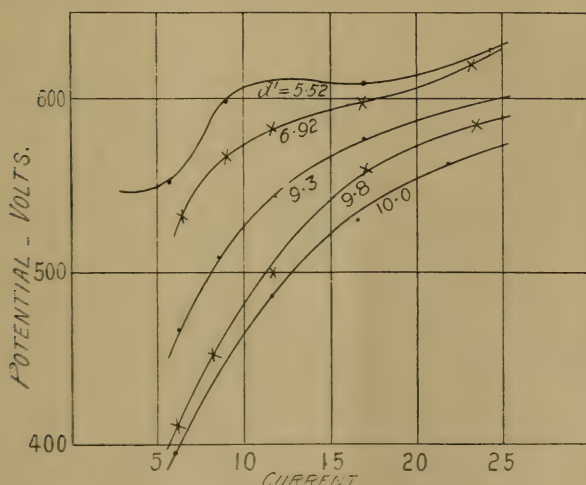
along the discharge and plotting the results as in fig. 2, it
was thence possible to plot the distribution of potential along
the discharge for a series of different values of the current,

as in fig. 3.   From these curves, for a temperature of about
1140° C., we see that the effect of increasing the current is to
continuously increase the steepness of the potential-gradient
at the cathode, and at the same time steadily to drive back

Fig. 3.

Temperature $\dfrac{R_T}{R_{20}} = \cdot 626$ ($\doteqdot 1140°$ C.)   Pressure $= 0\cdot 55$ mm.

DISTANCE ON SCALE (CATHODE AT $d = 10\cdot 9$ ANODE AT $d = 0\cdot 8$ cms.)
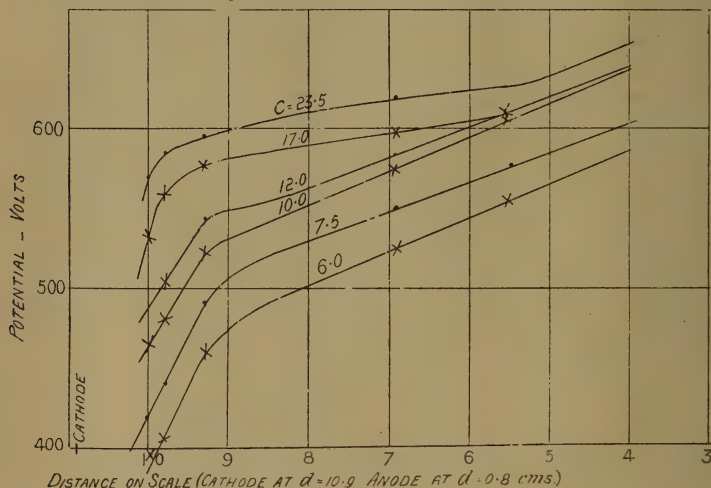
the positive column towards the anode.   Exactly similar
results were obtained at other temperatures.   This driving
back of the positive column was observed by Skinner [*].   It
is also exemplified for high temperatures in Table I., where
the bright bands seemed to disappear into the cathode.

My principal attention was, however, given to variations in
value of the Cathode Fall of Potential accompanying changes
in the current through the tube, and in the temperature of
the cathode.   Such a series of measurements is plotted in
fig. 4 (p. 202), which shows the connexion between current
and cathode-fall for a number of high temperatures.   (On the
same scale all those for lower temperatures would be crowded
together as almost vertical lines, making only a very small
angle with the axis of potential.   In order to be able to
include the measurements at the highest temperature, I have
had to contract the axis of current by cutting out certain
portions along which the slope of the curve was practically
uniform.)   The ratios of the resistances of the filament hot to

* Phil. Mag. [5] vol. 1. p. 563 (1900).

its value cold, marked (R) on the curves, correspond approximately to temperatures * of $1300°$, $1450°$, $1470°$, $1510°$, and $1670°$ respectively from left to right on the diagram.

TABLE I.

Temp. $\frac{R}{R_0}$	Pressure mm.	Current.		Cathode-Fall (C−K).	P.D. in rest of Tube (A−B).	Number of equi-distant Bright Bands.
		Deflexion observed.	28·75=10⁻³A.			
·552	0·38	16·6	56·3	(20)	123	5
		22·5	76·5	(80)	117	4
		26·0	88·3	200	97	3
		27·6	94·0	312	(90)	2½
		29·0	98·6	380	(60)	2
		31·1	105·8	460	......	1½
		37·7	128·3	606	......	⅜
·555	0·38	32·15	199·3	515	......	1½
		36·5	123·8	583	......	
		42·5	144·7	663	......	¾
		52·0	176·8	727	.. ...	½
·519	0·45	11·3	316	......	125	5
		20·3	568	250	95	3
		27·7	776	285	(95)	2½
		35·3	988	315	......	2
·541	0·40	7·2	201·6	(10)	103	4
		9·1	249	300	90	2
		11·4	319	550	......	1
·566	0·56	14·4	14·4	150	100	6
		19·8	19·8	270	98	5
		24·7	24·7	398	97	4
		29·1	29·1	461	97	3

These curves show a remarkable general resemblance (though on a much larger scale with respect to current) to those characteristic of all conduction through gases (*e. g.* to that produced by ultra-violet light striking a metal immersed in the gas), where we first attain a saturation current; and then, on further increasing the P.D., we find greater conductivity produced as the result of the ionizing collisions of the negative ions. But this is just the stage which is ordinarily regarded as initiating the luminous discharge. We have then here evidently reached a region of rapidly increasing ionization by collision, though such ionization must have been going on even during this comparatively "saturated" current (represented by the steep portions in

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* Extracted from Blaker's *résumé* of Le Chatelier's results, *loc. cit.*

Fig. 4.—Pressure = 0·45 mm.
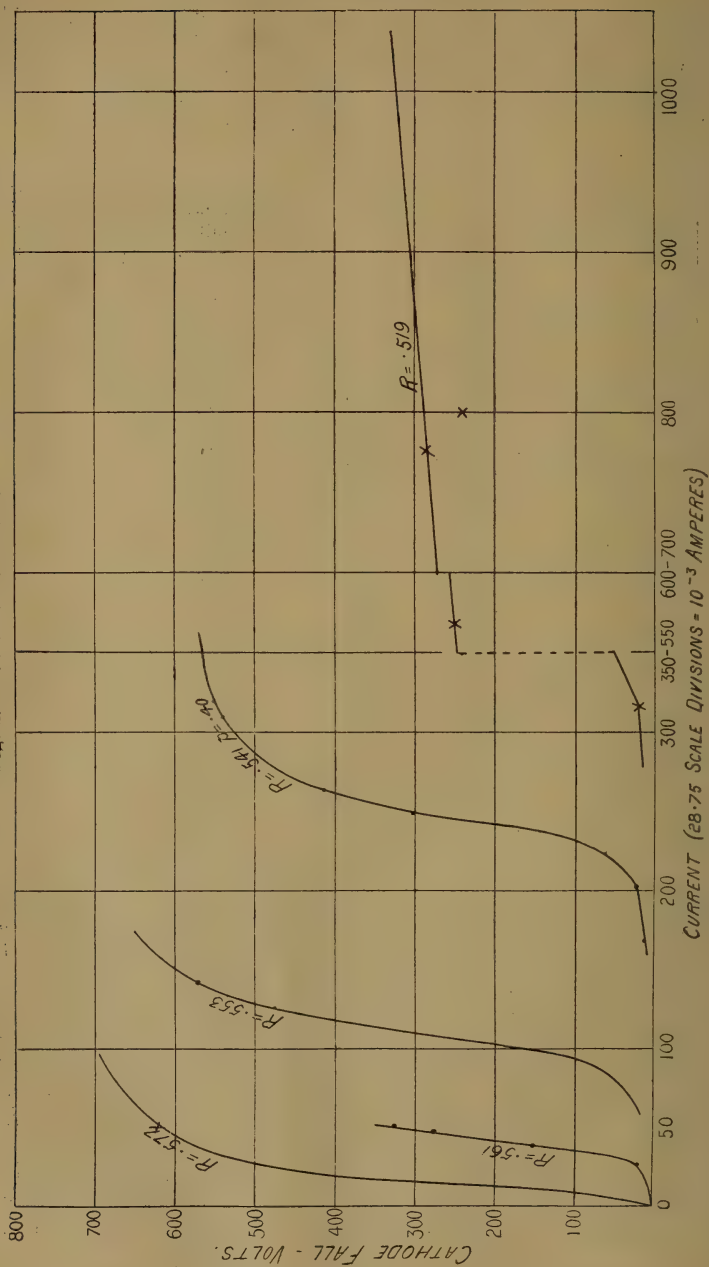
fig. 4) corresponding with normal cathode-fall, and in the case of low temperatures even before it. The complete current-P.D. curve for a gas at low pressure would thus be represented diagrammatically in fig. 5, where the abscissæ have a much greater proportional value for large currents than can be shown on the same figure without crowding the initial stages out of recognition. OA represents the first stage before all the ions are used in carrying the current, as they are in AB. From B to C new ions are produced, and

Fig. 5.
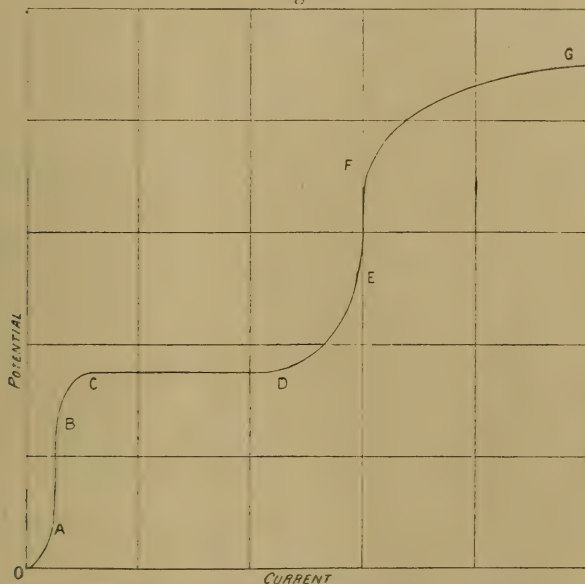
when their number reaches a certain value (C) the luminous discharge sets in and continues with constant cathode-fall along CD. As soon as the cathode is covered (D) the cathode-fall begins to rise again until (E) the current attains a second stage of saturation EF followed by a further increase of conductivity FG. At high temperatures, the first saturation stage (AB) is realized at a much smaller P.D. owing to the discharge of corpuscles from the cathode. The details of the early part of the diagram are accompanied by too small currents to be measurable by a galvanometer, and they have been completely studied by Townsend and others. They constitute the first "dark discharge;" and we have now a second dark discharge with very large currents when the positive column, and finally also the negative glow (only actually observed at high temperatures), is driven back upon the anode. The common characteristic of these two dark

discharges is that they are both accompanied by a low value
of $\frac{dV}{dC}$ (corresponding to "resistance" in metallic conduction,
where V = P.D. and C = current), or a high conductivity of
the gas. They are separated by a region of luminous dis-
charges accompanied by low conductivity of the gas. In
other words, we have here got to a second saturated current
succeeding a region of much ionization by collision, and suc-
ceeded in turn by another stage of further rapidly increasing
ionization.

There seem to be three possible explanations of this
phenomenon :—

(1) That a new sort of negative ions come into the field
which produce fresh ionization by collision at this particular
stage.

(2) That we have here reached a value for the energy of
the colliding corpuscle such that it knocks more than one
new corpuscle off the molecule which it strikes. Or

(3) That this second diminution of the value of $\frac{dV}{dC}$ marks
the point at which the positive ions also begin to produce
fresh ions by collision.

The objection to the first of these hypotheses is that, at the
low pressures here dealt with, it is not likely that the cor-
puscles coming from the cathode can get loaded up by
sticking to molecules of the gas, and so produce heavy
negative ions. And we have hitherto no evidence of two
kinds of corpuscles.

As regards the second hypothesis, J. J. Thomson [*] has
shown that the energy available for ionization by a moving
corpuscle attains a maximum for a certain value of its
velocity, and after that diminishes inversely as the square of
the velocity, without any possibility of a further increase, at
least if the corpuscles be assumed to repel each other with a
force inversely proportional to the square of their distance
apart.

The third seems, on the whole, the most likely hypothesis,
if we distinguish between the action of the positive ions, on
the one hand, *at* the surface of the cathode in helping out
the corpuscles which may be regarded as going on ordinarily [†]
with small currents not exceeding this second saturation
value ; and, on the other hand, their action in ionizing the gas
in the Crookes's dark space when the P.D. increases still further,
and the current in consequence rises again above its saturation
value. But the fact that this "supersaturation current"

* 'Conduction of Electricity through Gases,' pp. 344–345.
† J. J. Thomson, ibid. p. 480.

begins at a lower P.D. with increasing temperature of the cathode. might seem to indicate its being somehow directly due to the negative corpuscles which we know are given off in large numbers by the hot carbon. The effect of these, however. is to lessen the steepness of the gradient at the surface itself, and to diffuse the field, as it were, over a further distance, and so increase the region through which the positive ions may produce new ions by collision, while at the same time lessening the consumption of positive ions at the surface of the cathode.

These considerations give a special theoretical interest to that part of the curve where the current begins to increase again more rapidly than the cathode-fall rises. A number of measurements at this portion of the curve are shown in fig. 6 (p. 206), with the temperature $\left(R = \dfrac{R_T}{R_{28}} \right)$, pressure (p), and distance in cms. from the cathode (d) marked on each curve. In Tables II. and III., I have repeated the results for a couple of the curves which were taken beyond the limits of this figure.

<div style="display:flex">

TABLE II.

$\dfrac{R}{R_0} = {\cdot}664, \quad p = 0{\cdot}32$ mm.

Observed Deflexion.	Current. Deflexion, $28{\cdot}75 = 10^{-3}$ amp.	Cathode-Fall. Volts.
48·5	164·9	950
41·4	140·8	910
32·5	110·2	855
28·3	96·1	836
19·4	66·0	770
12·1	41·1	704
20·0	68·0	785
18·2	61·9	767
11·5	39·1	698
37·5	37·5	710
33·6	33·6	694
30·0	30·0	681
24·7	24·7	664
20·4	20·4	651
15·6	15·6	631
15·5	15·5	630
12·3	12·3	622
9·8	9·8	618
7·0	7·0	591
5·2	5·2	576
24·6	4·68	573
17·7	3·56	556
10·7	2·03	530

TABLE III.

$\dfrac{R}{R_0} = {\cdot}591, \quad p = 0{\cdot}43$ mm.

Observed Deflexion.	Current. Deflexion, $28{\cdot}75 = 10^{-3}$ amp.	Cathode-Fall. Volts.
16·8	16·8	570
11·8	11·8	543
8·8	8·8	520
7·0	7·0	501
10·3	10·3	535
29·3	29·3	607
23·3	23·3	599
35·5	35·5	630
44·5	44·5	677
51·5	51·5	691
15·3	52·0	690
21·5	74·5	741
28·1	95·4	785
18·6	63·2	735
30·4	103·3	824
34·0	115·5	852

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Fig. 6.
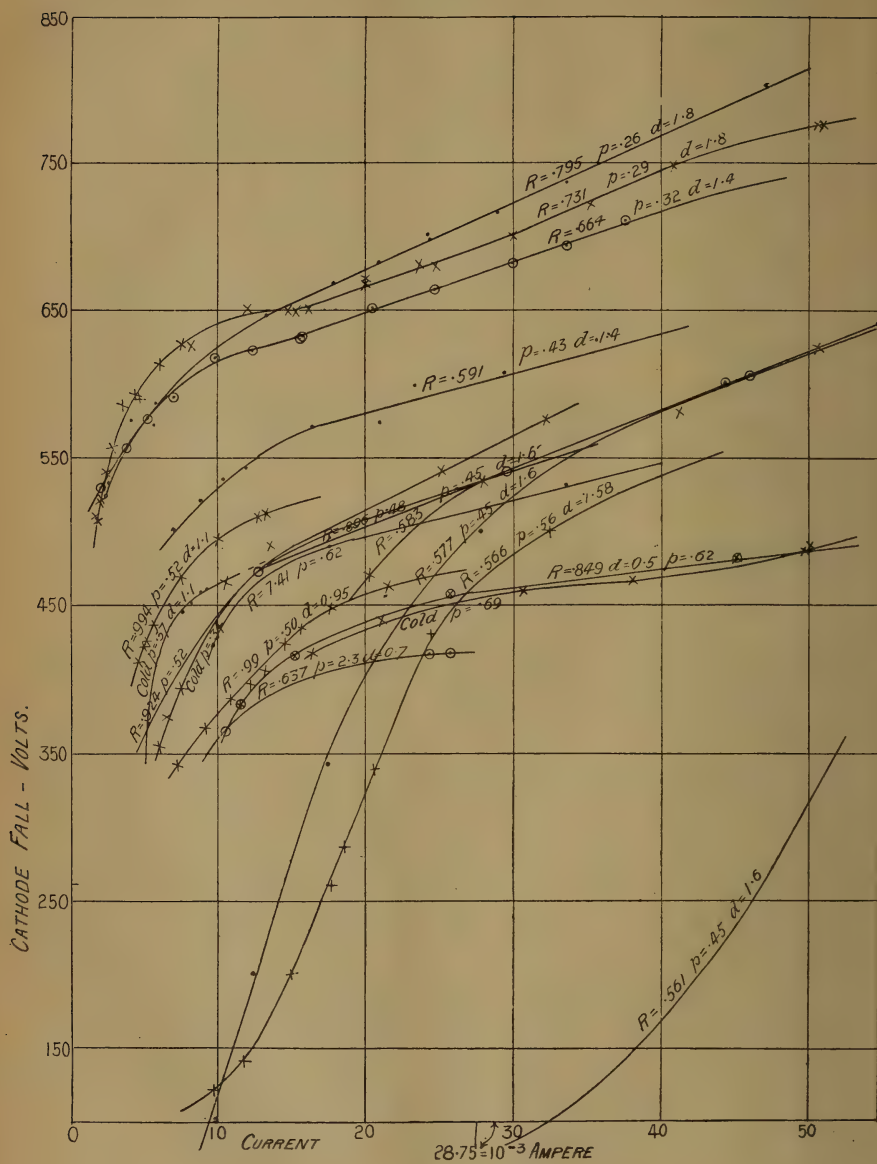
In fig. 7 I have plotted three curves (for different currents (C) in the same units) showing the very sudden diminution in cathode-fall as soon as the temperature of the cathode exceeds a certain value which is higher for larger currents.

Fig. 7.
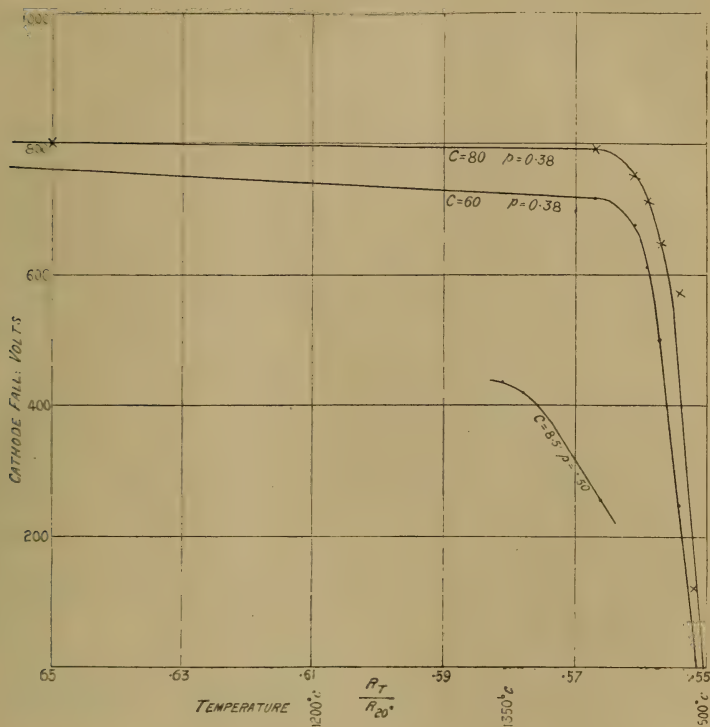
These are in entire agreement with the results published in my previous paper (*loc. cit.*), and need not be further dwelt upon here. It is often difficult to trace the values of the cathode-fall round this sharp bend in the curve.

In conclusion, I have much pleasure in acknowledging my gratitude to Professor J. J. Thomson for the kindly and suggestive interest which he took in the carrying out of these experiments.

XX. *On Certain Difficulties which are Encountered in the Study of Thermodynamics.* By EDGAR BUCKINGHAM[*].

THERE are three main difficulties which one meets when trying to develop the theory of Thermodynamics logically. Any one who has tried to get the subject ordered in his mind with the clearness and precision which are necessary if it is to be presented to students sufficiently advanced to be really critical, must have been impressed by the manner in which so many writers dodge these three difficulties, and will be glad that Professor Orr, in his very interesting article in the October number of the Philosophical Magazine, has pointed them out so distinctly. These three points are :—(1) the definition of temperature; (2) the definition of quantity of heat; and (3) the statement of the theorem of Clausius for irreversible processes. The last of these is ostensibly the subject of Prof. Orr's paper; but he touches critically on so many other points that his article should prove of great benefit, even to people who have reached a state of mind which satisfies themselves, at any rate, regarding the particular subject of the inequality of Clausius.

Aside from these three difficulties, the rest of " classical thermodynamics," as M. Duhem has named it, seems to be a fairly logical and satisfactory whole, but discussion on these points seems to me useful.

There are three things in Prof. Orr's article which stand out as of particular importance.

(1) He says in substance, though with great moderation, that all proofs of the theorem that $\int \frac{dQ}{\theta} < 0$, when the integral is taken round an irreversible cycle, are rubbish, if based on the two laws of thermodynamics as usually stated and on the usual definition of reversibility.

(2) He proposes a way to avoid the difficulty by a modification of the wording of the second law, as stated by Lord Kelvin or by Clausius.

(3) He makes a plea for the use, wherever it is practicable, of Carnot cycles, instead of entropy, free energy, or thermodynamic potentials, which are, mathematically at any rate, secondary ideas to the derivation of which the use of cycles is a necessary antecedent.

As regards the first of these points, any logical physicist always finds these so-called proofs worthless, and he always will so long as they rest on the bases mentioned. I say this with the fullest confidence and the most complete generality.

[*] Communicated by the Author.

There may have been hundreds of these " proofs " published, and I have probably not waded through more than a dozen or two at most, but I have reached the same state of mind that the French Academy reached about perpetual motion machines. They did not prove, and we cannot to-day *prove*, that no perpetual motion machine can ever be devised; but we have all stopped believing that it can, and we have taken the further step of concluding that there is a reason *in the nature of things* why it cannot be done. A man who reads thermodynamics only because he wants to follow the literature may not care much about the origin or the justification of a given theorem, if he finds it generally used and never with disastrous results. But the man who works up his thermodynamics for presentation in convincing form to a critical audience, has a different task; and such a man, if he be honest with himself, necessarily reaches the same conclusion as Professor Orr, namely, that the inequality of Clausius for irreversible cycles not only has not been, but cannot be deduced from the accepted forms of the second law.

The question at once arises: what are we to do about it? Probably no one doubts either the usefulness or the truth of the theorem, and no one, certainly, has ever found a case in which the correct application of it led to results which were contradicted by experiment—the final test. The theorem *is* correct for practical purposes and everyone knows that it is just as he knows that the principle of the conservation of energy is correct. What then? The proposition is true: the proofs of it are fallacious: shall we insist upon having a proof at all? To my mind it is very doubtful whether there is any advantage to be gained by having the proof.

At present, thermodynamics deals successfully and logically with states of equilibrium: it has advanced in a reasonably secure and satisfactory way over a field similar in extent to the statics of frictionless systems in pure dynamics. It has done this by means of reversible processes, first by using reversible Carnot cycles and later by using entropy, free energy, and thermodynamic potentials in perfectly legitimate ways and by using reasoning which is logical and satisfying when we have in view only reversible changes of state. The theory up to this point rests on the principle of the conservation of energy and on the second law as formulated by Lord Kelvin or by Clausius.

Thermodynamics now takes, or attempts to take, another step forward. Leaving pure statics, it attempts to say that if equilibrium does not subsist, whatever change takes place in the state of the system under consideration must take

place in a certain direction and not in the opposite one.
And how does it take this step?   By leaving the solid ground
of reversibility and making use of a new theorem—the one
under discussion.   The new step leads to results which
always turn out to be correct; but it is disquieting; we feel
that we may be making a mistake; we are not sure of our
ground, and why?   Because we demand of the original
bases of " reversible thermodynamics " more substance than
is in them.   It is new substance that we must have, no matter
how we get it.

Professor Orr suggests that we alter the wording of the
second law; in other words, he proposes that we insert the
substance of the new proposition into the old principles.   To
the mathematical mind, the one that regards logic and neat-
ness of form as in themselves admirable, this would seem the
best way out of our difficulty, as it is certainly the neatest.
But does it really do us any good?   It does if we are merely
trying to silence criticism, but is it the best way?   I think
not.

The immediate object of theoretical physics is to arrange
known facts under certain general principles using hypo-
theses only when necessary, and making the general prin-
ciples and the hypotheses as few, as simple, and as easily
comprehensible in statement as possible.   We are confronted
with the question whether we shall make an addition to one
of our generally accepted and familiar principles or set up a
new principle.   The old principle—Lord Kelvin's or Clausius'
statement of the second law—is well established and is
familiar to everyone who is interested in thermodynamics.
It has been used erroneously in the building up of fallacious
proofs.   Now it seems to me distinctly better, *at the present
time,* not to change the old statements by adding a little to
them, but to act in such a way as shall show clearly our
appreciation of their inadequacy to the new needs; in other
words, to set up a new principle and state that it is new, and
different from the old.   Finally, of course, it is all one: the
new substance has to be put in somewhere, and I am quite
agreed that in the end a single statement, even if it be a
little less simple than the usual statement of the second law,
may be preferable to two separate statements.   But I think
that at the present time we should do well to sacrifice our
feeling for neatness in the interest of the advance of the
knowledge and use of thermodynamics, an advance which is
impeded by the present obscurity of some of the fundamental
theorems of the subject.   My proposal is in the nature of
putting the foundations of thermodynamical reasoning into

words of one syllable, and I believe that at present that is a good thing to do. Less rudimentary statements of principles may come all in good time, just as in time Hertz's principles of mechanics may oust Newton's laws from the position they still hold as the basis not only for teaching but, what is more important, for learning the principles of dynamics.

I propose, then, that instead of inserting new substance into the statements of the second law, we leave the second law alone and put our substance into independent form as a new hypothesis. It will be an hypothesis only in the general sense in which any induction is hypothetical; it is really a conclusion, even though an indirect one, from a great accumulated mass of experience, and it is based on evidence of the same nature as the second law or the principle of the conservation of energy, though the evidence is not so extensive. The particular form we select for the new statement seems to me of small importance,—probably it would be best to start as early as possible and say that for all irreversible cycles $\int \frac{dQ}{\theta}$ is less than zero and not equal to zero. But this leads me to Professor Orr's last point, namely his plea for the use of cycles wherever they can be used, instead of dragging in entropy, or still worse, free energy and thermodynamic potentials.

This may be looked at from various points of view. In the first place, are we speaking of a journal article, of a text-book, or of a treatise? The object of a journal article is, in general, to influence as many people as may be, and to that end many people must read it. It is unquestionable that many men will read thermodynamics put in the form of cycles who would not read it if entropy or free energy were used in the reasoning. So far, then, as journal articles are concerned, I agree with Professor Orr as to the undesirability of using the more modern forms of the theory when they are not strictly necessary.

But suppose that we are considering how a text-book should be written. Let us grant at once that in an elementary book it is unwise to pile any further abstractions upon Carnot's cycle; we do not, therefore, admit the same proposition for a more advanced text-book. Thermodynamics seems to me to be a very difficult and abstract subject, and one only to be pursued with advantage by students who have already attained a considerable maturity of mind beside having been familiar with mathematical modes of thinking for several years. To such students, the mere mathematical difficulties encountered in dealing with free energy or the

thermodynamic potentials are of very little significance ; it is only a question whether the *ideas* are worth the trouble of mastering them.   Now one object of studying thermodynamics is to be able to follow the literature, and the current literature is written very largely by men who, wisely or unwisely, do use free energy or thermodynamic potentials, a usage which is characteristic of the modern developments of the subject.   For this reason, if for this alone, I think that any text-book of thermodynamics which is not professedly very elementary, should certainly, after showing how problems may be treated by the use of cycles, also show how they may be treated by other methods, and should, by the juxtaposition of different proofs of the same proposition, try to impress upon the student the fact that, though the mathematical form be different, the underlying idea is the same.

The question how a treatise should be written is not so easily answered.   We might turn to the treatises and compare the good and the bad ; but are there any treatises on thermodynamics ?   I do not know of a single book which today deserves the title of "Treatise on Thermodynamics," and the very few men who seem competent to write such a treatise show no intention of doing so.   We must leave the question of the proper method for a treatise to the future when the difficulties which now now beset us may have vanished.

But if each kind of writing should assume a form suitable to the objects it has in view, we may yet ask whether, in our private thinking about thermodynamics, each method may not have its advantages.   For the treatment of problems of pure equilibrium, where we do not have to consider irreversible processes, it seems to me merely a matter of personal taste and habit what method we use.   But when we come to the inevitable problems where we can no longer be content with the contemplation of states of equilibrium and nothing else, a treatment by cycles does not always give us the clearest view or the firmest grasp of the subject.

Every actual process *does* have a definite direction, and we know experimentally, by the whole experience of our lives, that if anything whatever happens in the physical world about us, the mere fact that it did happen in a certain way proves that it could not, under precisely the same conditions, have happened in the opposite way.   There is something in the nature of things which determines that any actual change of state of a material system *shall* have a certain particular direction and not the opposite one.   This statement is merely

another form of the new fundamental substance which we have to put into our reversible thermodynamics to make it applicable to real processes. After grasping this fact, the mind trained in quantitative reasoning seeks instinctively for some quantity—some function of the generalized coordinates of the system—which characterises the instantaneous state of the system, so that it may be possible to think of the changes in state as characterised by the changes in this quantity, which by its value expresses a certain property of the system. That is why we find men operating with entropy, free energy, and thermodynamic potentials; it is the desire to think of a change of state as measured by a single quantity without the complicated machinery of a cycle, with its four or more separate operations to be considered separately. And though it must be granted that these quantities are defined only by reference to cyclic processes, and though mathematically they are merely derived from something more elementary, they do, in spite of that, to the man who has once become familiar with them, present real advantages in simplicity, as I think must be admitted by any man who has constructed a model of the thermodynamic potential surface for the region of the triple point of ice, water, and steam, or for the critical point where the water-steam line comes to an end.

One man will prefer to work with entropy, as Professor Planck seems to do; another will use the free energy way of looking at things, as Professor Nernst does; still another will think altogether in terms of the thermodynamic potential, as M. Duhem does. For me, the potential idea is the natural one; I think of a natural process as going on because the system involved in the process has not yet reached the bottom of a potential slope; and I think of a state of stable equilibrium as a state such that the system is, so to speak, at the bottom of a hollow of potential so that it cannot change its state because to do so it would have to go up hill, which it cannot do *of itself*. But the particular form into which each man casts his ideas is of small moment. What I have tried to bring out is that beside regarding entropy, free energy, and thermodynamic potentials as mere mathematical fictions which complicate reasoning more simply performed by using only cyclic processes, they, and perhaps other similar functions, may have a real intrinsic value as aids to clear thinking, and may often be helpful in a way which is quite beyond the powers of the cycle.

I believe that it is for the benefit of thermodynamics that

people who have not yet surmounted all its difficulties nor forgotten the hard labour they went through in mastering the rudiments of it, should discuss just such points as are brought out so interestingly in the paper by Professor Orr which was the stimulus for the foregoing remarks.

Washington, D.C., U.S. Dept. of Agriculture,
            November 1, 1904.

XXI. *Recent Investigations bearing on the Theory of Electrolytic Dissociation. By* LOUIS KAHLENBERG, *Ph.D., Professor of Physical Chemistry in the University of Wisconsin*[*].

IN creating the theory of electrolytic dissociation the phenomena of actual electrolysis have played a minor part. The changes of concentration of the electrolyte around the electrodes, the elevation of the temperature observed during the passage of the current, and the fact that when polarization is prevented, electrolytes obey Ohm's law, were all explained many years before the existence of the theory of electrolytic dissociation. Indeed, the hypothesis of Clausius, which displaced the old Grotthus theory and brought conceptions of electrolysis into harmony with thermodynamic requirements, is by many chemists and physicists still regarded as the best mechanical explanation of the electrolytic process, when all facts are fully considered.

Briefly stated, the theory of electrolytic dissociation is based upon the observation that for a goodly number of aqueous conducting solutions the so-called molecular conductivity increases with the dilution ; that osmotic pressure, lowering of the freezing-point, elevation of the boiling-point, or lowering of the vapour-tension of such solutions is abnormally great (*i. e.*, that it is greater than for non-conducting solutions, the behaviour of which it is asserted conforms to the requirements of the gas equation); and that the so-called degree of electrolytic dissociation may be calculated from the electrical conductivity of the solution or from its osmotic pressure, its boiling-point, freezing-point, or vapour-tension. Indeed, the dissociation hypothesis is vitally connected with van't Hoff's theory of solution. The latter claims that dilute solutions behave according to the equation $PV = iRT$, in which the factor i is unity for non-electrolytes. The analogy between a gas and a solution has been recognized so long that it would be difficult to state when the idea originated ; but van't Hoff, by using Pfeffer's direct measurements of

* Communicated by the Faraday Society. Read before the Society, November 23, 1904.

osmotic pressure, sought to show that dilute solutions obey the gas laws. In this he was unsuccessful, in that he had to introduce the factor i above mentioned, which varies for different solutes in the same solvent, and even for different concentrations of the same solute in a given solvent. An examination of the original tables published by Arrhenius[*] shows that the factor i is by no means always unity for aqueous non-electrolytic solutions, nor does it always vary more from unity in the case of solutions of very considerable conductivity than in the case of non-electrolytes ; but for a goodly number of excellent electrolytes i is materially greater than unity. The latter fact was utilized by Arrhenius in his endeavour to show that the factor i exceeds unity for electrolytic conductors, and increases with the dilution as does the so-called molecular conductivity. And thus upon the assumption that free ions, part-molecules charged with electricity, exist in an electrolyte and act like so many molecules, the general statement which van't Hoff had made for solutions was upheld—namely, that dilute solutions having the same temperature and the same osmotic pressure contain the same number of dissolved molecules.

Now I wish to contend that the facts presented by Arrhenius in his original article are not sufficient to serve as a basis for the assumption of electrolytic dissociation in solutions ; and that while isolated facts may here and there be interpreted by means of the dissociation theory, the bulk of experimental evidence gathered since 1887 speaks strongly against the theory.

In 1901[†] I published a list of results of cryoscopic and ebullioscopic determinations made with typical aqueous solutions of electrolytes and non-electrolytes, and also a list of molecular-conductivity determinations of the same electrolytes at $0°$ and at $95°$. It is unnecessary to discuss again the details of these results, which are rather voluminous. Suffice it to state that a comparison of the freezing-point values with the molecular conductivity at $0°$, and also of the boiling-point values with the molecular conductivity at $95°$, revealed the fact that there is no such connexion between freezing-points and boiling-points of solutions on the one hand, and their conductivity on the other, as is claimed by the theory of Arrhenius. In numerous cases not even a qualitative agreement exists. The facts presented in the paper cited have since been corroborated by Smits in his careful vapour-tension measurements, and by H. C. Jones and co-workers in their

[*] *Zeit. phys. Chem.* i. p. 629 (1887).
[†] *Jour. phys. Chem.* v. p. 339 (1901).

molecular-weight determinations on solutions. The only objection which the adherents of the dissociation theory have made against using the data obtained as an argument against the dissociation hypothesis is that the solutions employed were not dilute enough, since the dissociation theory and the theory of van't Hoff hold only for infinitely dilute solutions. Now the work was performed with apparatus like that used by Beckmann, and the lowest concentrations examined were such as could very well be tested by means of apparatus of that kind without incurring too large an experimental error ; *i. e.*, the most dilute solutions tested were so chosen that the rise of the boiling-point or lowering of the freezing-point would as a rule be about one-tenth of a degree. But stronger solutions, in some cases almost up to the point of saturation, were also examined in order to observe the changes of boiling-point and freezing-point with increase of concentration. The objection that all the solutions employed were too concentrated to enable one to use the data to confute the dissociation theory, may be met by the statement that the adherents of the theory, among them Arrhenius himself, have repeatedly used solutions as strong as and even stronger than the most dilute solutions I employed in the investigation cited ; for it is a fact known to all that the bulk of cryoscopic and ebullioscopic data gathered to support Arrhenius's claim of the relation between molecular conductivity and boiling-points and freezing-points have been obtained with the Beckmann apparatus, and hence the solutions used were as a rule not more dilute than those of lesser concentration employed by my co-workers and me. Not all the adherents of the theory of electrolytic dissociation would be ready to admit with Whetham * that the only satisfactory cryoscopic measurements at the extreme dilutions necessary to test the theory are those of Mr. E. H. Griffiths for cane-sugar and potassium chloride. It certainly is demonstrated by Griffiths's work that the so-called molecular lowering of the freezing-point of potassium chloride is within very small limits of experimental error double that of cane-sugar. This is merely one isolated experimental fact, and it has not yet been shown that the curve representing the change of the molecular conductivity with the concentration runs parallel with the curve indicating the change of the freezing-point with the concentration in the case of potassium chloride within the range investigated by Griffiths. Nor has this parallelism between freezing-point and molecular conductivity been rigidly demonstrated for the

* Paper read before the **Faraday** Society, ' Electro-Chemist,' iii. July 1903, p. 16.

solutions cryoscopically examined by Loomis. If one were therefore to take the attitude of Whetham, there would be no unobjectionable experimental evidence whatever on hand at present upon which to base the claim that freezing-points and molecular conductivity are related, as held by Arrhenius.

It may be asserted that all data of freezing-points and boiling-points of solutions on the one hand, and of molecular conductivities on the other, when carefully scrutinized, show that there is no such relation between them as the theory of electrolytic dissociation requires. While here and there isolated values of the degree of electrolytic dissociation, as calculated from molecular conductivities on the one hand and molecular-weight determinations on the other, agree approximately, this agreement is generally not within the limits of experimental error, and holds only for a very limited range of concentration.

In order to show that I was fully cognizant of the fact that the theories of van't Hoff and Arrhenius hold strictly only for infinitely dilute solutions, and that my argument against these hypotheses was based upon the general trend of the results, as varying from what would be expected were moderately concentrated solutions to behave like compressed gases, I should like to quote the following paragraph from my paper above cited :—

" I am well aware that the gas equation is supposed to hold strictly only for infinitely dilute solutions, just as it holds only for ideal gases, and that the solutions with which Dieterici worked varied between 0·1 and 1·0 normal, and that those used in experiments detailed above frequently were much stronger than normal. That a normal solution is nevertheless for many of the practical purposes of life a rather dilute solution will hardly be disputed. No one expects the gas equation to hold strictly for a normal solution or even for one considerably more dilute ; but what one has a right to expect from the modern theory of solutions is, that with increasing concentration a solution should behave at least qualitatively as a gas does with increase of pressure. And this requirement is clearly not met, since while all gases behave alike under increase of pressure (so that van der Waals has been able to express their behaviour by means of his well-known equation) solutions, as has been shown, often behave in a manner opposite to that of gases, and this, too, frequently in solutions that cannot be termed concentrated. This demonstrates, then, that the van't Hoff law is at best only approximate and must be applied with great care. As Dieterici well says :—

" ' Raoult hat seine Gesetze der Dampfspannungs- und

Gefrierpunkts-depression durchaus nicht als absolut streng gültige Naturgesetze aufgestellt, sondern als nahezu zutreffende Erfahrungssätze, welche für die Zwecke der Molekulargewichtsbestimmung genau genug sind.'"

The molecular conductivity increases with the volume in very many cases; but in some instances, like that of the caustic alkalies, the molecular conductivity first increases and then decreases with increasing volume. In still other cases, the molecular conductivity diminishes or remains practically constant as the volume increases. The only cases which conform to the requirements of the dissociation theory, even in general trend, are those in which the molecular conductivity increases with the volume. Specific illustrations are given in the article cited above. The molecular conductivity at any volume v is expressed by the equation $\Lambda_v = C \times V$, where C is the specific conductance of the solution, *i. e.* the reciprocal of the resistance of a cube of the solution having an edge of one centimetre, and V is the volume in which one gramme-molecule is contained. Now, as the specific conductance of a solution always diminishes as the solution is diluted, it is clear that in order that Λ_v may increase as the solution is diluted, the increase of the factor V must outweigh the diminution of C. If C diminishes in the same ratio in which V increases, Λ_v remains constant; and if C diminishes more rapidly than V increases, Λ_v will decrease as the volume becomes greater. As already stated, examples of all three cases are well known in practice.

It has been contended that it is the high specific inductive capacity of the solvent that causes electrolytic dissociation. To demonstrate this, there are usually cited a few cases like benzene, chloroform, alcohol, and water, in which solvents, to be sure, the conductivity of solutions increases as the dielectric constant increases. However, in a series of articles published in the 'Journal of Physical Chemistry,' Schlundt and I have pointed out a goodly number of cases where solutions in solvents of low dielectric constant are better conductors than solutions in solvents with higher dielectric constants. Similar instances are constantly multiplying, now that methods of measuring specific inductive capacities have been simplified and perfected. In the face of the fact that numerous exceptions to the Nernst-Thomson rule are known, and that no quantitative relation between the conductivity of a solution and the dielectric constant of its solvent has ever been established, it would appear that the Nernst-Thomson rule is really untenable.

Liquid hydrocyanic acid has a dielectric constant of about

95, whereas the constant for water is about 80. It is known that strong acids conduct exceedingly well in aqueous solution, but it has been shown that in liquid hydrocyanic acid they are very poor conductors[*]. In general, salts conduct much better in water than in liquid hydrocyanic acid. There are some exceptions to this, however; so, for instance, the salts of potassium tested conduct better in liquid hydrocyanic acid than in water. We have here a striking illustration of what is found continually, namely, that whether a solution will conduct electrolytically or not depends upon the specific character of both solute and solvent, and is quite independent of the dielectric constant of the solvent.

The approximately additive properties which some solutions exhibit has frequently been regarded as evidence in favour of the dissociation theory. Additive properties, however, are known to occur where electrolytic dissociation is out of the question, and hence these properties cannot be used as an argument in favour of the Arrhenius hypothesis. In the case of solids, the molecular heat is approximately equal to the sum of the atomic heats; in liquids, insulators as well as electrolytes, the molecular volumes and molecular refractions are approximately equal to the sum of the atomic volumes and atomic refractions respectively, and yet who would assume dissociation in these cases on that account?

If the dissociation theory be held, all the properties of a solution—physical, chemical, and physiological—must be ascribed to the properties of the ions and the undissociated molecules; and if the dissociation be assumed as nearly complete, the properties exhibited by such solution are to be ascribed largely to the properties of the ions. So the colour of aqueous electrolytes has been ascribed to the colour of the ions. Copper ions are blue; nickel ions are green; cobalt ions are red, &c. But I have shown that benzene solutions of copper oleate, nickel oleate, and cobalt oleate are also blue, green, and red respectively, and that these solutions are as good insulators as benzene itself[†]. Furthermore, the cobalt-oleate solution turns blue when heated, and on cooling turns red again, exactly as aqueous conducting solutions of cobaltous salts. It has, therefore, been demonstrated that the colour of solutions is independent of their power to conduct the current, and cannot be used as an argument in favour of the dissociation hypothesis.

The reactions in aqueous solutions so commonly used in

[*] Compare Kahlenberg and Schlundt, *Jour. phys. Chem.* vi. p. 447 (1902).
[†] *Jour. phys. Chem.* vi. p. 1 (1902).

chemical analysis have been regarded as taking place between the ions in the solutions, and the fact that these reactions proceed practically instantaneously in most cases, has been ascribed to the peculiar electric properties which the ions are supposed to possess. But I have shown* that when copper oleate is dissolved in benzene and treated with HCl, PCl_3, $AsCl_3$, $SnCl_4$, or $SiCl_4$, also dissolved in the same solvent, cupric chloride is precipitated in each case exactly as silver chloride is thrown down from an aqueous silver-nitrate solution by soluble chlorides. And furthermore, a slight excess of the precipitant in the case of the benzene solutions mentioned, yields more complete precipitation, just as it does in aqueous solutions. The benzene solutions mentioned are all excellent insulators. It is therefore demonstrated that the instantaneous reactions observed in aqueous electrolytic solutions cannot be ascribed to the fact that these solutions conduct electricity—or, in the language of the dissociation theory, to the presence of free ions—for exactly similar reactions occur in solutions that are the best of insulators. Such views, then, as are presented in Ostwald's 'Scientific Foundations of Analytical Chemistry,' and are echoed by various other writers, cannot be maintained.

According to the dissociation hypothesis, solutions of acids are supposed to owe their characteristic properties to the presence of hydrogen ions. Now it is well known that trichloracetic acid, when dissolved in water, yields an excellent electrolyte, and exhibits the properties of a fairly strong acid. In the course of an investigation† on the electrical conductivity of solutions in sulphocyanates and mustard-oils, I showed that a normal solution of trichloracetic acid in allyl mustard-oil is a poorer conductor than the purest water which Kohlrausch ever prepared in contact with air, and yet this solution attacks dry magnesium rapidly, and decomposes the dry carbonates of sodium and potassium. It is well known that even ordinary distilled water attacks magnesium quite slowly, and of course does not evolve carbon dioxide from alkaline carbonates. However, judging from the conductivity test, water contains more hydrogen ions than the solution of trichloracetic acid in allyl mustard-oil, and ought to show more vigorously the characteristic action of the hydrogen ion.

In physiological investigations, too, the theory of electrolytic dissociation has proved inadequate; so, for instance, it has been found that dilute solutions of acid salts are sourer

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* *Journ. phys. Chem.* vi. p. 1 (1902).
† *Zeit. phys. Chem.* xlvi. p. 64 (1903).

in taste and more toxic towards plants than they ought to be according to their content of hydrogen ions, as determined by the methods in vogue[*].

That the hypothesis of Arrhenius cannot be harmonized with the facts established in the domain of thermal chemistry has repeatedly been emphasized by various writers, and has been well summed up by Reychler in his treatise on physico-chemical theories. The conclusion reached by Reychler is that "above everything else we notice that the hypothesis of free ions is in opposition to thermochemical observations."

It is a well-known fact that the dissociation theory cannot be harmonized with the law of mass action, and that this constitutes a strong argument against the hypothesis. For solutions of weak organic acids in water, to be sure, Ostwald succeeded in computing a series of constants by means of the formula expressing his well-known dilution law, growing out of an attempt to apply the law of mass action to electrolytic dissociation. But an examination of the constants obtained for these weak acids reveals the fact that the agreement is after all only rough; furthermore, the attempt to apply the dilution formula to electrolytes *par excellence* has utterly failed, and yet the advocates of the dissociation theory continue to calculate the so-called solubility product, which frequently involves the application of the law of mass action to strong electrolytes. Attempts made by Lincoln to apply the dilution formula to electrolytes in non-aqueous solutions of about the same order of conductivity as that of some of the solutions of weak organic acids in water, have also proved futile. How can the formula possibly hold in cases where the molecular conductivity remains practically constant, or decreases as the dilution increases? Rudolphi and also van't Hoff have sought to amend the formula by changing the exponents so that constants would result from the observed data in special cases. While thus better "constants" were obtained in these instances, the altered formulæ proved entirely unsatisfactory in other cases; and then, too, it must be added that the alterations made the formulæ simply empirical in character, without any theoretical basis.

The difficulties met with in applying the Nernst formula for calculating the electromotive forces of galvanic cells have been discussed in special papers to which the reader is referred[†]. It may be remarked here, however, that in the work on electromotive forces especially, many of the solutions

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[*] Compare *Jour. phys. Chem.* iv. p. 33 (1900); iv. p. 207; iv. p. 533; iv. p. 553.

[†] *Journ. phys. Chem.* iii. p. 379 (1899); also iv. p. 709 (1900).

used by advocates of the dissociation theory were far from dilute; and, at best, the agreement between calculated and observed values leaves much to be desired.

Many colloidal solutions coagulate when treated with solutions that conduct electrolytically. Attempts have been made to ascribe this coagulative power to the free ions which the electrolyte is supposed to contain. It is true, however, that the power to coagulate colloidal solutions is by no means confined to electrolytic solutions; and hence any one claiming, for example, that sodium-chloride solution coagulates colloids because it is an electrolyte, must assume the burden of proving his proposition.

The theory of electrolytic dissociation is unable to furnish a satisfactory explanation of the phenomena in aqueous solutions, but it is in the realm of non-aqueous solutions that the hypothesis proves especially impotent. When the theory was promulgated but few non-aqueous solutions had been examined as to their power to conduct the current, and those few solutions, in which ether, alcohol, chloroform, and hydrocarbons, for instance, were the solvents employed, were found to be poor conductors. So the idea that only aqueous solutions conduct electricity to any considerable extent gained ground and held sway for a number of years with remarkable tenacity, the view being strengthened by the fact that in the solvents just mentioned molecular weights as determined by boiling-point, or, where possible, by freezing-point methods, were not " abnormally " low, as in the case of solutions of many ordinary salts in water. But now we know of non-aqueous solutions which are as good and even better conductors of electricity than aqueous solutions; and these solutions, when tested by the methods of determining molecular weights just mentioned, yield results that are generally higher than those computed from the formulæ of the solutes; only exceptionally is the reverse the case. In the course of my work on non-aqueous solutions, I have pointed out numerous examples of this; and recently the splendid researches of Walden have established many more. His work on solutions in liquid sulphur dioxide especially shows that in this solvent " abnormally " *high* molecular weights are generally found in the case of solutions of excellent conducting power. The field of non-aqueous solutions is a very large one, and it is here that the theory of electrolytic dissociation can really best be tested. It is here that the fact that the theory is untenable becomes particularly apparent. Whetham's statement[*] that work with non-aqueous solutions is rendered difficult by the

* *L. c.* **p.** 19.

fact that with some solvents like alcohol freezing-point work is impossible, is hardly valid, for such solutions may be readily examined by vapour-tension or boiling-point methods.   It is true that in operating with many non-aqueous solutions, like liquid ammonia, liquid sulphur dioxide, liquid hydrocyanic acid, special difficulties and dangers present themselves; but these are generally quite surmountable, and indeed often it is fully as easy to investigate a non-aqueous solution as one in which water is the solvent.   It simply has not yet become generally fashionable to investigate non-aqueous electrolytic solutions.   When this work is taken in hand by more investigators, the fact that the dissociation theory is useless and misleading will gain ground more rapidly.

It has been argued by some that since the dissociation theory has been developed by use of aqueous solutions, the fact that it cannot explain the phenomena in non-aqueous solutions should not be used against it.   It is well known, however, that defenders of the ionic hypothesis have continually sought to support their views by citing some phases of the behaviour of non-aqueous solutions.   Thus in trying to find facts to support the Nernst-Thomson rule, in comparing chemical activity in electrolytes and non-electrolytes and in considering molecular weights in conducting and nonconducting solutions, the behaviour of non-aqueous solutions has always been compared with that of aqueous solutions. By so doing it has been virtually admitted that it is legitimate to apply the dissociation theory to non-aqueous solutions. It is only since the discovery of excellent electrolytes in solvents other than water, the behaviour of which constitutes a strong argument against the theory, that objections have been made to testing the theory in the field of non-aqueous solutions.

But the dissociation hypothesis has been applied even to electrolytes that are not solutions at all; to fused salts, for instance, to pure solvents, and to gases.   The line of thought in this procedure is that, since the electrolytic conductivity of solutions is assumed to be due to free ions, free ions must occur wherever electrolytic conduction is found.   And so we find the idea established that pure substances dissociate themselves electrolytically without the aid of any solvent whatever. Now since a pure substance may be considered as a solution 100 per cent. strong, it might be well for the advocates of the dissociation theory to pause to consider whether such solutions are sufficiently dilute to warrant the attempt to apply to them a theory which it is admitted holds strictly only for infinitely dilute solutions.   Even conservative advocates

of the theory of electrolytic dissociation who question whether there is electrolytic dissociation in the case of fused salts, are ready to consider that the fact that the determination of the so-called degree of electrolytic dissociation of pure water by conductivity and electromotive-force measurements leads to approximately the same result, constitutes an argument in favour of the theory in question.

The theory of electrolytic dissociation and the van't Hoff theory of solutions have done much toward stimulating inquiry; but the discovery of many new facts has made them entirely untenable. I do not share the opinion that these hypotheses may be retained in a somewhat modified form. The observed facts that speak against them are by far too numerous; and furthermore, any theory of solution which holds strictly only for infinitely dilute solutions cannot be expected to maintain itself, for in reality it does not apply at all to any solutions with which we work in practice.

I have thus briefly rehearsed the reasons why the theories of Arrhenius and van't Hoff are untenable, to me a task far from agreeable, inasmuch as during my student days and for several years afterwards I was quite enthusiastic over these hypotheses, and it was while seeking to establish new experimental facts to further support the theories that contradictory phenomena were continually observed.

I should not like to close these pages without indicating the direction which in my opinion further investigations on solutions must take. All treatises on physics and chemistry seek to draw a sharp distinction between the processes of solution and chemical action; the former is commonly described as purely physical in character, the latter as more deep seated and as caused by a specific attraction termed chemical affinity. It is well known that some substances react chemically (using that term in its usual sense) with each other, and again others do not; and that the rate of interaction and the final condition of equilibrium reached are subject to influences of temperature especially, but also to pressure and concentration. The same may be stated of the process of solution. Some substances dissolve each other, others do not; and again, the process, like that of chemical action, is subject to influences of temperature, pressure, and concentration. Energy changes, such as thermal changes, electrical changes, expansions and contractions, and evolutions of light, are known to accompany the processes of solution as well as those of chemical action. With the exception of the mass, the properties of chemical compounds are never exactly equal to the sum of the properties of the

constituents entering into them ; the same is true of solutions. Furthermore, those properties which are approximately additive in character in the case of chemical compounds are also approximately additive in the case of solutions. Examples of such properties are the molecular volume, molecular refractive power, and molecular heat. Again, other properties like, for example, the optical rotatory power, the specific inductive capacity, which are not additive in chemical compounds, are also not additive in the case of solutions. In fact, the only distinction between a solution and a chemical compound is that the latter conforms to the laws of definite and multiple proportions and the former does not. But it must be borne in mind that so-called pure chemical compounds are obtained only by subjecting the total reaction products to certain purifying processes, such as evaporation, distillation, sublimation, crystallization, washing, extraction, &c. All of these processes put the reaction products, which are in general solutions (using that word in its broader sense) under special duress, as a result of which there are obtained as cleavage pieces of these solutions, as it were, phases whose composition remains constant through a greater or lesser range of temperature, pressure, and contact with other phases. *The processes of solution and chemical action are then identical in character, and chemical compounds are merely the cleavage pieces of solutions placed under special stress or duress represented by the so-called purifying processes.* The process of solution is thus the general case of the interaction of bodies, union resulting when the specific attraction, commonly called chemical affinity, between them is under the existing conditions sufficient to cause an interpenetration, a fusion or blending of their masses as it were. Furthermore, adhesion, absorption, adsorption, and imbibition are also due to the same specific attraction which causes solution and chemical action. Adhesion is really to be regarded as an unsuccessful attempt at solution. Thomas Graham was unquestionably right when he stated that from adhesion to solution and chemical action there is every stage of gradation. It is scarcely necessary to add that these views were also entertained by numerous other scientists of note, among whom Bunsen was especially prominent. The truth of this view will force itself upon the mind of any one who seeks to work out the problem of solutions in the laboratory rather than at the writing-desk. It needs hardly to be emphasized that the recognition of the importance of the laws of definite and multiple proportions, which have been found to hold in so many cases of phases that maintain their composition for

considerable ranges of alterations of temperature, pressure, and nature of co-existing phases, is not at all affected by the above considerations. The discovery of these laws has been of inestimable value in analytical and synthetic work and in systematizing our knowledge; but it is a mistake to think that they necessitate the conclusion that the processes of solution and chemical action are different in character and are caused by different agencies. The fact that the quantities of similar solutes which must be added to equal quantities of a given solvent to produce solutions of the same vapour-tension, are to each other as the quantities in which such solutes unite with other substances to form stereotyped chemical compounds, constitutes a strong argument for regarding the processes of solution and chemical action as identical in character.

It thus becomes evident that in investigating solutions we must begin with the most concentrated and end with the most dilute; the latter will appear simply as a limiting case. So, for instance, the change of the vapour-tension of solutions with the concentration must be studied from the strongest solutions obtainable to the dilutest that can still be measured throughout the range of temperatures at which the solutions can exist at all. And this work must be done for a large number of solutes in a large number of solvents. Such data being at hand, the equations expressing the changes of vapour-tension with temperature and concentration may be written. Work of this character has only barely begun; but judging from the results at hand, analogous substances will exhibit similar behaviour; and though it is not to be expected that one equation will serve for all solutions, similarities between the equations holding for different solutions will not be lacking.

According to the views here advanced—or, I had better say, revived—the molecular weights of, for instance, sodium and potassium salts in aqueous solutions as determined by the diminution of the vapour-tension, elevation of the boiling-point, and lowering of the freezing-point, are not abnormally low because these salts are dissociated, but rather because they have great affinity for water. On the other hand, these salts have less affinity for liquid sulphur dioxide, as is indicated by the relatively high vapour-tension of the solutions and the correspondingly high molecular weights, though the solutions conduct well nevertheless. As an example of the other extreme, the molecular weights of colloids in water are not enormous; the results come out high simply because the affinity between water and these colloids is relatively slight. Furthermore, it is also easy to see why colloids are

readily coagulated by substances which lower the vapour-tension greatly, and that their coagulating power stands in relation to their ability to lower the vapour-tension ; but this coagulating power has nothing to do with electrolytic conduction. From the vapour-tension of a solution no one can predict whether that solution will conduct electricity or not, much less tell how well it will conduct. This statement is fully warranted by the facts as known at present.

Many salts are very good electrolytes when in the molten state ; others are poor conductors, and still others are insulators. From the views just advanced, it is clear that it is perfectly correct to consider electrolytic solutions and fused electrolytes together : for conducting solutions are electrolytes which are in a fused condition at ordinary temperatures. Faraday selected the experiment in which a current is passed through fused silver chloride between two silver electrodes as representing the simplest case of electrolysis ; for here there are but two chemical elements involved, and yet all the phenomena accompanying electrolysis are here exhibited. Who would question the correctness of this view? The theory of electrolytic dissociation has led Whetham to consider the case of fused electrolytes as entirely different from that of electrolytic solutions, a position which is clearly not warranted by experimental facts.

Take water and pure acetic acid. Both are very poor conductors. Add a little water to some of the acid ; and again add a little acid to some of the water. In both cases the resulting solutions conduct. Now is it the acid that conducts in the water, or is it the water that conducts in the acid ? Years ago it was customary to say that we acidulate the water to make it conduct better ; now it is fashionable to say that it is the acid which in the dissociated state conducts in the water, the latter serving merely as a medium in which the ions of the acid may exist and migrate. Furthermore, the original conductivity of the pure water is frequently deducted from that of the solution, in order to determine the conductivity of the acid. As a matter of fact, however, it is the combination of the acetic acid and water that conducts. The water has just as much to do with the conduction as the acid, and it is absurd to subtract either the conductivity of the water or that of the acid from the conductivity of the solution. But the calculation of the molecular conductivity is founded upon the supposition that the solvent plays no part in the conduction. This supposition being untenable, it follows that the comparison of the molecular conductivities at different concentrations is a process which is quite

unjustifiable. This becomes particularly apparent from the researches of Patten [*], and recently C. J. Reed [†] has presented the subject in a very able manner before the American Electrochemical Society. The Kohlrausch law of the additive character of the molecular conductivity of solutions at very high dilutions holds at best only for a limited class of aqueous solutions : it is, of course, founded upon the calculations of molecular conductivity, which, it has been pointed out, are objectionable. The whole matter of the calculation of so-called molecular conductivities of solutions at different concentrations being open to serious objection, it is not at all surprising that such a relation between electrical conductivity of solutions and their freezing-points or boiling-points as is claimed by the theory of electrolytic dissociation does not exist.

Faraday continually emphasized the view that in electrolysis the products do not appear at the electrodes because they are attracted there by the latter. His idea was that the products are ejected at the electrodes by the electrical energy acting as an axis of force driving the products out in the two directions, the electrodes serving simply as doors. In a special paper on electrochemical theories, to which the reader is referred [‡], I have entered somewhat more fully into a consideration of theories of electrolysis.

During the past year I have been engaged in osmotic experiments, which included both qualitative investigations and quantitative measurements of osmotic pressures. The latter determinations were made with semi-permeable membranes. The details of the apparatus used, and the results obtained, will soon be published in a special paper. Here I would like to state, however, that it was found that whether osmosis will take place or not depends on the nature of the membrane and the liquids bathing it; and the direction and magnitude of the osmotic pressure is also dependent upon the nature of the membrane and the liquids in contact with it. The quantitative measurements of osmotic pressure made with semi-permeable membranes yielded results which are not such as the gas laws require. It is especially noteworthy that cases have been found in which at constant temperature the osmotic pressure diminishes very rapidly as the solution is diluted, becoming practically *nil* for solutions about one-twentieth normal. The results of the osmotic experiments are readily explained by the view that the membrane imbibes or unites with the solvent, because of the affinity existing between them,

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[*] *Journ. phys. Chem.* vi. p. 554 (1904).
[†] Trans. Amer. Electrochem. Soc. v. p. 103 (1904).
[‡] Trans. Amer. Electrochem. Soc. i. p. 119 (1902).

and then the solution extracts solvent from the membrane. The latter part of the process can only take place when the affinity of the solution for additional solvent exceeds that of the saturated membrane for solvent. It is clear that, as the solution is diluted, a strength will finally be reached where the affinity of the solution for additional solvent is just equal to that between membrane and solvent, and then osmosis will stop. *i. e.*, the osmotic pressure will become *nil*. This critical strength of the solution varies with the nature of the membrane and that of the solution; but in general it is far from infinitely dilute, being quite within the pale of what is readily measurable. The classical copper ferrocyanide membrane, being a colloid, has but slight affinity for water, which is readily extracted from it by a fairly dilute solution of sugar. Thus osmotic pressure so-called is due to the same affinity which causes adhesion, imbibition, absorption, adsorption, solution, and chemical action.

Of late years work in physical chemistry has been largely directed to the study of the effects of temperature, pressure, and concentration on the progress of chemical reactions; and properly so. It has at times been forgotten, however, that affinity must exist before union can take place; and that temperature, pressure, and concentration are simply to be regarded as modifying factors aiding or retarding the tendency of affinity. By some it is even deemed as somewhat old-fashioned to talk about affinity. But upon the distinct recognition of affinity, and a careful investigation of the laws governing it, clearly depends the future progress of chemistry, physical chemistry, and physiology. We must learn to measure affinity quantitatively; but in order to take into account the phenomena of morphology as they confront us in crystals, but particularly in living beings, we need to learn to study the direction as well as the strength with which affinity acts under given conditions. Finally, the question why certain solutions, molten salts, &c., conduct electricity and others do not, will probably not be answered until we can tell why a stick of silver conducts electricity and a stick of sulphur does not. These questions really involve a better understanding of the relation between electricity and gross matter, a problem which is apparently being attacked with promise by **J. J. Thomson** and his co-workers. Until we have more light on this subject, we can hardly hope for very material improvements of our views of the nature of electrolytic processes.

Laboratory of Physical Chemistry,
University of Wisconsin, Madison,
August, 1904.

XXII. *On Secondary Radiation.* By J. A. McClelland, *M.A., Professor of Experimental Physics, University College, Dublin* *.

THE following paper gives an account of an investigation of the secondary radiation given off by bodies when they are exposed to a strong primary radiation from radium.

In the present paper, the secondary radiation produced by the more penetrating radium rays only—the β and γ rays—has been considered ; the nature of this secondary radiation and its relative intensity in the case of different substances have been studied. The subject of secondary radiation seems of some importance and promise, as there are many problems on which it may have a possible bearing ; it is closely related to the subject of spontaneous radioactivity and to all phenomena produced by the discharge from bodies of particles charged with electricity.

Secondary radiation under the action of cathode rays has been investigated by Starke †, Austin and Starke ‡, Swinton §, and others. Secondary radiation under the action of Röntgen rays has been studied by Perrin ‖, Townsend¶ , and others ; while secondary radiation due to radium rays has been detected and investigated to some extent by Becquerel ** and others.††

Apparatus.

The secondary radiation was detected and measured by its ionizing power. Fig. 1 gives a sketch of the apparatus used. T is a brass tube 20 cms. in length and 3·2 cms. internal diameter, connected to small storage-cells by which it could be kept at any required potential ; a metal rod is fixed along the axis of the tube, being insulated by paraffin and joined to a Dolezalek electrometer, the joining wire being suitably

* Communicated by the Author. From Royal Dublin Society's Transactions, 1904 : read Dec. 20th, 1904.

† Wied. *Ann.* vol. lxvi. p. 49 (1898) ; Drude's *Ann.* vol. iii. p. 75 (1900).

‡ Drude, *Ann.* vol. ix. p. 271 (1902).

§ Proc. Roy. Soc. vol. lxiv. p. 377 (1899).

‖ *Annales de Chimie et de Physique,* 1897, p. 496.

¶ Proc. Camb. Phil. Soc. 1899, p. 217.

** *Comptes Rendus,* 1901, pp. 371, 734, 1286.

†† When the work described in this paper was completed and the paper partly written, one on the same subject by Mr. Eve appeared in the Philosophical Magazine, Dec. 1904 ; the points discussed in the two papers are not, however, by any means always the same. In cases where the same points are discussed the agreement is, on the whole, good, although in some cases somewhat different deductions have been made.

protected. The radium is placed at R so as to send a pencil of rays through a hole in a thick lead screen, other lead screens being placed to protect the tube T from direct radiation.

Fig. 1.
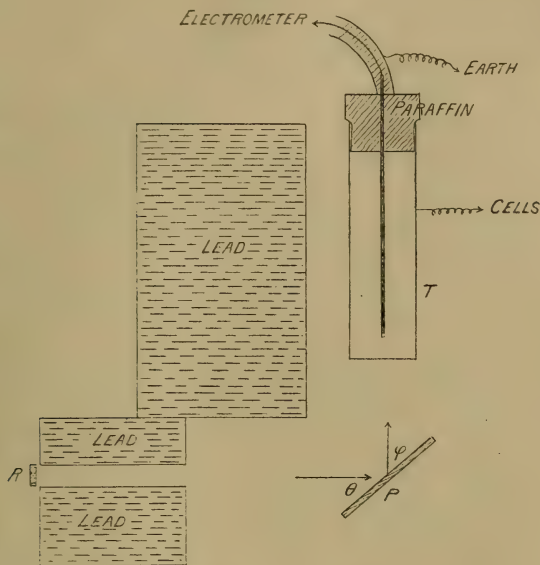
A plate of the substance under examination is placed at P, and the secondary radiation from it enters the tube T and produces ionization, which is measured in the usual way by the electrometer. The end of the tube T is covered with a single sheet of tinfoil. The distance from the radium to the plate P was in most of the experiments about 26 cms., and from P to the nearest point of T was usually about 9 cms. The distances are given in every case where they are of importance.

In every experiment the ionization observed in T is corrected for whatever small conductivity existed between the tube T and its inner terminal when the plate P is not in position; this small conductivity being due partly to the normal ionization inside T, and partly to insufficient screening of direct radiation from the radium, and also to secondary radiation from the air traversed by the primary rays.

Fifty milligrams of radium bromide were used, enclosed in a vessel which stopped the α radiation.

Effect of the Position of the Plate.

With the apparatus described it is easy to detect a secondary radiation which travels in all directions from the part of the plate struck by the primary rays. The first point investigated was whether the amount of secondary radiation depended on the angle of incidence of the primary rays, and on the angle that the testing-tube T made with the plate ; whether, in fact, there was anything of the nature of " reflexion " of the primary rays. This is a point which has been investigated by several observers using cathode rays as the primary rays; and while some observers have found a well-marked maximum of secondary radiation in the direction obeying the law of reflexion, others have not obtained such a result.

The point was first tested by keeping the direction of the primary rays and that of the tube T constant and at right angles, as in the figure, while the lead plate P was tilted ; and this experiment showed a well-marked maximum when θ was 45°.

The following observations were made, the secondary radiation being expressed in an arbitrary scale.

	Secondary Radiation.
$\theta = 22\frac{1}{2}°$	73
$\theta = 45°$	100
$\theta = 67\frac{1}{2}°$	71

The observed numbers are reduced so as to express the maximum by 100.

The plate P was then set so that ϕ was 45°, and this angle was kept constant, the direction of the primary rays only being changed. We then get as follows :—

	Secondary Radiation.
$\theta = 25°$	93
$\theta = 45°$	100
$\theta = 65°$	97
$\theta = 80°$	92

In the third case, the primary rays and the plate are both tilted, keeping the angle between them constant and equal to 45°, while the angle ϕ changes. We have then :—

	Secondary Radiation.
$\phi = 20°$	84
$\phi = 45°$	100
$\phi = 70°$	98

The first experiment shows a decided maximum when

$\theta = 45°$. The other two experiments can be interpreted as showing a maximum when the law of reflexion is fulfilled, with the secondary radiation falling off unequally on the two sides of the maximum position.

The numbers indicate that, superposed on an effect of the nature of reflexion, we have other effects ; the law of reflexion being fulfilled, we get more secondary radiation the more we approach the normal, which is to be expected, as then the radiation coming from some depth in the plate has a smaller thickness to penetrate.

The general effect is therefore a radiation in all directions from the part of the plate struck by the primary rays, with a decided maximum in one direction, due probably to a sort of reflexion of the impinging particles. Many points described later show that the whole effect is not due to a simple scattering of the primary rays, but that it consists of a true secondary radiation from the plate.

It may also be stated here that experiments with a magnetic field, to be described later, show that the secondary rays we are now dealing with consist of negatively charged particles. The rays we are now dealing with travel through several centimetres of air, and also penetrate the tinfoil covering the end of the ionization-tube T before they produce the effect by which they are detected ; there may be also a more easily absorbed radiation from the plate, which has not been studied in the present paper.

Secondary Radiation not merely a Surface Effect.

The secondary rays are not produced merely at the surface of the plate struck by the primary rays ; it is easy to show that they come from all parts of a layer of considerable depth, the depth depending on the substance of which the plate is composed.

This point was tested by using for the plate in fig. 1 a single sheet of tinfoil (about ·013 mm. thick) in the first place, then two sheets, and so on. The secondary radiation was as follows :—

Sheets of Foil.	Secondary Radiation.
1	13
2	24
4	44·5
8	60·5
16	73·5
32	77

Fig. 2 is plotted from these numbers.

Fig. 2.
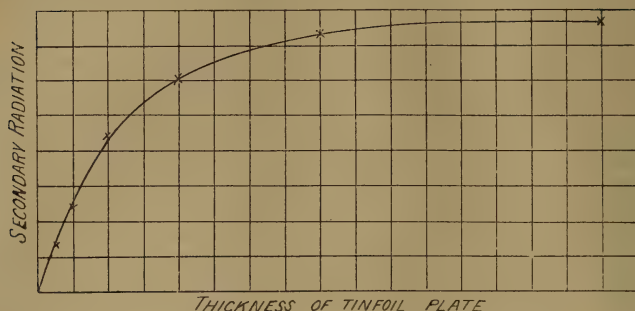
We should of course expect the secondary rays to come from all depths reached by the primary, unless the penetrating power of the secondary be less than that of the primary, in which case the effective thickness of plate will be determined by the penetrating power of the secondary.

The penetrating power of the secondary rays will be discussed later, but it may be pointed out here that the above numbers indicate that the secondary rays must at least approximate in penetrating power to the primary. The second sheet of tinfoil contributes almost as much secondary radiation as the first, and the rays from the second sheet must pass through the first before reaching the testing-tube T. The secondary rays must therefore suffer no very great loss of intensity in passing through the sheet of foil.

Analysing the Primary Radiation (β and γ rays) with respect to its power of producing Secondary Rays.

The primary rays used in the above experiments were a pencil of β and γ rays : it is necessary to investigate what part of the secondary rays is due to the more penetrating part of the primary and what is due to the less penetrating primary rays.

This was done by making two separate experiments. In the first the apparatus was used as in fig. 1, and successive layers of tinfoil were placed in front of the lead screen so that the primary rays passed through them, and the intensity of the secondary rays was measured at each step.

In the second experiment, the radium with the thick lead screen was placed as in fig. 4, so that a pencil of primary rays fell directly on the testing-tube T travelling parallel to its axis, and as before sheets of tinfoil were placed in front

of the radium and the ionization measured at each stage. We thus get from one experiment the intensity of the primary rays after they have passed through various thicknesses of foil; and from the other experiment we get a measure of the secondary rays produced by the primary after passing through the same thickness of foil as before.

The results are as follows, the primary and secondary radiation being expressed so as to have a maximum of 100 in each case.

Sheets of foil.	Primary Radiation.	Secondary Radiation.
0	100	100
1	61	56
2	47·5	41
4	40	31
8	33	19
16	27·5	10
32	23	7

Fig. 3.
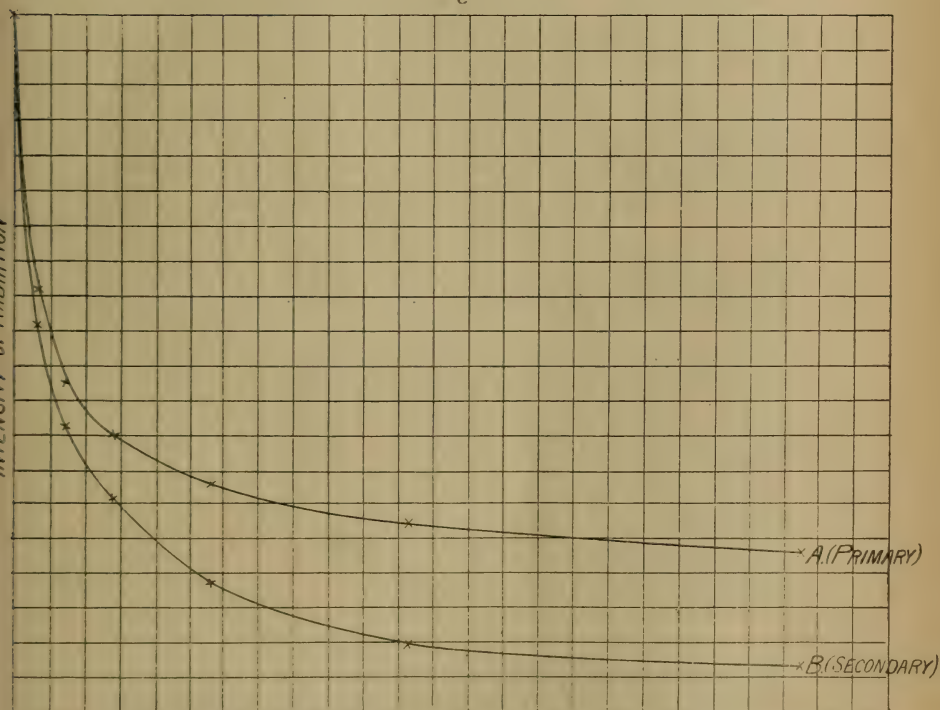
The curves in fig. 3 are plotted from the above numbers. The curves show at a glance the relative efficiency of the

various parts of the primary pencil in producing secondary radiation. The less penetrating β rays are more efficient than the more penetrating, and the efficiency falls off more and more as we get into that part of the curve A corresponding to the γ rays. But even when the curve A has become practically horizontal—that is, when there is nothing left in the primary pencil but γ rays—there is still some secondary radiation.

The magnetic experiments described later show that this part of the secondary radiation—the part produced by γ rays—also consists of negatively charged particles.

Different Substances as sources of Secondary Rays.

The experiments described in this paper have all been carried out (unless when otherwise stated) with a lead plate as the source of secondary rays.

A large number of substances have been tested as sources of secondary rays. The substances were placed in the form of plates at the position P, fig. 1. The experiment described above in which tinfoil was used, shows that it is important when comparing substances to use in each case a thickness sufficient to give the maximum effect ; this was done when comparing different substances.

The following list shows the results obtained, the radiation from lead being taken as 100.

Substance.	Secondary Radiation.	Density.	Atomic weight.	Secondary Radiation / Atomic weight
Lead	100	11·4	206·4	·48
Platinum	92	21·5	194·3	·47
Tin	80	7·3	118·8	·67
Brass.................	75	8·4		
Zinc	65	7·2	65	1·00
Copper	64	8·9	63·2	1·01
Iron	62	7·8	56	1·10
Glass.................	31	2·5		
Aluminium	33	2·7	27	1·22
Ebonite...............	27	1·15		
Cardboard	23	·80		
Wood	19	·52		
Paraffin	18	·90		

The numbers show that, taken generally, the substances of greater density produce the greater secondary radiation; but there are many exceptions to the rule, and the secondary radiation is by no means proportional to the density. Platinum,

for example, produces less radiation than lead, and tin more than copper.

It gives a more interesting result to compare the secondary radiation with the atomic weight of the substance: the atomic weight and the ratio of the secondary radiation to the atomic weight is given in the table. It will be seen that throughout the table the greater the atomic weight the greater is the secondary radiation. The ratio is not constant, but it will be noticed that the substances fall into groups with respect to this ratio. There is a well-marked group for which the ratio is unity, and another for which the ratio is one-half. Of course, the absolute value of the ratio is merely a question of the scale in which the secondary radiation is expressed.

It is intended to largely extend the list of substances examined, to see if the dependence of the radiation on the atomic weight holds throughout, and to follow out the apparent grouping. The radioactive substances, radium, thorium, and uranium, are all substances of high atomic weight, and the results now before us indicate that the greater the atomic weight, the greater is the secondary radiation under the action of β rays.

The secondary radiation seems therefore to be closely allied to the spontaneous radiation from radioactive substances. The impact of the β rays produces a disturbance sufficient to cause the disintegration of the atom in substances which in the normal state are in stable equilibrium.

The dependence of the secondary radiation on the atomic weight and the close relation between it and radioactivity, suggests that the activity of radium might be increased by the action of its own radiations ; and Voller (*Physikalische Zeitschrift*, Dec. 1, 1904) has obtained indications of such an action. We must remember, however, that the secondary radiation we are considering is one of β particles, so that in extending our table to radium we should consider a transformation product of radium which emits β particles.

It is interesting to compare the table given above with the results of observers who have studied the apparent small radioactivity of ordinary substances, when there is no artificially applied exciting cause. McLennan found that lead was about twice as active as zinc, while tin gave an intermediate value ; this is the same order as in our table of substances. A comparison with numbers given by Strutt for the activity of a number of substances does not, however, show much agreement.

In the above comparison of different substances, the secondary rays passed through a single sheet of tinfoil, ·013 mm.

thick ; another set of observations was taken with the end of the tube T covered with a sheet of aluminium ·3 mm. thick in addition to the tinfoil. No difference in the order of the substances took place, but if we again express the secondary radiation from lead as 100, the corresponding number for each substance is less than before when the rays passed through tinfoil only; the relative diminution is different for different substances, and is particularly great for ebonite, cardboard, and paraffin. In other words, of all the substances examined the secondary rays from lead are the most penetrating, those from paraffin, cardboard, and ebonite the least penetrating. The difference in penetrating power of the rays from the different substances is not very marked.

A rather rough experiment was tried to see if the temperature of the substance had any effect on its power of giving off this secondary radiation. A plate of copper was used, and its temperature increased by a strong bunsen-flame playing on the back of the plate ; no change in the amount of secondary radiation was noticed. It should also be stated that no particular care was taken with the preparation of the surface of the different plates used ; the surface was always well-cleaned, but the degree of polish was not attended to and did not seem to be of any importance.

As all numbers in this paper are given in arbitrary units, it may be useful to state a result specifically to give an idea of the absolute magnitude of the radiation we are dealing with ; 50 milligrams of radium bromide are placed 26 cms. from a lead plate as in fig. 1, the rays from the radium passing through a hole 1·2 cms. in diameter in a lead screen ; the lead plate is 9 cm. from the ionization-tube T ; the electrometer gives a deflexion of 2000 scale-divisions per volt, and the total capacity of the system charged is ·001 microfarad ; the secondary radiation then produces a deflexion of 115 scale-divisions per minute.

Relative Importance of Secondary β rays.
Penetrating Power of Secondary β rays.

In making some experiments on the penetrating power of the secondary β rays, and comparing it with that of the primary rays, some results were obtained which show that very different values may be deduced for the absorption of β rays according to the apparatus used, and which also show very forcibly the importance of secondary effects when dealing with this type of radiation. Suppose we place the radium so that a pencil of β (and γ) rays falls directly on

the ionization-tube T, as in fig. 4. If now we proceed to measure the absorption produced by say a screen of cardboard, the result we get depends very much on where we place the cardboard in the path of the rays.

Fig. 4.
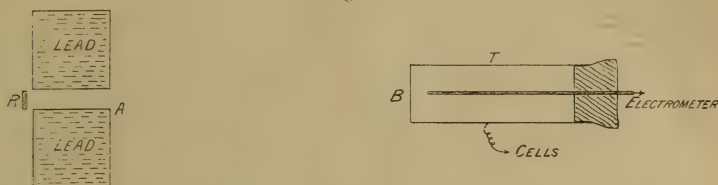
In an experiment the distance AB, fig. 4, was 26 cms., the internal diameter of the tube T being 3·2 cms. The result was as follows :—

	Ionization in T.
No screen	100
Screen at B	71
Screen 5·5 cms. from B	62
Screen 10·5 cms. from B	54
Screen 26·0 cms. from B	36

The screen produces a different effect in different positions, because it is a source of a radiation which travels in all directions from the part struck by the primary rays; and therefore the nearer the screen is to the testing-vessel T the greater is the observed ionization. If the secondary rays were of much less penetrating power than the primary, the observed ionization would of course be increased by moving the screen towards the vessel T; but we have seen there is no very great difference in penetrating power between the primary and secondary rays, so that the chief cause of the effect noted must be the fact that the screen becomes a source of radiation spreading out in all directions. Of course, we have also tertiary radiation from the air and from the end of the testing-tube.

The numbers given above show that after passing through the sheet of cardboard, the secondary rays are of as great importance as the primary rays, for the ionization with the screen close to the testing-vessel is twice what it is with the screen 26 cms. away.

Another example of the importance of the secondary rays is given by results obtained in comparing the penetrating power of the secondary rays with that of the primary pencil consisting chiefly of β rays.

The radium was first placed as in fig. 4, and the primary

rays examined by placing different thicknesses of tinfoil first at B and then at A. The apparatus was then arranged as in fig. 1, and the absorption of the secondary rays measured by placing sheets of tinfoil at the end of the tube T.

The results are as follows :—

Sheets of foil.	Primary rays.		Secondary rays.
Each ·013 mm. thick.	Tinfoil 26 cms. from ionization-tube T.	Tinfoil at ionization-tube T.	Tinfoil at ionization-tube T.
No foil	100	100	100
1	60	96	—
2	47	90	80
4	39	77	64
8	32	66	44
16	27	49	26
32	22	35	9

Fig. 5.
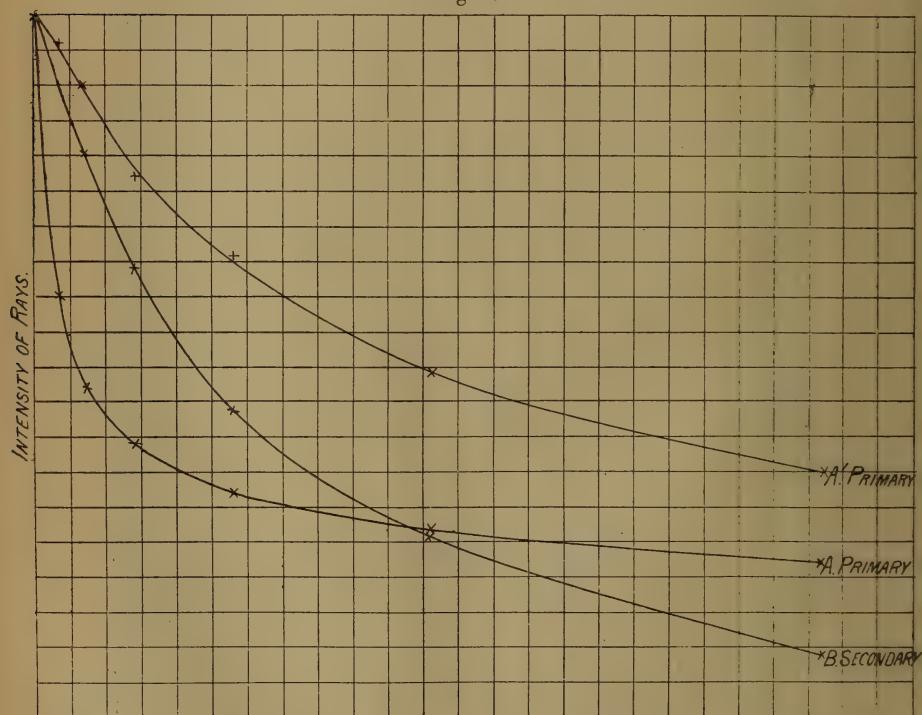
Curve A, fig. 5, refers to the primary rays when the absorbing tinfoil is placed 26 cms. from the ionization-tube :

Curve A′ to the primary rays with the foil placed close to the ionization-tube; Curve B to the secondary rays. Curves A and A′ show the importance of the part played by the secondary rays, and how they may influence the value calculated for the coefficient of absorption. Comparing Curves A′ and B, in both of which the foil is placed close to the ionization-tube, the secondary rays appear to be more easily absorbed than the primary; but the conditions of experiments are not quite the same in the two cases, as the secondary rays are not confined to a narrow pencil as in the case of the primary, and therefore the relative position of the absorbing tinfoil is not so important. The difference in penetrating power is probably, therefore, not so great as a comparison of the Curves A′ and B would suggest.

It will be observed that the primary Curves A and A′ become practically horizontal, which is of course due to the presence in the primary pencil of the very penetrating γ rays. The secondary curve B, on the other hand, continues to descend with increasing thicknesses of foil; this is because the secondary pencil consists entirely of β rays. The magnetic experiments showed that it was possible to deflect all the secondary rays. The experiments, therefore, did not detect any γ rays produced by the stoppage of the primary β particles, or by the starting of the secondary β particles; such rays may have existed in small amount and escaped detection, as we must remember that in every case a correction has to be made for the small normal ionization in the tube T.

The importance of the secondary rays in the above experiments suggest a more detailed consideration of the manner in which matter is penetrated by a stream of charged particles like β rays; whether, in fact, there is great penetration by the primary particles, or whether it is not a successive stopping of one set and starting of another set of particles. The secondary rays from the above experiments are not very different in penetrating power from the rays that produce them, and the magnetic experiments described below indicate that the secondary rays contain some even more penetrating than the primary. The point is under further consideration.

Magnetic Deflexion of the Secondary Rays.

The nature of the secondary rays has been examined by passing them through a strong magnetic field. The apparatus (fig. 1, p. 231) was modified by interposing between the plate P and the ionization-vessel T the poles of an electromagnet so

that the rays had to pass for a distance of 7·7 cms. through a
horizontal magnetic field ; the total distance from plate P to
vessel T was 15 cms. If the secondary rays consist of charged
particles, they should be deflected up or down and thrown off
the end of the tube T. It was found that as the magnetic field
was gradually increased the ionization in T gradually dimi-
nished, reaching a minimum which was not reduced by a
further increase of the field.   When this stage was reached,
if the poles were allowed to come close together, thus inter-
posing in the path of the rays a thickness of 7·7 cms. of iron,
no further reduction of the ionization took place, showing that
this residual ionization was not due to radiation from P, but
simply to the normal ionization in T, perhaps increased by
some direct radiation from the radium.   This residual ioni-
zation was therefore subtracted from that observed at each
stage of increase of the magnetic field, and the remainder
represented the effect due to the radiation from the lead
plate P.

The result is shown in column 2 of the following table :—

Magnetic Field.	Secondary β Rays.	Primary β Rays.
No Field	100	100
160	73	81
310	52	39
460	23	12
600	16	4
960	7	0
1250	0	0
2300	0	0

It was easy to show, by placing a thick lead screen in the
path of the rays so as to cut off the upper or lower half of the
pencil falling on the tube T, and then applying a magnetic
field first in one direction and then in the opposite, that the
direction of deflexion of the rays showed them to be nega-
tively charged.   The fact that the residual ionization with
strong magnetic fields was not diminished by interposing
several centims. of iron, showed that there was no non-
deflectable radiation of the nature of γ rays, or at least that
it was too small to be detected by the apparatus used.

The primary β rays were examined in the same way, so as
to get a comparison of the velocities of the primary and
secondary, the mass and the charge being no doubt the same
in the two cases.   The radium with the accompanying lead
screen was placed so that the primary rays fell directly on
the tube T, passing between the poles of the magnet on the
way.   In this case a correction had to be applied for the

γ rays which could not be deflected, and for the normal ionization in T.

The result is given in column 3 of the above table; the maximum when no magnetic field is applied is represented by 100 for both primary and secondary rays. The numbers show that while the first small field removes more secondary than primary, it requires a stronger field to remove completely the secondary. The secondary pencil seems to include some which are travelling with a greater velocity than any of the primary.

XXIII. *On the Theory of Electric Inertia.* By S. H. BURBURY, *F.R.S.*[*]

1. SIR O. LODGE, in his address on Electrons read before the Institution of Electrical Engineers, 1903, calculates the inertia which a small sphere, charged with electricity e, and moving with velocity u, has by virtue of its charge as a function of the charge. Inertia may be defined as resistance to acceleration. If the sphere have mass m in the ordinary sense of that term, its kinetic energy if without charge is $E = \frac{1}{2}mu^2$, and its inertia if it be moving freely is $2E/u^2 = m$. But when charged it has electric energy in addition to E, namely,

$$E' = \frac{\mu}{8\pi} \int H^2 d\tau,$$

where H is the magnetic force in the volume element $d\tau$ due to the motion of the sphere, μ is a known constant, which I shall treat as unity, and the integration is throughout all space outside of the moving sphere. Further, he takes $H = ue\dfrac{\sin\theta}{r^2}$, in which r is the distance from S, the present position of the sphere, to the point P where the element $d\tau$ is, and θ is the angle between SP and the axis, that is the direction of motion of the sphere. Substituting this value for H, he obtains

$$E' = \frac{e^2 u^2}{4} \int_c^{\infty} dr \int_0^{\pi} d\theta \frac{\sin^3\theta}{r^2},$$

and the additional inertia due to the sphere being charged is in this theory $\dfrac{2E'}{u^2}$, which is, as thus expressed, a function of e and the radius c of the sphere.

2. The value of H here given is not generally exact,

R 2

because owing to the finite velocity of light, which shall be denoted by v, r and θ relate strictly, not to the present position, S, of the sphere, but to the position, S′, in which it was t seconds ago. And S′ and t are determined by the conditions S′P $=vt$, and S′S $=ut$ if u be constant, or S′S $=\displaystyle\int_{0}^{t} u\,dt$ if u be variable. In like manner, if e be supposed variable with the time, the value of e in the expression $\dfrac{eu\sin\theta}{r^2}$ is the value of e when the sphere was at S′, but I shall not further consider the possible variation of e. If, as is generally the case, u be very small in comparison with v, the result given is a very near approximation, unless the rate of time variation of u be very great.

3. It is noteworthy, however, that if u be constant, being less than v (which is the case considered by Lodge), the result given is exact. For, S being the present position of the sphere, let S_t be its position t seconds ago, S_1 its position $t-dt$ seconds ago, S_2 its position $t-2dt$ seconds ago, and so on. About each of the points S , S_1, S_2, &c. as centres describe spherical surfaces having radii, vt for S_t, $v(t-dt)$ for S_1, $v(t-2dt)$ for S_2, &c. Then, since u is less than v, no two of these surfaces intersect each other. The sphere described about S_t includes that described about S_1, and so on. Then the magnetic force at P at this instant due to the motion of the charged sphere through S_t t seconds ago is H $=ue\dfrac{\sin\theta}{r^2}$. The normal distance between the two successive spherical surfaces described about S_t and S_1 is at the point (r, θ) is $(v-u\cos\theta)dt$. And the integral of H$^2/8\pi$ throughout the spherical shell between these surfaces is

$$dt\,\frac{e^2u^2}{8\pi}\int_{0}^{\pi}\frac{2\pi r^2\sin^3\theta\,(v-u\cos\theta)\,d\theta}{r^4},$$

that is

$$dt\,\frac{e^2u^2}{4}\int_{0}^{\pi}\frac{\sin^3\theta\,(v-u\cos\theta)\,d\theta}{v^2t^2}.$$

The term in $\cos\theta$ disappears in the integration, so when we integrate for t the result is

$$\frac{e^2u^2}{4}\int_{\frac{c}{v}}^{\infty}dt\int_{0}^{\pi}d\theta\,\frac{\sin^3\theta}{vt^2},$$

which agrees with Lodge's integral.

4. If, however, u be variable, we must put u^2 under the

sign of integration for t. And since u may in this case be written

$$u = u_0 - t\frac{du}{dt} + \tfrac{1}{2}t^2\frac{d^2u}{dt^2} - \&c.,$$

the result must contain not only $-e$ but also u_0, $\dfrac{du}{dt}$, &c. It may, indeed, still be a very near approximation, but whether it is so or not will depend on the rate at which u varies with the time. The electric inertia calculated by this method cannot in general be completely expressed as a function of e only as proposed.

5. Lodge subsequently employs another method of calculating the inertia, the method, namely, of *the magnetic reactions*. This rests on the principle that any change of velocity, as ∂u, impressed on the moving sphere while at S, and also the change of its position while moving at S with the velocity u, gives rise at any point P in space after the interval of time SP/v to a change in the magnetic force H at P. And this change of magnetic force gives rise after the further interval of time $\dfrac{\mathrm{PS}'}{v}$ to an electric force at another point S′. And if S′ be suitably chosen on the axis, this electric force will take effect at S′ at the instant when the moving sphere arrives at S′, and so affect the sphere.

6. S′ being chosen to satisfy this condition, let us describe an ellipse with S and S′ for foci and $vt = \tfrac{1}{2}(\mathrm{SP} + \mathrm{PS}') = \dfrac{v}{u}\mathrm{SS}'$ for semi-major axis, and let it revolve about the axis SS′ describing a prolate ellipsoid of revolution. Then all the electric forces arising from the change in the magnetic force taking place on the surface of the ellipsoid affect the charged sphere as it passes S′. And if S″ be a further point in the axis distant udt from S′, we may describe in like manner another ellipsoid of revolution having S and S″ for foci, and $v(t + dt)$ for semi-major axis. Then, since $v > u$, the two ellipsoids cannot intersect each other. And all the electric forces arising from changes in the magnetic force at points in the ellipsoidal shell between the two ellipsoids affect the charged sphere in its course between S′ and S″, that is in the limit at S′. And no other of the electric forces considered affects it at that point or between S′ and S″.

7. If P be any point on the ellipsoid and PN the perpendicular from P on the axis, $\mathrm{PN} = r\sin\theta$. And if we describe a circle with N as centre and radius NP, the magnetic force

has the same value at all points on that circle.   Let ϕ be the angle between the plane SPS′ and a fixed plane through SS′. Then we may express an element of volume of the ellipsoidal shell at P as follows :—

Fig. 1.

In the figure let SP, SP′ be two neighbouring values of r, making the angle PSP′$=d\theta$ with each other, and let PP′ be the corresponding elementary arc of the inner ellipse. Let SQ, SQ′ be the corresponding values of r, and QQ′ the corresponding arc for the outer ellipse.   If P′R be drawn at right angles to SP, P′R$=rd\theta$.   An element of volume of the ellipsoidal shell is the area PQQ′P′ multiplied by $r\sin\theta\,d\phi$.   But the area PQQ′P′ is equal to $rd\theta$ multiplied by PQ, that is by $vdt+u\cos\theta\,dt$.   We may therefore take for an element of volume of the ellipsoidal shell

$$r^2\sin\theta\,d\theta\,d\phi\,vdt\Big(1+\frac{u}{v}\cos\theta\Big)=d\omega.$$

8. I shall now make the hypothesis that $\dfrac{u}{v}$, the eccentricity of the ellipse, is very small, so that its square and higher powers may be neglected.   Let us write for brevity $\dfrac{u}{v}=\kappa$.

9. We have now to find the electric force at S′ due to the change of magnetic force H in the element of volume at P. To this end let us replace H at P by a small circular electric current.   Suppose about P in the plane SPS′ a circle described of very small radius a, and in that circle an electric current i, such that the magnetic force due to it at P is H. Then $\dfrac{i}{a}$ has the dimensions of H, and a^2i has the dimensions of H$d\omega$.   We may therefore write, since a is an arbitrary radius subject to a numerical factor,

$$a^2i=\mathrm{H}r^2\sin\theta\,d\theta\,d\phi\,vdt(1+\kappa\cos\theta),$$

and

$$a^2\frac{di}{dt}=\frac{d\mathrm{H}}{dt}\,r^2\sin\theta\,d\theta\,d\phi\,vdt\,(1+\kappa\cos\theta).$$

10. Again, the electric force at S' due to $\dfrac{d\mathrm{H}}{dt}$ at P is equal to the electric force at S' due to $\dfrac{di}{dt}$, which can be found as follows :—

Fig. 2.
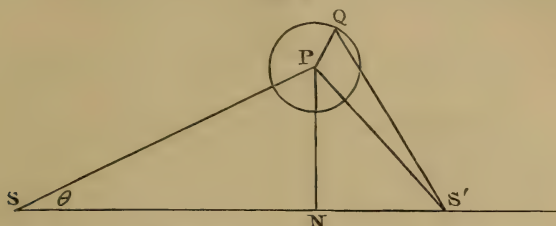
In the figure (2) the circle is the "a" circle described about P. $SP = r$, $PS' = r'$, $\angle PSN = \theta$, $\angle PS'N = \theta'$. Q is a point on the circumference of the a circle. $\angle QPN = \pi - \beta$. Let $S'Q = \rho$.

Then $\rho^2 = (r' \cos \theta' - a \sin \beta)^2 + (r' \sin \theta' + a \cos \beta)^2$
$\qquad = r'^2 + a^2 + 2r'a \sin (\theta' - \beta)$,

and, a being very small compared with r',

$$\frac{1}{\rho} = \frac{1}{r'} - \frac{r'a \sin (\theta' - \beta)}{r'^3}.$$

Again, the component parallel to the axis of the circular current i at Q is $-i \cos \beta$. And the electric force at S' in direction SS' due to the variation of the circular current, denoted by $\dfrac{di}{dt}$ is

$$\frac{di}{dt} \int_0^{2\pi} a \cos \beta \frac{1}{\rho} d\beta = \frac{di}{dt} \int_0^{2\pi} a \cos \beta \left(\frac{1}{r'} - \frac{a \sin (\theta' - \beta)}{r'^2} \right) d\beta$$

$$= -\pi a^2 \frac{di}{dt} \frac{\sin \theta'}{r'^2}$$

$$= -\pi \frac{d\mathrm{H}}{di} r^2 \sin \theta \, d\theta \, d\phi \, vdt \, (1 + \kappa \cos \theta) \frac{\sin \theta'}{r'^2}.$$

11. In order to effect the integration for the ellipsoidal shell, we shall have to express θ', r', and r in terms of θ. We have first

$$\sin \theta' = \frac{r}{r'} \sin \theta.$$

Again, using the ordinary equation to the ellipse, in which $x = r \cos \theta - \kappa vt$, $y = r \sin \theta$, and remembering that the minor

axis is $vt\sqrt{1-\kappa^2}$, we obtain easily a quadratic equation in r, the solution of which is

$$r = vt(1-\kappa^2)\frac{\kappa\cos\theta \pm 1}{1-\kappa^2\cos^2\theta};$$

and taking the upper sign, which is sufficient,

$$r = vt\frac{1-\kappa^2}{1-\kappa\cos\theta},$$

also

$$r' = 2vt - r = vt\frac{1+\kappa^2-2\kappa\cos\theta}{1-\kappa\cos\theta}:$$

whence, neglecting κ^2 and higher powers,

$$r = vt(1+\kappa\cos\theta), \quad r' = vt(1-\kappa\cos\theta).$$

12. Our integral for the ellipsoidal shell then becomes

$$\mathrm{X} = -\int_0^{\pi}\frac{d\mathrm{H}}{dt}2\pi^2 r^2\sin\theta\,d\theta\,v\,dt(1+\kappa\cos\theta)\frac{\sin\theta'}{r'^2}\,; \quad (\mathrm{A})$$

or substituting for r, r', and θ' their values above given,

$$\mathrm{X} = -\int_0^{\pi}\frac{d\mathrm{H}}{dt}2\pi^2\frac{r^3\sin^2\theta\,d\theta}{r'^3}v\,dt(1+\kappa\cos\theta)$$

$$= -\int_0^{\pi}\frac{d\mathrm{H}}{dt}2\pi^2 v\,dt(1+7\kappa\cos\theta)\sin^2\theta\,d\theta.$$

13. Before proceeding further with this integration we have to give to $\dfrac{d\mathrm{H}}{dt}$ its value according to the nature of the problem to be considered. I will take two cases.

Case I.—The change of magnetic force $\dfrac{d\mathrm{H}}{dt}$ is due simply to the motion of the charged sphere, u being maintained constant. In this case $\dfrac{d\mathrm{H}}{dt} = eu\dfrac{d}{dt}\left(\dfrac{\sin\theta}{r^2}\right)$

$$= -2eu\frac{\sin\theta}{r^3}\frac{dr}{dt} + eu\cdot\frac{\cos\theta}{r^2}\frac{d\theta}{dt}.$$

But, due to the motion with constant u,

$$\frac{dr}{dt} = -u\cos\theta, \quad \text{and} \quad \frac{d\theta}{dt} = \frac{u\sin\theta}{r}.$$

Therefore

$$\frac{d\mathrm{H}}{dt} = 3eu^2\frac{\sin\theta\cos\theta}{r^3}.$$

And substituting this value of $\dfrac{d\mathrm{H}}{dt}$ in the integral A, we get for that integral

$$\mathrm{X} = -3eu^2 \int_0^\pi 2\pi^2 r^2 \frac{\sin^2\theta\cos\theta}{r^3} \frac{\sin\theta'}{r'^2} d\theta\, r\, dt(1+\kappa\cos\theta),$$

$$= -6\pi^2 eu^2 \int_0^\pi \frac{\sin^3\theta\cos\theta\, d\theta}{r'^3} v\, dt(1+\kappa\cos\theta),$$

$$= -6\pi^2 eu^2 \int_0^\pi \frac{\sin^3\theta\cos\theta(1+4\kappa\cos\theta)d\theta v\, dt}{v^2 t^3},$$

$$= -\frac{32\pi^2}{5} \frac{\kappa eu^2 dt}{v^2 t^3}.$$

And the integral of this force for the whole course of the sphere beginning at S is

$$\frac{32\pi^2}{5} \kappa eu^2 \int_{t_0}^\infty \frac{dt}{v^2 t^3} = -\frac{16\pi^2}{5} \kappa eu^2 \frac{1}{v^2 t_0^2}$$

if we neglect the value at the upper limit, which for any considerable value of t must be inappreciable. This may also be put in the form $\dfrac{16\pi^2}{5}\dfrac{\kappa^3 e}{t_0^2}$. It may be appreciable if t_0 can be made small enough.

14. Case II.—Suppose that when at S the sphere receives a change of velocity ∂u, that is that its velocity is $u-\partial u$ before it reaches S, and there becomes u, and is thereafter maintained constant. Then, due to the change of velocity ∂u at S considered by itself, $\dfrac{d\mathrm{H}}{dt} = e\partial u\dfrac{\sin\theta}{r^2}$. And using this in the expression for the electric force E at S', we get for X

$$\mathrm{X} = -\int_0^\pi e\partial u \frac{\sin\theta}{r^2} 2\pi^2 r^2 \sin\theta\, d\theta \frac{\sin\theta'}{r'^2} v\, dt(1+\kappa\cos\theta)$$

$$= -2\pi^2 \int_0^\pi e\partial u \frac{r\sin^3\theta\, d\theta}{r'^3} v\, dt(1+\kappa\cos\theta)$$

$$= -2\pi^2 \int_0^\pi e\partial u \frac{(1+5\kappa\cos\theta)\sin^3\theta\, d\theta}{vt^2}.$$

$$= -\frac{8\pi^2}{3} e\partial u \frac{1}{vt^2},$$

κ now disappearing along with the first power of $\cos\theta$.

15. The interpretation of this result considered alone is that, the velocity having been increased or diminished at S by ∂u, the effect of the magnetic reactions is at every point in the subsequent course partially to restore the velocity gained or lost, as the case may be, when the sphere was at S. To maintain u constant we should have to apply a force in the direction of the motion equal at each instant t to $\dfrac{8\pi^2}{3} e\partial u \dfrac{dt}{vt^2}$. Or we should have to do work in the whole equal to $\dfrac{8\pi^2}{3} \dfrac{e\partial u}{vt}$ from the lower to the higher limit of t. Neglecting as we may the value of $\dfrac{1}{vt}$ at the higher limit, we have for the whole work required $\dfrac{8\pi^2}{3} \dfrac{e\partial u}{vt_0}$. In fact, there will be strictly two forces to be applied at every point. One which is proportional to ∂u and of opposite sign, and does not contain the factor κ. The other, which is always in the direction of the motion, and contains the factor κ^3, and which is therefore, except perhaps for very small values of t_0, inappreciable compared with the first. The first force, which is proportional to ∂u, expresses the effect of the magnetic reactions against the acceleration denoted by ∂u. Divided by ∂u it represents the electric inertia. It does not appear, however, from this investigation that the sphere having charge e and velocity u can correctly be said to possess inertia as a function of e only, because the magnetic reactions do not depend only on the instantaneous velocity u.

16. For the lower limit of the integration for t, which I have denoted by t_0, Lodge takes $\dfrac{c}{v}$, where c is the radius of the charged sphere. It may be admitted that c is the least distance for which the formulæ employed have any meaning. But I would here ask whether it would not be safer to take for c the least distance for which the formulæ employed $\left(\text{e. g., } H = eu\dfrac{\sin\theta}{r^2}\right)$ are capable of being verified experimentally? In like manner in the ellipsoidal integration, art. 13, should not the limits be, instead of $\theta = 0$ and $\theta = \pi$, $\theta = \dfrac{c}{r}$ and $\theta = \pi - \dfrac{c}{r}$, where c is the least distance for which the formula, electric force $= -\dfrac{e\dfrac{du}{dt}}{r}$ can be verified experimentally?

XXIV. *Experiments with Rotating Viscous Liquids.* By
JOHN BUCHANAN, *D.Sc.* (*Lond.*), *and* HENRY W. MALCOLM,
B.Sc.. Carnegie Scholar of Aberdeen University *.

[Plate IV.]

THE object of these experiments was to study the behaviour
of some viscous liquids when placed in a cylindrical
drum which is subjected to angular acceleration about a
horizontal axis.

Let us imagine such a drum to be mounted similarly to
the wheel of an Atwood's machine, and, like the wheel, to be
caused to rotate about a horizontal axis by means of a falling
load attached to a cord wrapped round the axle. Provided
the liquid fills the drum completely, it is not difficult to
see beforehand, in a general way, what kind of results may be
expected.

When the drum is undergoing angular acceleration by the
pull of the load on the cord, the more viscous is the liquid,
the more closely must it follow the motion of the drum. The
less viscous is the liquid, however, so much the more slowly
will the various parts of the liquid take up the motion of the
drum. In other words, there will be more relative motion
between the neighbouring parts of the liquid, and therefore a
greater retarding effect, in the latter case.

Experiment bears out these conclusions. The motion of
a drum filled with castor-oil could hardly be distinguished
from the motion of a solid body; with the drum full of water,
the presence of the liquid showed itself in a very marked
degree.

The conditions are completely altered, however, if the
drum be only partially filled with liquid. As the rotation of
the drum round a horizontal axis proceeds, there will be a
continual lifting of the centre of gravity of the liquid by
an amount depending on the viscosity of the liquid, as
the various layers cling to one another and to the walls of the
drum. Clearly, the greater is the viscosity of the liquid, the
greater will be the retardation due to this cause,—except,
indeed, in the extreme case where the motion is so rapid that
the liquid is maintained, by centrifugal force, as a ring of
invariable form carried bodily round with the walls of the
drum.

Under ordinary circumstances the motion inside the drum
is so exceedingly complex that only by experiment can the
amount of retardation due to a specified volume of a given

* Communicated by the Authors.

liquid be determined. The results obtained by us in the time, and by the means, at our disposal, appear to us to be new and of sufficient interest to warrant publication.

Apparatus used.

Two drums were tried, a small and a large. These were made of brass, and for convenience of cleaning can be separated circumferentially into two portions. When either drum is in use, the two parts can be pressed together by a brass nut which engages with a screw-thread cut on the steel spindle passing axially through the drum. Leather washers keep the joints tight. For admission of the liquid to the interior of the drums each has provided on it a small nipple closed by a nut; a dummy nipple and nut are so placed as to balance the inertia of the other.

The same steel spindle and its bearings serve for one or the other drum. The spindle has coned ends which turn on fixed set-screws of steel furnished with lock-nuts, so that the spindle can turn freely and yet have scarcely any sensible end play. The other parts, comprising side-brackets and base, are made of stout cast iron for the sake of rigidity.

The principal approximate dimensions are as follows:—

Mean depth of small drum inside ....=	3·5 cms.
Mean depth of large drum inside ....=	19·0 „
Mean internal diameter of both ....=	12·04 „
Diameter of spindle to top of thread......=	1·25 „
Diameter of axle on which cord wraps, to ⎫ centre of cord ....⎭ =	5·15 „

Theory of the Experiments.

In the case of such extremely complex internal motions as we have to deal with here, it appears to be hopeless to attempt quantitative measurements other than those of a kind which will serve to bring out the *average* effect of such motions.

It would seem that the most convenient means of expressing this average effect is to take it as measured by *the amount of retardation of the falling load over and above what the retardation would have been if the drum and its contents were moving as a solid.*

As will be seen presently, this quantity—the additional retardation due to the fact that the interior of the drum contains a liquid, and not a rigid material—can be measured without much difficulty by allowing the falling load to run down to the full extent of the cord, and then permitting the inertia of the rotating mass to rewind the cord on the axle,

and so to raise again the load at the end of the cord until it comes to rest.

Let us put:—

I = average moment of inertia of the rotating mass;
W = mass of the load in grammes hung on the cord;
ϖ = angular velocity;
v = linear velocity of the falling mass;
g = acceleration of gravity;
H = total descent of the load;
h = greatest height to which the load is raised;
t = time of descent;
f = total retardation due to friction in grammes weight;
  = friction at axle + fluid friction.

For the descent we have

$$(W-f)gH = \tfrac{1}{2}I\varpi^2 + \tfrac{1}{2}Wv^2. \quad . \quad . \quad . \quad . \quad (1)$$

For the upward motion,

$$\tfrac{1}{2}I\varpi^2 = (W+f)gh. \quad . \quad . \quad . \quad . \quad (2)$$

By elimination of $\tfrac{1}{2}I\varpi^2$ between (1) and (2), and by putting $v = \dfrac{2H}{t}$, we get, after rearranging,

$$f = \frac{W}{H+h}\Big(H - h - \frac{2H^2}{gt^2}\Big). \quad . \quad . \quad . \quad . \quad (3)$$

It will be observed that the assumption is made in equation (2) that the kinetic energy of the falling load W has been absorbed by the viscosity of the cord before the upward motion begins—an assumption which cannot be far from the truth; and in any case does not affect the value of f to any great extent. The cord used was chosen so as to show as little " spring " as possible. It is worth noticing that the kinetic energy of the load, when at the bottom of the descent, was not expended in reducing the speed of the rotating mass, because at this stage of the motion the cord was all run off, and consequently its point of attachment to the axle was vertically under the axis of rotation. The cord wound itself on the axle, and unwound, quite evenly.

Methods of Observation.

The quantities in the right-hand member of (3) above which offer most difficulty in measuring are t and h.

The time of fall from rest of the load was measured by help of an electrical device of a well-known type. On a moving strip of paper two pens placed side by side drew continuous traces. Each pen was attached to the armature of a small

electromagnet. The current through one electromagnet was controlled by a metronome approximately beating seconds. The metronome was compared from time to time with a chronometer. The current through the other electromagnet was " made " by the horizontal movement of the trigger which released the drum; and was " broken " by the falling load touching a trigger when almost at the bottom of its descent.

The observed times of fall never differed by more than one-tenth of a second; usually the differences were of the order of two or three hundredths of a second.

In order to measure h, a common pin to serve as a mark was pushed at right angles through the cord near to the load, and the highest point to which it rose was read off against a vertical scale. A telescope was used to avoid parallax. The observations were repeated until the mean height was certainly known to a millimetre.

Three liquids—water, Price's " cycle-lamp oil," and castor-oil—were chosen so as to give a wide range of viscosity.

In making a set of observations, the values of H, h, and t were found for a series of loads, first with the drum empty. The loads extended from 50 grms. to a maximum of 300 grms. A larger load than this maximum was not considered safe to use, on account of the severe jerk experienced by the cord when the bottom of the descent was reached.

A measured quantity of the liquid under examination was next run into the drum, and H, h, and t again observed for the same series of loads as before. And so on, until the drum was completely filled with the liquid. Each time the drum was empty, check observations were taken.

Sometimes the order of the observations was altered, by beginning with a full drum, and gradually emptying it in order to get a series of readings.

The bearings of the drum were kept well lubricated, and were not tampered with during the taking of each set of readings.

The value of f evidently depended on the extent of the tightening up of the end cones. But with the drum empty f was found to be independent, within the errors of observation, of the load used. Moreover, it was found that f was not greatly different whether the drum was empty or was completely full.

As is well known, in quantitative observations connected with viscosity, the difficulties arising from varying temperature are by far the most serious. Unless very elaborate precautions be taken, they render nugatory any attempts at

very close consistency between series of observations taken on different days. In our experiments, as a matter of fact, the temperature did not vary more than 1° C. during a series of observations made on any particular day. As far as possible, each series was completed on the same day as it was begun. If this was not possible, the last reading was repeated. This accounts for one or two small discrepancies between two determinations of the same point on some of the graphs given below.

Here is a sample of observations taken quite at random from a laboratory notebook, in order to show the degree of consistency between consecutive readings.

Small drum with cycle-lamp oil; temp. 15°·4 C.

> Volume of oil in drum = 120 c. c.
> Load hung on cord   = 100 grms.
> Load at top : scale-reading of index-pin = 218·8 cms.
> Load at bottom :   ,,   ,,   = 38·4 cms.

$$\therefore \quad H = 218 \cdot 8 - 38 \cdot 4$$
$$= 180 \cdot 4 \text{ cms.}$$

Consecutive scale-readings of height of rise of load :—

> 101·2 cms.
> 101·1 ,,
> 101·1 ,,
> Mean = 101·1 cm.

$$\therefore \quad h = 101 \cdot 1 - 38 \cdot 4$$
$$= 62 \cdot 7 \text{ cms.}$$

For the times of fall were found :—

> 5·30 (metronome) secs.
> 5·36   ,,   ,,
> Mean = 5·33   ,,   ,,

The factor of reduction to mean solar seconds is 1·058. From the above data we get

$$f = 47 \cdot 6 \text{ grms.-weight.}$$

The value of f for the empty drum was previously found to be

$$= 10 \cdot 6 \text{ grms.-weight.}$$

$\therefore$ friction due to presence of liquid in the drum
$$= 37 \cdot 0 \text{ grms.-weight, approximately.}$$

Experimental Results.

So far as concerns the general *form*, the results exhibited in the graphs may be taken as correctly representing the behaviour of the viscous liquids used in these experiments.

The *position* of these graphs relatively to one another, depending so intimately on temperature as it does, can, however, only be considered as on the whole substantially correct.

The most complete observations were made with the small drum. In figs. 1, 2, and 3 (Pl. IV.) are shown the graphs connecting the liquid friction and the load on the cord, with a specified constant volume of liquid in the drum. These graphs have been reduced to approximately a common zero by subtracting from the total friction the friction of the empty drum.

When the drum was completely full of liquid, it will be noticed that (*a*) the graphs for all three liquids are approximately straight lines ; (*b*) the liquid friction for the same load is greatest for water and least for castor-oil ; (*c*) the approximately horizontal straight line for castor-oil shows that the drum when full of this oil behaves almost as would a solid body.

When the drum was not full these graphs show clearly that the normal result is that the liquid friction for castor-oil diminishes continuously as the load increases ; for water the rise is continuous and nearly linear ; for lamp-oil it rises to a maximum, then diminishes.

In fig. 4 the graphs refer to similar experiments with the large drum. Inspection shows (*d*) the resemblance in form between those for water and for lamp-oil; (*e*) this form is similar to that of the left-hand portions of the lamp-oil graphs of fig. 2.

The experiments with the large drum were projected in order to discover whether the results would present any fresh points not shown by the small drum. The graphs show, however, that the general results are of the same nature with the two drums. The large drum gives results which appear to be merely exaggerated versions of those given by the small drum.

The behaviour of castor-oil under the action of small loads is somewhat striking, and is indeed a consequence of the fact that the liquid friction with this oil decreases with increased acceleration. For example, with 320 c.c. in the small drum, loads of 50 grms., 75 grms., 100 grms. respectively gave simply a slow, apparently uniform, creep-down of the load to the bottom of the descent, where it stopped dead. Here, of course, the total friction was equal to the load. This action is indicated by the straight line AB in fig. 1 ; it is brought out still more fully in figs. 5 and 6.

Again, when for instance the large drum contained 89 c.c. of castor-oil, a load of 150 grms. gave initially a very slow, seemingly uniform, motion, until about a revolution of the drum had been completed; at this stage the accelerated motion began and continued thereafter in the normal manner. An effect of this kind frequently occurred with small loads, but it does not appear to be possible to represent the effect by points on such graphs as we have here.

The graphs representing the relation between the total friction and volume of liquid in each drum, with a specified load hung on the cord, are shown in figs. 5 and 6. In order not to confuse the graphs, the ordinates here represent the total friction = axle friction + fluid friction.

The axle friction is very nearly that for the empty drum.

The liquids were poured into the drums from a measuring-cylinder. With the oils especially it was very difficult to measure out exactly a definite volume, so that there was an accumulation of small errors in measurement when the drum was full. Due to this cause are the discrepancies between the right-hand terminations of the graphs in figs. 5 and 6.

These graphs show clearly that (f) the type of graph is the same for the two oils used. They indicate an increase—at one stage it may be a very rapid increase—of friction with quantity of liquid in the drums. This is followed afterwards by a decrease, more or less rapid, until the drums are full; (g) for small quantities of liquid, and the same load, the friction is less with castor-oil than with the lamp-oil. There is a certain volume for which the friction is the same with both oils; (h) in the case of the castor-oil, the effect referred to above, where the friction is equal to the load, is brought out by these graphs in a very marked way. Thus, the graph in fig. 6 marked for 100 grms. load, with castor-oil in the small drum, rises up to the 100 grms. ordinate; that in fig. 5 for 300 grms. load, with the same oil in the large drum, rises up to the 300 grms. ordinate.

With the large drum the effect in question was made still more striking by beginning with the large drum quite full; a load of 300 grms. caused a rapid acceleration. About one-eighth of the contents of the drum being withdrawn, there ensued a slow creep-down of the load.

Aberdeen,
    31st October, 1904.

XXV. *On a Compact Apparatus for determining Young's Modulus for Thin Wires.* By CHARLES H. LEES, *D.Sc.,* and ROGER E. GRIME, *B.Sc.**

MANY who have determined Young's Modulus for a material both by the bending of a beam and by the stretching of a wire of the material, must have contrasted the compactness of the apparatus used in the former with the bulkiness of that used in the latter method. It was this contrast which led us to attempt to devise a simple and compact apparatus for testing wires. The initial difficulties met with in making the apparatus at the same time compact and reliable proved much less serious than we anticipated, and as the apparatus in its final form proved easy to use and satisfactory, it seemed to us worth while calling the attention of physicists to a method of determining Young's Modulus for thin wires which does not appear to have received in the past the attention it deserves.

Theory of the Method.

The method utilizes the depression produced in the middle of a straight horizontal length of thin wire †, supported rigidly at its ends, by a load applied at its middle point.

If AB be a wire whose resistance to bending may be neglected, of length $2l$, supported at A and B, and if a mass

Fig. 1.
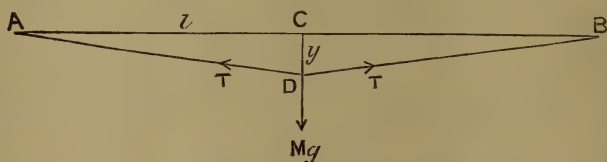
M, suspended at its middle point C, depress that point to D, where $CD = y$, the downward force Mg at D is balanced by the vertical components of the pull T in DA and DB;

$$i.e.\ \mathrm{M}g = 2\mathrm{T}\cos\mathrm{C\hat{D}B} = 2\mathrm{T}\sin\mathrm{C\hat{B}D},$$

$$\text{or}\quad \mathrm{T} = \frac{\mathrm{M}g}{2}\operatorname{cosec}\theta \quad \text{where}\quad \theta = \mathrm{C\hat{B}D}.$$

When $\mathrm{C\hat{B}D} = 0$, and therefore $M = 0$, let $T = T_0$. Then the change of length of a length originally l, due to the increase of the pull of the wire from T_0 to T is equal to

* Communicated by the Authors.

† Up to about No. 27 S.W.G. in the apparatus used. For thicker wires the flexural rigidity renders it necessary, if an accuracy of 1 per cent. is required, to treat the wire as an *elastica*. It is neither straight nor is the tension in it constant, and the simple theory gives too high values for the modulus.

$l\dfrac{T-T_0}{a\epsilon}$, where a is the cross section of the wire and ϵ is the Young's modulus of its material, Hooke's law being supposed to hold up to the maximum value of T used.

Therefore

$$1+\frac{T-T_0}{a\epsilon}=\sec \hat{CBD}$$

$$\therefore\quad 1+\frac{\dfrac{Mg}{2}\operatorname{cosec}\theta-T_0}{a\epsilon}=\sec\theta.$$

$$\frac{Mg}{2}\operatorname{cosec}\theta-T_0=a\epsilon(\sec\theta-1)\quad . \quad . \quad (1)$$

From equation (1) it is evident that T_0 should be taken as small as possible to secure the maximum accuracy of determination of ϵ.

If a second mass M_1 suspended from C produce a deflexion θ_1 at B,

$$\frac{M_1 g}{2}\operatorname{cosec}\theta_1-T_0=a\epsilon(\sec\theta_1-1).$$

Hence eliminating T_0,

$$\frac{g}{2}(M_1\operatorname{cosec}\theta_1-M\operatorname{cosec}\theta)=a\epsilon(\sec\theta_1-\sec\theta)$$

or

$$\epsilon=\frac{g}{2a}\cdot\frac{M_1\operatorname{cosec}\theta_1-M\operatorname{cosec}\theta}{\sec\theta_1-\sec\theta}.$$

Or, in terms of the half-length l of the wire and the depressions $y\ y_1$ at the centre

$$\epsilon=\frac{g}{2a}\frac{M_1\sqrt{1+\left(\dfrac{l}{y_1}\right)^2}-M\sqrt{1+\left(\dfrac{l}{y}\right)^2}}{\sqrt{1+\left(\dfrac{y_1}{l}\right)^2}-\sqrt{1+\left(\dfrac{y}{l}\right)^2}}.$$

or

$$\epsilon=\frac{gl}{2a}\cdot\frac{\dfrac{M_1}{y_1}\sqrt{\dfrac{1+(y_1/l)^2}{1+(y/l)^2}}-\dfrac{M}{y}}{\sqrt{\dfrac{1+(y_1/l)^2}{1+(y/l)^2}}-1}\quad . \quad . \quad . \quad (2)$$

In all the experiments which follow $\dfrac{y}{l}$ is sufficiently small to allow the expression under the root to be written 1 in the numerator and $1+\dfrac{y_1^2-y^2}{2l^2}$ in the denominator; and the equation then takes the simpler approximate form

$$\epsilon=\frac{gl^3}{a}\cdot\frac{\dfrac{M_1}{y_1}-\dfrac{M}{y}}{y_1^2-y^2}\quad . \quad . \quad . \quad . \quad . \quad (3)$$

Description of the Apparatus.

The wire was supported rigidly in an iron frame AB (fig. 2) by two clamps C, D whose distance apart could be adjusted

Fig. 2.
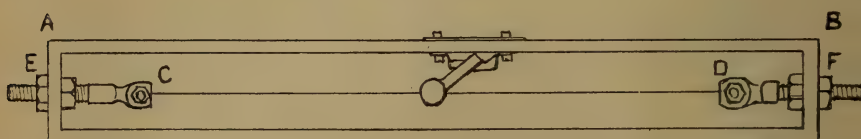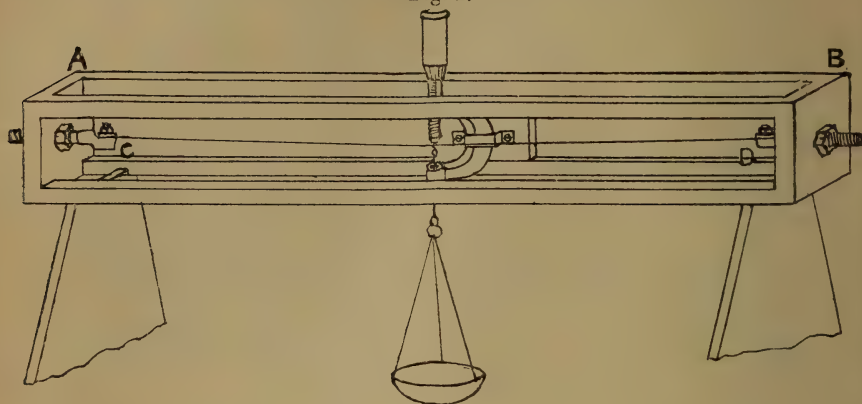
by means of the nuts E, F. These were adjusted till the tension in the wire was rather more than that necessary to straighten it when unloaded. In order to be able to reproduce this state of tension readily in successive experiments, a subsidiary wire half the length of the one experimented on, and cut from the same piece, was suspended vertically and loaded at the lower end (fig. 3) till the note it gave on being plucked with a pointed brass wire was identical with that given by the wire experimented on when held at its centre and plucked in the same way. The accuracy of the determination of T_0 by this method was not, however, sufficient to admit of its being used in equation (1) to determine ϵ, and equation (2) was therefore used throughout.

Fig. 3.

The load was applied at the centre of the length of the wire by means of a very small hook of steel wire, from which a scale-pan could be suspended. The vertical descent of the hook was measured by a Brown & Sharp micrometer screw-gauge with a pitch of ·5 mm., and head graduated in ·01 mm., admitting of estimation down to ·001 mm. The screw-stop in the lower jaw of the screw-gauge was removed to allow the wire suspending the

scale-pan to pass downwards along the axis of the screw. The gauge was fixed rigidly in the iron frame by metal straps, and was placed obliquely to save space.

The end surface of the screw of the gauge was polished in order to give a reflexion of the hook with which it had to be brought into contact, and thus enable the point of contact to be determined with greater accuracy. A small magnifying-glass was used to increase still further the accuracy of the setting.

The depression due to the load was determined both while the load was being increased and while it was being diminished. The zeros at no load were taken before and after the loading, and the mean taken if they were in close agreement. If not, the observations were rejected.

Equation (2) is obtained on the assumption that the supports of the wire are rigid, and in some of the earlier experiments this was tested by placing a second wire in the clamps alongside the wire experimented on, loading it lightly, and observing whether on loading the experimental wire above the limit to which it was proposed to go in the actual experiment, any depression of the subsidiary wire was produced. This was found not to be the case.

The temperature throughout was that of the air of the room (16° C.). It was not found necessary to take any special precautions to maintain the apparatus at constant temperature. So long as the wire was not touched during an experiment, it came back at the end to the same position, showing that its temperature had remained sufficiently constant.

The following tables give the measurements made on a few wires, and show the degree of accuracy which can be readily obtained with the apparatus.

Iron wire, No. 34 S.W.G. $l = 24 \cdot 05$ cms., mean diameter $\cdot 0230$ cm. $\therefore$ area a of cross-section $= \cdot 000415$ sq. cm., and $\dfrac{gl^3}{a} = 3 \cdot 28 \times 10^{10}$.

Initial stretching force in dynes.	M. gr.	y mean. cms.	$\dfrac{M}{y}$	y^2.	$\dfrac{M_1}{y_1} - \dfrac{M}{y}$	$y_1^2 - y^2$.	$\dfrac{\frac{M_1}{y_1} - \frac{M}{y}}{y_1^2 - y^2}$	ϵ. dynes/sq. cm.
Not recorded.	15·50	·2461	62·98	·0605				
	25·50	·3759	67·84	·1413	4·86	·0808	60·2	$1 \cdot 97 \times 10^{12}$
	35·50	·4845	73·27	·2347	10·29	·1742	59·1	$1 \cdot 94 \times 10^{12}$

Mean $\epsilon = 1 \cdot 95 \times 10^{12}$

Iron wire, No. 34 S.W.G. $l = 26\cdot50$ cms., mean diameter $\cdot02414$ cm. $\therefore$ area a of cross-section $= \cdot0004577$ sq. cm., and $\dfrac{gl^3}{a} = 3\cdot99 \times 10^{10}$.

Initial stretching force in dynes.	M. gr.	y mean. cms.	$\dfrac{M}{y}$	y^2.	$\dfrac{M_1}{y_1} - \dfrac{M}{y}$	$y_1{}^2 - y^2$.	$\dfrac{\frac{M_1}{y_1} - \frac{M}{y}}{y_1{}^2 - y^2}$	ϵ. dynes/sq. cm.
$440 \times g$	15.0	·3755	39·95	·1410				
	25.0	·5310	47·08	·2820	7·13	·1410	50·6	$2\cdot02 \times 10^{12}$
	35.0	·6479	54·00	·4198	14·05	·2788	50·4	2·01

Mean $\epsilon = 2\cdot01 \times 10^{12}$.

Iron wire, No. 29 S.W.G. $l = 26\cdot50$ cms., mean diameter $\cdot03228$ cm. $\therefore$ area a of cross-section $= \cdot0008184$ sq. cm., and $\dfrac{gl^3}{a} = 2\cdot23 \times 10^{10}$.

Initial stretching force in dynes.	M. gr.	y mean cms.	$\dfrac{M}{y}$	y^2.	$\dfrac{M_1}{y_1} - \dfrac{M}{y}$	$y_1{}^2 - y^2$.	$\dfrac{\frac{M_1}{y_1} - \frac{M}{y}}{y_1{}^2 - y^2}$	ϵ. dynes/sq. cm.
$790 \times g$	15.0	·2251	66·37	·0507				
	25	·3446	72·54	·1187				
	35	·4412	79·33	·1947	12·96	·1440	90·0	$2\cdot00 \times 10^{12}$
	45	·5219	86·24	·2724	13·70	·1537	89·2	1·99
$950 \times g$	15.0	·1992	75·30	·0397				
	25	·3112	80·33	·0968				
	35	·4053	86·36	·1643	11·06	·1246	88·80	$1\cdot98 \times 10^{12}$
	45	·4845	92·88	·2347	12·55	·1379	91·02	2·03

Mean $\epsilon = 2\cdot00 \times 10^{12}$.

Steel wire (pianoforte-wire), No. 29 S.W.G. $l = 26\cdot40$ cms., mean diameter $\cdot03500$ cm. $\therefore$ area a of cross-section $= \cdot000962$ sq. cm., and $\dfrac{gl^3}{a} = 1\cdot88 \times 10^{10}$.

Initial stretching force in dynes.	M. gr.	y mean. cms.	$\dfrac{M}{y}$	y^2.	$\dfrac{M_1}{y_1} - \dfrac{M}{y}$	$y_1{}^2 - y^2$.	$\dfrac{\frac{M_1}{y_1} - \frac{M}{y}}{y_1{}^2 - y^2}$	ϵ. dynes/sq. cm.
$1830 \times g$	25.0	·1712	146·02	·0293				
	35	·2339	149·63	·0547				
	45	·2919	154·16	·0852	8·14	·0559	146	$2\cdot74 \times 10^{12}$
	55	·3462	158·86	·1198	9·23	·0651	142	2·67

Mean $\epsilon = 2\cdot70 \times 10^{12}$.

A second similar wire. $l = 26\cdot40$ cms., mean diameter ·03550 cm. ∴ area a of cross-section $= \cdot000990$, and $\dfrac{gl^3}{a}$ $= 1\cdot82 \times 10^{10}$.

Initial stretching force in dynes.	M. gr.	y mean. cms.	$\dfrac{M.}{y}$	y^2.	$\dfrac{M_1}{y_1} - \dfrac{M}{y}$.	$y_1{}^2 - y^2$.	$\dfrac{\dfrac{M_1}{y_1} - \dfrac{M}{y}}{y_1{}^2 - y^2}$	ϵ. dynes/sq. cm.
$1870 \times g$	25·0	·1614	154·90	·0260				
	35	·2208	158·51	·0487				
	45	·2767	162·63	·0765	7·73	·0505	153	$2\cdot78 \times 10^{12}$
	55	·3284	167·47	·1078	8·96	·0591	152	2·77

Mean $\epsilon = 2\cdot78 \times 10^{12}$.

Nickel wire, No. 30 S.W.G. $l = 26\cdot50$ cms., mean diameter $= \cdot03184$ cm. ∴ area a of cross-section $= \cdot000796$ sq. cm., and $\dfrac{gl^3}{a} = 2\cdot29 \times 10^{10}$.

Initial stretching force in dynes.	M. gr.	y mean. cms.	$\dfrac{M.}{y}$	y^2.	$\dfrac{M_1}{y_1} - \dfrac{M}{y}$.	$y_1{}^2 - y^2$.	$\dfrac{\dfrac{M_1}{y_1} - \dfrac{M}{y}}{y_1{}^2 - y^2}$	ϵ. dynes/sq. cm.
$330 \times g$	15	·3909	38·37	·1528	10·09	·1134	88·9	$2\cdot04 \times 10^{12}$
	25	·5159	48·46	·2662	9·19	·1024	89·8	2·06
	35	·6071	57·65	·3686	19·28	·2158	89·3	2·05
$450 \times g$	15	·3566	42·06	·1272	9·42	·1086	86·8	$1\cdot99 \times 10^{12}$
	25	·4856	51·48	·2358	9·00	·0991	90·8	2·08
	35	·5787	60·48	·3349	18·42	·2077	88·6	2·03
$530 \times g$	15	·3033	49·46	·0920	8·30	·0953	87·1	$1\cdot99 \times 10^{12}$
	25	·4328	57·76	·1873	8·35	·0930	89·8	2·06
	35	·5294	66·11	·2803	16·65	·1883	88·4	2·02

Mean $\epsilon = 2\cdot04 \times 10^{12}$.

Aluminium wires Nos. 25 and 30 S.W.G. were tried, but neither gave consistent results, owing to permanent set being produced by the small loads used.

Copper wire, No. 27 S.W.G. $l = 26\cdot50$ cms., mean diameter $\cdot0412$ cm. $\therefore$ area a of cross-section $= \cdot00133$ sq. cm., and $\dfrac{gl^3}{a} = 1\cdot37 \times 10^{10}$.

Initial stretching force in dynes.	M. gr.	y mean. cms.	$\dfrac{M}{y}$.	y^2.	$\dfrac{M_1}{y_1} - \dfrac{M}{y}$.	$y_1{}^2 - y^2$.	$\dfrac{\dfrac{M_1}{y_1} - \dfrac{M}{y}}{y_1{}^2 - y^2}$.	ϵ. dynes/sq.cm.
$650\,g$	15	$\cdot2749$	$54\cdot56$	$\cdot0756$	$7\cdot47$	$\cdot0863$	$86\cdot6$	$1\cdot19 \times 10^{12}$
	25	$\cdot4024$	$62\cdot03$	$\cdot1619$	$7\cdot50$	$\cdot0915$	$82\cdot0$	$1\cdot12$
	35	$\cdot5034$	$69\cdot53$	$\cdot2534$	$14\cdot97$	$\cdot1778$	$84\cdot2$	$1\cdot15$
$380\,g$	15	$\cdot3630$	$41\cdot32$	$\cdot1318$	$9\cdot44$	$\cdot1108$	$85\cdot2$	$1\cdot17$
	25	$\cdot4925$	$50\cdot76$	$\cdot2426$	$8\cdot80$	$\cdot1027$	$85\cdot7$	$1\cdot17$
	35	$\cdot5876$	$59\cdot55$	$\cdot3453$	$18\cdot24$	$\cdot2135$	$85\cdot4$	$1\cdot17$
$630\,g$	15	$\cdot2774$	$54\cdot07$	$\cdot0769$	$7\cdot75$	$\cdot0866$	$89\cdot5$	$1\cdot23$
	25	$\cdot4044$	$61\cdot82$	$\cdot1635$	$7\cdot83$	$\cdot0890$	$88\cdot0$	$1\cdot21$
	35	$\cdot5025$	$69\cdot65$	$\cdot2525$	$15\cdot58$	$\cdot1756$	$88\cdot7$	$1\cdot22$

Mean $\epsilon = 1\cdot18 \times 10^{12}$.

TABLE OF RESULTS.

	Young's Modulus at 16° C.
	dynes per sq. cm.
Iron wire, No. 34 A	$1\cdot95 \times 10^{12}$
,,　　　No. 34 B	$2\cdot01$
,,　　　No. 29	$2\cdot00$
Steel pianoforte-wire, No. 29 A.	$2\cdot70$
,,　　　,,　　　No. 29 B.	$2\cdot78$
Nickel wire, No. 30	$2\cdot04$
Copper wire, No. 27	$1\cdot18$

XXVI. *On the Application of Legendre's Functions to the Theory of the Jacobian Elliptic Integrals.* By J. W. NICHOLSON, *B.Sc.(Lond. & Vict.), Trinity College, Cambridge* [*].

THE object of this paper is to indicate a method of some generality, of expanding (1) functions expressible in terms of the complete elliptic integrals of Jacobi; and (2) definite integrals containing incomplete elliptic integrals in their integrand, in series of Legendre coefficients of increasing

* Communicated by the Author.

order, and to deduce from the theory certain integrals which are otherwise difficult to evaluate. Most of the results of this latter class, also, are new.

We shall employ the two expressions for $P_n(\mu)$, where $\mu = \cos\theta$, which are due to Mehler, viz. :

$$P_n(\mu) = \frac{2}{\pi} \int_0^\theta \frac{\cos\left(n+\frac{1}{2}\right)\phi \, d\phi}{\sqrt{2(\cos\phi - \cos\theta)}} \quad . \quad . \quad . \quad (1)$$

$$P_n(\mu) = \frac{2}{\pi} \int_\theta^\pi \frac{\sin\left(n+\frac{1}{2}\right)\phi \, d\phi}{\sqrt{2(\cos\theta - \cos\phi)}}, \quad . \quad . \quad (2)$$

and in all future work θ is considered to be a positive angle, lying between 0 and π.

Let $F(z)$ be a function of z, satisfying the conditions for a Fourier expansion between $z = z_1$, $z = z_2$, the expansion having one of the two forms :—

$$F(z) = \sum_{n=0}^\infty a_n \cos(2n+1)z,$$

$$F(z) = \sum_{n=0}^\infty a_n \sin(2n+1)z.$$

Then between the limits $2z_1$, $2z_2$, $F\left(\dfrac{\phi}{2}\right)$ has one of the forms

$$F\left(\frac{\phi}{2}\right) = \sum_0^\infty a_n \cos\left(n+\frac{1}{2}\right)\phi, \quad . \quad . \quad . \quad (3\,a)$$

$$F\left(\frac{\phi}{2}\right) = \sum_0^\infty a_n \sin\left(n+\frac{1}{2}\right)\phi. \quad . \quad . \quad . \quad (3\,b)$$

Multiplying the first form by $\dfrac{2}{\pi} \cdot \dfrac{1}{\sqrt{2(\cos\phi - \cos\theta)}}$ and

integrating from 0 to θ, supposed not to include $2z_1$, or $2z_2$,

$$\therefore \quad \frac{2}{\pi} \int_0^\theta \frac{F\left(\dfrac{\phi}{2}\right) d\phi}{\sqrt{2(\cos\phi - \cos\theta)}} = \sum_0^\infty a_n P_n(\mu). \quad . \quad . \quad (4)$$

Similarly, the second form gives

$$\frac{2}{\pi} \int_\theta^\pi \frac{F\left(\dfrac{\phi}{2}\right) d\phi}{\sqrt{2(\cos\theta - \cos\phi)}} = \sum_0^\infty a_n P_n(\mu). \quad . \quad . \quad (5)$$

These formulæ will be applied to several Fourier expansions. The latter will be freely quoted, and are all to be found in

Whittaker, 'Modern Analysis,' Chap. vii., or a similar work.

The simplest case is the series

$$\cos\frac{\phi}{2} - \cos\frac{3\phi}{2} + \cos\frac{5\phi}{2}\ldots\ldots$$

which we shall not dwell upon.

The expansion produced is

$$\frac{1}{2\cos\dfrac{\theta}{2}} = P_0(\mu) - P_1(\mu) + P_2(\mu)\ldots + (-)^n P_n(\mu) + \ldots \quad (6)$$

where θ is not zero or π, but may have any intermediate value. Therefore, by the integral properties of Legendre's functions, the integral on left being finite at both limits,

$$\frac{1}{2}\int_{-1}^{1}\frac{P_n(\mu)d\mu}{\cos\dfrac{\theta}{2}} = (-)^n\frac{2}{2n+1}$$

or

$$\int_0^\pi P_n(\cos\theta)\sin\frac{\theta}{2}\,d\theta = (-)^n\frac{2}{2n+1}. \quad . \quad . \quad (7)$$

The next case is the result:

$$\frac{\pi}{4} = \sin z + \tfrac{1}{3}\sin 3z + \tfrac{1}{5}\sin 5z + \ldots\ldots$$

holding from $z=0$ to π, both exclusive.

Therefore between $\phi=0$ and 2π,

$$\frac{\pi}{4} = \sum_0^\infty\frac{\sin(2n+1)\dfrac{\phi}{2}}{2n+1}.$$

Hence

$$\sum_0^\infty\frac{P_n(\mu)}{2n+1} = \frac{2}{\pi}\sum_0^\infty\frac{1}{2n+1}\int_\theta^\pi\frac{\sin(2n+1)\dfrac{\phi}{2}}{\sqrt{2(\cos\theta-\cos\phi)}}$$

$$= \frac{1}{4}\int_\theta^\pi\frac{d\phi}{\sqrt{\cos^2\dfrac{\theta}{2}-\cos^2\dfrac{\phi}{2}}};$$

the excluded values of ϕ being outside the range of integration.

Put
$$\tan\frac{\phi}{2} = \tan\frac{\theta}{2}\operatorname{cosec}\lambda,$$

$$\therefore \quad d\phi = -\frac{2\tan\dfrac{\theta}{2}\,.\,\operatorname{cosec}\lambda\cot\lambda\,d\lambda}{1+\tan^2\dfrac{\theta}{2}\operatorname{cosec}^2\lambda}$$

$$\cos^2\frac{\theta}{2} - \cos^2\frac{\phi}{2} = \frac{\sin^2\dfrac{\theta}{2}\cot^2\lambda}{1+\tan^2\dfrac{\theta}{2}\operatorname{cosec}^2\lambda}.$$

$$\therefore \int_\theta^\pi \frac{d\phi}{\sqrt{\cos^2\dfrac{\theta}{2}-\cos^2\dfrac{\phi}{2}}} = 2\int_0^{\frac{\pi}{2}} \frac{\sec\dfrac{\theta}{2}\operatorname{cosec}\lambda\,d\lambda}{\sqrt{1+\tan^2\dfrac{\theta}{2}\operatorname{cosec}^2\lambda}}$$

$$= 2\mathrm{K}\left(\cos\frac{\theta}{2}\right),$$

where $\mathrm{K}(k)$ is Jacobi's complete integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\lambda}{\sqrt{1-k^2\sin^2\lambda}}$.

Hence

$$\mathrm{K}\left(\cos\frac{\theta}{2}\right) = 2\sum_0^\infty \frac{\mathrm{P}_n(\mu)}{2n+1} \quad . \quad . \quad . \quad . \quad (8)$$

where $\mu = \cos\theta$.

The formula
$$\frac{\pi}{4} = \cos z - \tfrac{1}{3}\cos 3z + \tfrac{1}{5}\cos 5z\ldots\ldots$$

holds from $z = -\dfrac{\pi}{2}$ to $z = +\dfrac{\pi}{2}$, both exclusive.

$$\therefore \quad \frac{2}{\pi}\int_0^\theta \frac{\pi}{4}\,.\,\frac{d\phi}{\sqrt{2(\cos\phi-\cos\theta)}} = \sum_0^\infty \frac{(-)^n}{2n+1}\mathrm{P}_n(\mu).$$

To transform this integral, put $\tan\dfrac{\phi}{2} = \tan\dfrac{\theta}{2}\cos\lambda$.

$$\therefore \quad d\phi = -\frac{2\tan\dfrac{\theta}{2}\,.\,\sin\lambda\,d\lambda}{1+\tan^2\dfrac{\theta}{2}\cos^2\lambda}$$

$$\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2} = \frac{\sin^2\dfrac{\theta}{2}\sin^2\lambda}{1+\tan^2\dfrac{\theta}{2}\cos^2\lambda}.$$

$$\therefore \quad \int_0^\theta \frac{d\phi}{\sqrt{\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}}} = 2\int_0^{\frac{\pi}{2}} \frac{d\lambda}{\sqrt{1 - \sin^2\frac{\theta}{2}\sin^2\lambda}}$$

$$= 2\mathrm{K}\left(\sin\frac{\theta}{2}\right).$$

Hence

$$\mathrm{K}'\left(\cos\frac{\theta}{2}\right) = \mathrm{K}\left(\sin\frac{\theta}{2}\right) = 2\sum_0^\infty \frac{(-)^n}{2n+1}\mathrm{P}_n(\mu). \quad . \quad (9)$$

It follows from (8) that

$$\int_{-1}^1 \mathrm{P}_n(\mu)\mathrm{K}\left(\sqrt{\frac{1+\mu}{2}}\right)d\mu = \frac{4}{(2n+1)^2} \quad . \quad . \quad (10)$$

or

$$\int_0^1 k\mathrm{K}\mathrm{P}_n(2k^2-1)dk = \frac{1}{(2n+1)^2}. \quad . \quad . \quad . \quad (11)$$

Now when n is large,

$$\mathrm{P}_n(\mu) = \sqrt{\frac{2}{n\pi\sin\theta}} \cdot \cos\left\{(n+\tfrac{1}{2})\theta + \frac{\pi}{4}\right\} \quad . \quad (12)$$

and the series (8) and (9) are accordingly absolutely convergent between $\theta = 0$ and π, but *at* these limits the theorems are not true. The integrals further deduced, however, may be taken over the entire range, the integrals being finite at each limit.

Squaring (8) and integrating with respect to μ,

$$\int_{-1}^1 \mathrm{K}^2(k)d\mu = 4\sum_0^\infty \int_{-1}^1 \frac{\mathrm{P}_n^2(\mu)d\mu}{(2n+1)^2} + 4\sum_0^\infty \int_{-1}^1 \frac{\mathrm{P}_n(\mu)\mathrm{P}_m(\mu)d\mu}{(2n+1)(2m+1)}$$

$$= 8\sum_0^\infty \frac{1}{(2n+1)^3}.$$

Let

$$\sigma_{2m+1} = \frac{1}{1^{2m+1}} + \frac{1}{2^{2m+1}} + \frac{1}{3^{2m+1}} + \cdots \quad . \quad . \quad (13)$$

$$\therefore \quad \int_0^1 k\mathrm{K}^2 dk = \frac{7}{4}\sigma_3. \quad . \quad . \quad . \quad (14)$$

The formulæ arising in a similar way from (9) are

$$\int_{-1}^1 \mathrm{P}_n(\mu)\cdot\mathrm{K}'\left(\sqrt{\frac{1+\mu}{2}}\right)d\mu = (-)^n\frac{4}{(2n+1)^2} \quad . \quad (15)$$

$$\int_0^1 k\mathrm{K}'\mathrm{P}_n(2k^2-1)dk = \frac{(-)^n}{(2n+1)^2} \quad . \quad . \quad . \quad (16)$$

and

$$\int_0^1 kK'^2 dk = \frac{7}{4}\sigma_3 \quad . \quad . \quad . \quad . \quad (17)$$

and we note that *if n is even,*

$$\int_0^1 k(K-K')P_n(2k^2-1)dk = 0 \quad . \quad . \quad . \quad (18)$$

and also that

$$\int_0^1 k(K^2-K'^2)dk = 0 \quad . \quad . \quad . \quad . \quad (19)$$

Combining (7) and (8)

$$\int_0^\pi K\left(\cos\frac{\theta}{2}\right)\sin\frac{\theta}{2}d\theta = 2\sum_0^\infty \frac{1}{2n+1}\int_0^\pi P_n(\cos\theta)\sin\frac{\theta}{2}d\theta$$

$$= 4\sum_0^\infty \frac{(-)^n}{(2n+1)^2}$$

$$\therefore \quad \int_0^1 K dk = 2\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \dots\right) \quad . \quad . \quad (20)$$

while

$$\int_0^1 K'dk = \frac{1}{2}\sum_0^\infty \frac{4}{(2n+1)^2},$$

$$\therefore \quad \int_0^1 K'dk = \frac{\pi^2}{4}. \quad . \quad . \quad . \quad . \quad (21)$$

Again,

$$\int_{-1}^1 KK'd\mu = \sum_0^\infty \frac{2}{2n+1}\int_{-1}^1 P_n(\mu)K'\left(\sqrt{\frac{1+\mu}{2}}\right)d\mu$$

$$= 8\left\{1 - \frac{1}{3^3} + \frac{1}{5^3}\dots\right\}$$

$$= 8\left(\frac{\pi}{2}\right)^3 \frac{E_1}{2\,2\,!}, \quad \text{by the use of Euler's numbers}$$

$$= \frac{\pi^3}{4};$$

$$\therefore \quad \int_0^1 kKK'dk = \frac{\pi^3}{16}. \quad . \quad . \quad . \quad . \quad (22)$$

Expressions may be found for integrals containing higher powers of K, K′, but cannot be concisely exhibited.

A theorem will be used, which will be stated as follows, and is easy to prove.

If m is equal to, or less than n,

$$P_m(z)P_n(z) = \sum_{r=0}^{m} \frac{A_r A_{m-r} A_{n-r}}{A_{m+n-r}} \left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right) P_{n+m-2r}(z) \quad (23)$$

where

$$A_r = \frac{2m\,!}{2^m (m\,!)^2} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Now

$$\int_{-1}^{1} K^2\left(\cos\frac{\theta}{2}\right) P_n(\mu) d\mu = \sum_{s=0}^{\infty} \frac{2}{2s+1} \int_{-1}^{1} K\left(\cos\frac{\theta}{2}\right) P_s(\mu) P_n(\mu) d\mu.$$

Write (23) in the form

$$P_m(z)P_n(z) = \sum_{r=0}^{m} \left(\lambda_r\right)_n^m P_{(n+m-2r)}(z) \quad . \quad . \quad (25)$$

where $m \angle n$;

$$\therefore \int_{-1}^{1} K^2\left(\cos\frac{\theta}{2}\right) P_n(\mu) d\mu$$

$$= \sum_{s=0}^{n} \frac{2}{2s+1} \int_{-1}^{1} K\left(\cos\frac{\theta}{2}\right) \sum_{r=0}^{s} \left(\lambda_r\right)_n^s P_{n+s-2r}(\cos\theta) d\mu$$

$$+ \sum_{s=n+1}^{s=\infty} \frac{2}{2s+1} \int_{-1}^{1} K\left(\cos\frac{\theta}{2}\right) \sum_{r=0}^{n} \left(\lambda_r\right)_s^n P_{n+s-2r}(\cos\theta) d\mu$$

$$= \sum_{s=0}^{n} \sum_{r=0}^{s} \frac{2}{2s+1} \left(\lambda_r\right)_n^s \cdot \frac{2}{2n+2s-4r+1}$$

$$+ \sum_{s=n+1}^{\infty} \sum_{r=0}^{n} \frac{2}{2s+1} \left(\lambda_r\right)_s^n \frac{2}{2n+2s-4r+1}.$$

$$\therefore \int_{0}^{1} k K^2 P_n(2k^2-1) dk = \sum_{s=0}^{n} \sum_{r=0}^{s} \frac{(\lambda_r)_n^s}{(2s+1)(2n+2s-4r+1)}$$

$$+ \sum_{s=n+1}^{\infty} \sum_{r=0}^{n} \frac{(\lambda_r)_s^n}{(2s+1)(2n+2s-4r+1)} \quad . \quad . \quad . \quad (26)$$

and

$$\int_{0}^{1} k K^3 dk = \frac{1}{4} \int_{-1}^{1} K^2(k) \cdot \sum_{0}^{\infty} \frac{2}{2n+1} P_n(\mu) d\mu.$$

$$\therefore \int_{0}^{1} k K^3 dk = 2 \sum_{n=0}^{\infty} \sum_{s=0}^{n} \sum_{r=0}^{s} \frac{(\lambda_r)_n^s}{(2s+1)(2n+1)(2n+2s-4r+1)}$$

$$+ 2 \sum_{n=0}^{\infty} \sum_{s=n+1}^{\infty} \sum_{r=0}^{n} \frac{(\lambda_r)_s^n}{(2s+1)(2n+1)(2n+2s-4r+1)} \quad . \quad (27)$$

where $(\lambda_r)_n^s = (\lambda_r)_s^n$, and is defined in (25).

This process is general, and we can theoretically find a general expression for $\int_0^1 k\mathrm{K}^r\mathrm{K}'^s dk$, which, however, is of no importance.

Results will now be obtained involving the complete elliptic integral

$$\mathrm{E}(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\phi}\, d\phi \quad . \quad . \quad . \quad (28)$$

The formulæ connecting E and K are

$$k\frac{d\mathrm{E}}{dk} = \mathrm{E} - \mathrm{K}, \quad . \quad . \quad . \quad . \quad (29\,a)$$

$$\frac{\mathrm{E}}{1 - k^2} = \mathrm{K} + k\frac{d\mathrm{K}}{dk} \quad . \quad . \quad . \quad . \quad (29\,b)$$

Integrating (10) by parts,

$$\int_{-1}^1 \mathrm{P}_n(\mu)\mathrm{K}(k)d\mu$$

$$= \left[-\frac{\mathrm{K}(k)}{n(n+1)} (1-\mu^2)\frac{d\mathrm{P}_n}{d\mu} \right]_{-1}^1 + \frac{1}{n(n+1)}\int_{-1}^1 (1-\mu^2)\frac{d\mathrm{P}_n}{d\mu} \cdot \frac{d\mathrm{K}(k)}{dk}\, d\mu$$

$$= \frac{4}{n(n+1)}\int_0^1 \mathrm{P}_n'(2k^2-1) \cdot k^2 k'^2 \frac{d\mathrm{K}}{dk}\, dk$$

$$= \frac{4}{n(n+1)}\int_0^1 \mathrm{P}_n(2k^2-1)\{\mathrm{E} - k'^2\mathrm{K}\}k\,dk.$$

$$\therefore \int_0^1 k\{\mathrm{E} - k'^2\mathrm{K}\}\mathrm{P}_n'(2k^2-1)dk = \frac{n(n+1)}{(2n+1)^2} \cdot \quad . \quad (30)$$

Therefore changing the variables k, k',

$$\int_0^1 k\{\mathrm{E}' - k^2\mathrm{K}'\}\mathrm{P}_n'(2k^2-1)dk = (-)^{n-1}\frac{n(n+1)}{(2n+1)^2} \quad . \quad (31)$$

An integration of (30) by parts gives

$$\frac{4n(n+1)}{(2n+1)^2} = \left[(\mathrm{E} - k'^2\mathrm{K})\mathrm{P}_n(\mu) \right]_{-1}^1 - \int_{-1}^1 \mathrm{P}_n(\mu)\frac{d}{4k\,dk}\left\{ \mathrm{E} - \mathrm{K}k'^2 \right\}d\mu.$$

The integrated term is zero, for $\mathrm{lt}_{k'=0}\, k'^2 \log\frac{4}{k'} = 0$,

and, again, $\qquad \dfrac{d}{kdk}\left\{ \mathrm{E} - \mathrm{K}k'^2 \right\} = \mathrm{K}$ by (29).

$$\therefore \quad \frac{4n(n+1)}{(2n+1)^2} = \mathrm{E}(1)\mathrm{P}_n(1) - \int_0^1 k\mathrm{K}\mathrm{P}_n(2k^2-1)dk,$$

giving

$$\int_0^1 k \mathrm{K} \mathrm{P}_n(2k^2-1)dk = \frac{1}{(2n+1)^2}$$

which is equation (10) again.

We now proceed to treat (15) in the same way as (10). The formulæ connecting E' and K' are

$$\frac{d\mathrm{E}'}{dk} = -\frac{k}{k'^2}(\mathrm{E}'-\mathrm{K}') \quad . \quad . \quad . \quad . \quad (32\,a)$$

$$\frac{d\mathrm{K}'}{dk} = -\frac{1}{kk'^2}\left\{\mathrm{E}'-k^2\mathrm{K}'\right\} \quad . \quad . \quad . \quad (32\,b)$$

(15) gives

$$(-)^n \frac{4}{(2n+1)^2} = \left[\frac{-\mathrm{K}'(k)}{n(n+1)}(1-\mu^2)\frac{d\mathrm{P}_n}{d\mu}\right]_{-1}^1 + \int_{-1}^1 \frac{(1-\mu^2)}{n(n+1)}\frac{d\mathrm{P}_n}{d\mu}\frac{d}{d\mu}(\mathrm{K}')d\mu.$$

$$\therefore \int_0^1 (1-\mu^2)\frac{d\mathrm{P}_n}{d\mu}\frac{d\mathrm{K}'}{dk}dk = (-)^n \frac{4n(n+1)}{(2n+1)^2},$$

or, by (32 b),

$$\int_0^1 k\{k^2\mathrm{K}'-\mathrm{E}'\}\mathrm{P}_n'(2k^2-1)dk = (-)^n \frac{n(n+1)}{(2n+1)^2},$$

a repetition of (31) by a different method.

We now consider another expansion in a series of Legendre functions, commencing with the equation

$$1 + \cos 2z = \frac{4}{\pi}\left\{\frac{\cos z}{1.3} + \frac{\cos 3z}{1.3.5} - \frac{\cos 5z}{3.5.7} \cdots + \frac{(-)^{n-1}\cos(2n+1)z}{(2n-1)(2n+1)(2n+3)} + \cdots\right\}$$

$$\cdots \cdots (33$$

It leads to

$$\frac{1}{2\sqrt{2}}\int_0^\theta \frac{(1+\cos\phi)d\phi}{\sqrt{\cos\phi-\cos\theta}} = \frac{1}{3}\mathrm{P}_0(\mu) + \sum_1^\infty \frac{(-)^{n-1}\mathrm{P}_n(\mu)}{(2n-1)(2n+1)(2n+3)}.$$

But

$$\frac{1+\cos\phi}{\sqrt{\cos\phi-\cos\theta}} = \sqrt{\cos\phi-\cos\theta} + \frac{1+\cos\theta}{\sqrt{\cos\phi-\cos\theta}},$$

$$\int_0^\theta \frac{d\phi}{\sqrt{\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}}} = 2\mathrm{K}'\left(\cos\frac{\theta}{2}\right),$$

$$\int_0^\theta \sqrt{\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}}\,d\phi = 2\int_0^{\frac{\pi}{2}} \frac{\sin\frac{\theta}{2}.\tan\frac{\theta}{2}.\sin^2\lambda\,d\lambda}{\left(1+\tan^2\frac{\theta}{2}\cos^2\lambda\right)^{\frac{3}{2}}},$$

with the usual substitution.

If $\cos\dfrac{\theta}{2} = k$, this becomes

$$2\int_0^{\frac{\pi}{2}} \frac{k^2 k'^2 \sin^2\lambda \, d\lambda}{(1 - k'^2 \sin^2\lambda)^{\frac{3}{2}}} = -4k^2 k'^2 \frac{d\mathrm{K}'}{dk^2}$$

$$= 2(\mathrm{E}' - k^2\mathrm{K}') \text{ by } (32\,b).$$

$$\therefore \quad \tfrac{1}{3}\mathrm{P}_0(\mu) + \sum_1^\infty \frac{(-)^{n-1}\mathrm{P}_n(\mu)}{(2n-1)(2n+1)(2n+3)}$$

$$= \frac{1}{2\sqrt{2}} \cdot \left\{ 2\sqrt{2}\,(\mathrm{E}' - k^2\mathrm{K}') + \frac{(1+\cos\theta)}{\sqrt{2}} \dots 2\mathrm{K}' \right\}$$

$$= \mathrm{E}' - k^2\mathrm{K}' + \frac{2k^2 \cdot \sqrt{2}}{2\sqrt{2}}\mathrm{K}'$$

$$= \mathrm{E}'.$$

$$\therefore \quad \mathrm{E}'\!\left(\cos\frac{\theta}{2}\right) = \sum_0^\infty \frac{(-)^{n-1}\mathrm{P}_n(\mu)}{(2n-1)(2n+1)(2n+3)} \quad \text{if } \mu = \cos\theta \quad . \quad (34)$$

The series being absolutely convergent.
It follows that

$$\int_{-1}^1 \mathrm{E}'(k)\mathrm{P}_n(\mu)d\mu = \frac{(-)^{n-1}\,2}{(2n-1)(2n+1)^2(2n+3)} \quad \cdot \quad \cdot \quad \cdot \quad (35)$$

$$\therefore \quad \int_0^\pi \mathrm{E}'\!\left(\cos\frac{\theta}{2}\right)\mathrm{P}_n(\cos\theta)\sin\theta\,d\theta = \frac{(-)^{n-1}\,2}{(2n-1)(2n+3)(2n+1)^2}. \quad (36)$$

Combining (7) and (34)

$$\int_0^\pi \mathrm{E}'\!\left(\cos\frac{\theta}{2}\right)\sin\frac{\theta}{2}\,d\theta = -2\sum_0^\infty \frac{1}{(2n-1)(2n+3)(2n+1)^2}$$

$$= 2\sum_0^\infty \frac{1}{(2n+1)^2} \text{ on reduction}$$

$$= \frac{\pi^2}{4}.$$

$$\therefore \quad \int_0^1 \mathrm{E}'(k)dk = \frac{\pi^2}{8}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (37)$$

Squaring (34) and integrating,

$$\int_{-1}^1 \mathrm{E}'^2(k)d\mu = 2\sum_0^\infty \frac{1}{(2n-1)^2(2n+3)^2(2n+1)^3}.$$

Now

$$\left(\frac{1}{2n-1} - \frac{2}{2n+1} + \frac{1}{2n+3}\right)^2 = \frac{64}{(2n-1)^2(2n+1)^2(2n+3)^2}.$$

$$\therefore \quad \int_0^1 (\mathrm{E}'(k))^2 k\, dk$$

$$= \frac{1}{128} \overset{\infty}{\underset{0}{\Sigma}} \frac{1}{2n+1} \left(\frac{1}{2n-1} + \frac{1}{2n+3} - \frac{2}{2n+1}\right)^2$$

$$= \frac{1}{128} \overset{\infty}{\underset{0}{\Sigma}} \left[\frac{1}{2n+1}\left(\frac{1}{(2n-1)^2} + \frac{1}{(2n+3)^2} + \frac{4}{(2n+1)^2}\right) + \frac{3}{2}\right]$$

$$= \frac{1}{32} \overset{\infty}{\underset{0}{\Sigma}} \frac{1}{(2n+1)^3},$$

and if $\sigma_3 = \dfrac{1}{1^3} + \dfrac{1}{2^3} + \dfrac{1}{3^3} + \dots\dots$

$$\therefore \quad \int_0^1 (\mathrm{E}'(k))^2 k\, dk = \frac{7}{256}\sigma_3. \quad \dots \quad \dots \quad (38)$$

We now integrate (35) by parts,

$$\therefore \quad \frac{(-)^{n-1}2}{(2n-1)(2n+3)(2n+1)^2} = \left[\frac{-\mathrm{E}'(k)}{n(n+1)}(1-\mu^2)\frac{d\mathrm{P}_n}{d\mu}\right]_{-1}^{1}$$

$$+ \frac{1}{n(n+1)}\int_{-1}^{1}(1-\mu^2)\frac{d\mathrm{P}_n}{d\mu}\frac{d\mathrm{E}'(k)}{d\mu}\,d\mu,$$

$$\therefore \quad \int_0^1 (1-\mu^2)\frac{d\mathrm{P}_n}{d\mu}\frac{d\mathrm{E}'}{dk}\,dk = \frac{(-)^{n-1}2n(n+1)}{(2n-1)(2n+3)(2n+1)^2},$$

or

$$\int_0^1 k^3(\mathrm{E}'-\mathrm{K}')\mathrm{P}_n'(2k^2-1)dk$$

$$= \frac{(-)^n}{2}\cdot\frac{n(n+1)}{(2n-1)(2n+3)(2n+1)^2}. \quad \cdot \quad (39)$$

$\therefore$ Changing the variable from k to k',

$$\int_0^1 k(1-k^2)(\mathrm{E}-\mathrm{K})\mathrm{P}_n'(2k^2-1)dk$$

$$= \frac{1}{2}\cdot\frac{n(n+1)}{(2n-1)(2n+3)(2n+1)^2}. \quad \cdot \quad (40)$$

Combining (40) with (30), and (39) with (31) respectively, we find

$$\int_0^1 k^3\mathrm{E}\,\mathrm{P}_n'(2k^2-1)dk = \frac{n(n+1)}{(2n+1)^2}\left\{1 - \frac{1}{2(2n+3)(2n-1)}\right\}$$

$$= \frac{n(n+1)(8n^2+8n-7)}{2(2n+3)(2n-1)(2n+1)^2}, \quad \cdot \quad (41)$$

and

$$\int_0^1 kk'^2 E' P_n'(2k^2-1)dk = \frac{(-)^{n-1}}{2} \cdot \frac{n(n+1)}{(2n+1)^2} \cdot \frac{8n^2+8n-5}{(2n-1)(2n+3)} \cdot \quad (42)$$

Now in (34), write $(\pi-\theta)$ for θ.

$$\therefore \quad E\left(\cos\frac{\theta}{2}\right) = -\Sigma_0^\infty \frac{P_n(\mu)}{(2n+1)(2n-1)(2n+3)}, \quad \cdot \quad (43)$$

whence

$$\int_0^\pi E\left(\cos\frac{\theta}{2}\right) P_n(\cos\theta) \sin\theta d\theta = \frac{2}{(2n-1)(2n+3)(2n+1)^2}. \quad (44)$$

Combining (7) and (43)

$$\int_0^\pi E\left(\cos\frac{\theta}{2}\right) \sin\frac{\theta}{2} d\theta = -2\Sigma_0^\infty \frac{(-)^n}{(2n-1)(2n+3)(2n+1)^2}$$

$$= \left\{ 3 - \frac{2}{3^2} + \frac{2}{5^2} \dots \right\};$$

$$\therefore \quad \int_0^1 E(k)dk = \left\{ 1 - \frac{1}{3^2} + \frac{1}{5^2} \dots \right\} + \frac{1}{2}. \quad \cdot \quad (45)$$

We note that

$$\int_0^1 Edk = \frac{1}{2}\int_0^1 Kdk + \frac{1}{2}, \quad \cdot \quad \cdot \quad \cdot \quad (46)$$

while

$$\int_0^1 E'dk = \frac{1}{2}\int_0^1 K'dk \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (47)$$

These might be obtained by integration by parts.

By (43)

$$\int_{-1}^1 E^2\left(\cos\frac{\theta}{2}\right)d\mu = 2\Sigma_0^\infty \frac{1}{(2n+1)^3(2n-1)^2(2n+3)^2}$$

$$= \int_{-1}^1 E'^2\left(\cos\frac{\theta}{2}\right)d\mu,$$

$\therefore$ by previous result

$$\int_0^1 (E(k))^2 kdk = \frac{7}{256}\sigma_3. \quad \cdot \quad \cdot \quad \cdot \quad (48)$$

Again

$$\int_{-1}^1 EE'd\mu = \Sigma_0^\infty \frac{2}{2n+1} \cdot \frac{(-)^n}{(2n-1)^2(2n+1)^2(2n+3)^2}$$

T 2

$$\therefore \quad \int_0^1 k\mathrm{E}\mathrm{E}'dk = \frac{1}{2}\Sigma_0^\infty \frac{(-)^n}{(2n-1)^2(2n+1)^3(2n+3)^2}$$

$$= \frac{1}{32}\Sigma_0^\infty \frac{(-)^n}{(2n+1)^3}$$

$$= \frac{\pi^3}{1024}.$$

Hence

$$\int_0^1 k\mathrm{E}\mathrm{E}'dk = \frac{\pi^3}{1024} = \frac{1}{64}\int_0^1 k\mathrm{K}\mathrm{K}'dk. \quad . \quad . \quad (49)$$

Again, by (43) and (9),

$$\int_0^1 k\mathrm{E}\mathrm{K}'dk = -\Sigma_0^\infty \frac{1}{(2n+1)^2}\frac{(-1)^n}{(2n+1)(2n-1)(2n+3)}$$

$$= \Sigma_0^\infty \frac{(-1)^{n-1}}{(2n-1)(2n+3)(2n+1)^3},$$

and

$$\int_0^1 k\mathrm{E}'\mathrm{K}dk = \Sigma_0^\infty \frac{(-)^{n-1}}{(2n-1)(2n+3)(2n+1)^3}$$

$$= \int_0^1 k\mathrm{E}\mathrm{K}'dk \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Now

$$\mathrm{E}\mathrm{K}' + \mathrm{E}'\mathrm{K} = \mathrm{K}\mathrm{K}' + \frac{\pi}{2}$$

(Cayley, Elliptic Functions, p. 48)

$$\int_0^1 k\mathrm{E}\mathrm{K}'dk = \frac{1}{2}\int_0^1 k\left\{\mathrm{K}\mathrm{K}' + \frac{\pi}{2}\right\}dk$$

$$= \frac{1}{2}\left(\frac{\pi^3}{16} + \frac{\pi}{4}\right),$$

$$\therefore \quad \int_0^1 k\mathrm{E}\mathrm{K}'dk = \int_0^1 k\mathrm{E}'\mathrm{K}dk = \frac{\pi^3}{32} + \frac{\pi}{8}. \quad . \quad . \quad (51)$$

Again,

$$\int_0^1 k\mathrm{K}\mathrm{E}dk = -\Sigma_0^\infty \frac{1}{(2n-1)(2n+3)(2n+1)^3},$$

and

$$\int_0^1 k\mathrm{K}'\mathrm{E}'dk = -\Sigma_0^\infty \frac{1}{(2n-1)(2n+3)(2n+1)^3};$$

$$\therefore \quad \int_0^1 k\mathrm{K}\mathrm{E}dk = \int_0^1 k\mathrm{K}'\mathrm{E}'dk \quad . \quad . \quad . \quad . \quad (52)$$

The value of each

$$= -\frac{1}{4} \overset{\infty}{\underset{0}{\Sigma}} \left(\frac{1}{2n-1} - \frac{1}{2n+3} \right) \cdot \frac{1}{(2n+1)^3}$$

$$= -\frac{1}{4} \overset{\infty}{\underset{0}{\Sigma}} \frac{1}{8} \left(\frac{1}{2n-1} + \frac{1}{2n+3} \right) - \frac{1}{8} \left(\frac{4}{(2n+1)^3} + \frac{2}{(2n+1)^2} + \frac{1}{(2n+1)} \right)$$
$$- \frac{1}{8} \left\{ \frac{4}{(2n+1)^3} - \frac{2}{(2n+1)^2} + \frac{1}{2n+1} \right\}$$

$$= \frac{1}{4} \overset{\infty}{\underset{0}{\Sigma}} \frac{1}{(2n+1)^3} + \frac{1}{16}$$

$$= \frac{7}{32} \sigma_3 + \frac{1}{16};$$

$$\therefore \qquad \int_0^1 k\mathrm{K}\mathrm{E} dk = \int_0^1 k\mathrm{K'E'} dk = \frac{7}{32} \sigma_3 + \frac{1}{16} \quad . \quad . \quad (53)$$

We may employ the result

$$\int_{-1}^1 (1-\mu^2)^r \frac{d^r\mathrm{P}_m(\mu)}{d\mu^r} \frac{d^r\mathrm{P}_n(\mu)}{d\mu^r} d\mu = 0, \quad \text{or} \quad \frac{2}{2n+1} \frac{n+r\,!}{n-r\,!}, (54)$$

according as n and m are unequal or equal, provided r be a small enough integer. *E. g.*, we deduce, if $r=1$,

$$\int_{-1}^1 (1-\mu^2) \frac{d}{d\mu} \mathrm{K}(k) \cdot \frac{d\mathrm{P}_n}{d\mu} d\mu = \frac{2n(n+1)}{(2n+1)^2},$$

which coincides with a previous result,
and also

$$\int_{-1}^1 (1-\mu^2) \frac{d\mathrm{E}}{d\mu} \frac{d\mathrm{P}_n}{d\mu} d\mu = - \frac{2n(n+1)}{(2n-1)(2n+1)^2(2n+3)},$$

also a previous result.

If $r=2$,

$$\int_{-1}^1 (1-\mu^2)^2 \frac{d^2\mathrm{E}}{d\mu^2} \frac{d^2\mathrm{P}_n}{d\mu^2} d\mu = - \frac{2(n-1)(n)(n+1)(n+2)}{(2n-1)(2n+1)^2(2n+3)},$$

or

$$\int_0^1 k^3 k'^2 \mathrm{E} \cdot \mathrm{P}_n''(2k^2-1) dk = - \frac{1}{4} \cdot \frac{(n-1)(n)(n+1)(n+2)}{(2n-1)(2n+1)^2(2n+3)} \quad (55)$$

and further particular theorems follow in a similar way, the expansion for E′ giving rise to an equal number.

Some applications to incomplete elliptic integrals will now be treated. The elliptic integrals in question occur in the

integrands of other integrals, and the integration is now with respect to their amplitude, and not their modulus as in the above treatment. $E(\lambda)$ now will denote

$$\int_0^\lambda \sqrt{1 - k^2 \sin^2 \lambda}\, d\lambda, \qquad \text{and } F(\lambda) = \int_0^\lambda \frac{d\lambda}{\sqrt{1 - k^2 \sin^2 \lambda}}.$$

In all cases, the modulus is $k = \cos\dfrac{\theta}{2}$ for E, F, and $\sin\dfrac{\theta}{2} = k'$ for E', F'.

If z is between 0 and π,

$$\frac{\pi}{8}(\pi - 2z) = \cos z + \frac{1}{3^2}\cos 3z + \ldots\ldots ;$$

$$\therefore \quad \Sigma_0^\infty \frac{P_n(\mu)}{(2n+1)^2} = \frac{1}{8}\int_0^\theta \frac{(\pi - \phi)\,d\phi}{\sqrt{\cos^2\dfrac{\phi}{2} - \cos^2\dfrac{\theta}{2}}}.$$

But

$$\int_0^\theta \frac{\pi d\phi}{\sqrt{\cos^2\dfrac{\phi}{2} - \cos^2\dfrac{\theta}{2}}} = 2\pi K'\left(\cos\dfrac{\theta}{2}\right)$$

$$= 4\pi \Sigma_0^\infty \frac{(-)^n}{(2n+1)} P_n(\mu) \quad \text{by (9)};$$

$$\therefore \quad \int_0^\theta \frac{\phi d\phi}{\sqrt{\cos^2\dfrac{\phi}{2} - \cos^2\dfrac{\theta}{2}}} = 4\pi \Sigma_0^\infty \frac{(-)^n}{2n+1} P_n(\mu) - 8\Sigma_0^\infty \frac{P_n(\mu)}{(2n+1)^2} \quad (56)$$

Put

$$\tan\frac{\phi}{2} = \tan\frac{\theta}{2} \cdot \cos\lambda ;$$

$$\therefore \quad \int_0^\phi \frac{d\phi}{\sqrt{\cos^2\dfrac{\phi}{2} - \cos^2\dfrac{\theta}{2}}} = 2F'(\lambda, k \; :$$

$$\therefore \quad \int_0^\theta \frac{\phi d\phi}{\sqrt{\cos^2\dfrac{\phi}{2} - \cos^2\dfrac{\theta}{2}}} = \left[2\phi F'(\lambda)\right]_0^\theta - 2\int_0^{\frac{\pi}{2}} \frac{d\phi}{d\lambda} \cdot F'(\lambda)\,d\lambda$$

$$= 4\int_0^{\frac{\pi}{2}} \frac{F'(\lambda)\sin\lambda\,d\lambda}{1 + \tan^2\dfrac{\theta}{2}\cos^2\lambda} \cdot \tan\frac{\theta}{2},$$

since

$$\frac{d\phi}{d\lambda} = \frac{-2\sin\lambda \cdot \tan\dfrac{\theta}{2}}{1 + \tan^2\dfrac{\theta}{2}\cos^2\lambda} ;$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \frac{\mathrm{F}'(\lambda)\sin\lambda d\lambda}{k^2 + k'^2\cos^2\lambda} = \frac{\pi}{kk'}\sum_0^\infty \frac{(-)^n}{2n+1}\mathrm{P}_n(\mu) - \frac{2}{kk'}\sum_0^\infty \frac{\mathrm{P}_n(\mu)}{(2n+1)^2}, \quad (57)$$

if $k = \cos\dfrac{\theta}{2}, \quad \mu = \cos\theta.$

It follows that

$$\int_0^1 k^2 k' dk \int_0^{\frac{\pi}{2}} \frac{\mathrm{F}'(\lambda)\sin\lambda d\lambda}{k^2 + k'^2\cos^2\lambda} = \int_{-1}^1 \frac{\pi-2}{4} d\mu$$

$$= \frac{\pi}{2} - 1. \quad . \quad . \quad . \quad (58)$$

Also

$$\int_0^1 k^2 k' \mathrm{P}_n(2k^2-1) dk \int_0^{\frac{\pi}{2}} \frac{\mathrm{F}'(\lambda)\sin\lambda d\lambda}{k^2 + k'^2\cos^2\lambda} = \frac{(-)^n\pi}{2(2n+1)^2} - \frac{1}{(2n+1)^3}. \quad (59)$$

While squaring (57), and calling the integral on the left, I,

$$\int_{-1}^1 \mathrm{I} k^2 k'^2 d\mu = \sum_0^\infty \left(\frac{\pi(-)^n}{2n+1} - \frac{2}{(2n+1)^2} \right)^2 \frac{2}{2n+1}$$

$$= 2\pi^2 \cdot \frac{7}{8}\sigma_3 + \frac{31}{4}\sigma_5 - 8\pi\left(1 - \frac{1}{3^4} + \frac{1}{5^4}\cdots \right);$$

$$\therefore \quad \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{k^3 k'^2 \mathrm{F}'(\lambda)\mathrm{F}'(\mu)\,\sin\lambda\,\sin\mu\,dk\,d\lambda\,d\mu}{(k^2 + k'^2\cos^2\lambda)(k^2 + k'^2\cos^2\mu)}$$

$$= \frac{7}{16}\pi^2\sigma_3 + \frac{31}{8}\sigma_5 - 2\pi\left(1 - \frac{1}{3^4} + \frac{1}{5^4}\cdots \right). \quad (60)$$

It may be noted that if

$$\mathrm{G}(\theta) = \int_0^\theta \frac{(\pi-\phi)d\phi}{\sqrt{\cos^2\dfrac{\phi}{2} - \cos^2\dfrac{\theta}{2}}};$$

then

$$\int_0^\pi \mathrm{G}^2(\theta)\sin\theta d\theta = 64\sum_0^\infty \frac{2}{(2n+1)^5}$$

$$= 128\left(1 - \frac{1}{2^5} \right)\sigma_5;$$

$$\therefore \quad \int_0^\pi \mathrm{G}^2(\theta)\sin\theta d\theta = 124\sigma_5 \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Similar results follow from the series

$$\cos z - \frac{1}{3^3}\cos 3z + \frac{1}{5^3}\cos 5z\ldots \quad = \frac{\pi}{8}(\pi^2 - z^2) \quad z \text{ between } \pm\frac{\pi}{2};$$

for

$$\sum_0^\infty \frac{(-)^n}{(2n+1)^3} P_n(\mu) = \frac{1}{32} \int_0^\theta \frac{(4\pi^2 - \phi^2) d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}}$$

$$= \frac{\pi^2}{8} \cdot 2K' - \frac{1}{32} \int_0^\theta \frac{\phi^2 d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}}.$$

But by previous substitution

$$\int_0^\theta \frac{\phi^2 d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} = \left[2\phi^2 F'\left(\lambda, \cos\frac{\theta}{2}\right) \right]_{\phi=0}^\theta$$

$$+ 16 \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin\lambda \tan\frac{\theta}{2} \tan^{-1}\left(\frac{k'}{k}\cos\lambda\right) d\lambda}{1 + \tan^2\frac{\theta}{2}\cos^2\lambda}$$

$$= 16 k k' \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin\lambda \tan^{-1}\left(\frac{k'}{k}\cos\lambda\right) d\lambda}{k^2 + k'^2 \cos^2\lambda};$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \tan^{-1}\left(\frac{k'}{k}\cos\lambda\right) \sin\lambda \, d\lambda}{k^2 + k'^2 \cos^2\lambda} = \frac{\pi^2}{kk'} \sum_0^\infty \frac{(-)^n}{(2n+1)} P_n(\mu)$$

$$+ \frac{2}{kk'} \sum_0^\infty \frac{(-)^{n-1}}{(2n+1)^3} P_n(\mu), \quad (62)$$

and if

$$G_1(\theta) = \int_0^\theta \frac{(4\pi^2 - \phi^2) d\phi}{\sqrt{\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}}};$$

$$\therefore \int_0^\pi G_1^2(\theta) \sin\theta \, d\theta = (32)^2 \sum_0^\infty \frac{2}{(2n+1)^7}$$

$$= 2 \times (32)^2 \cdot \sigma_7\left(1 - \frac{1}{2^7}\right)$$

$$= 2032 \, \sigma_7, \quad . \quad . \quad . \quad . \quad (63)$$

and

$$\int_0^1 k^2 k' P_n(2k^2 - 1) dk \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin\lambda \tan^{-1}\left(\frac{k'\cos\lambda}{k}\right) d\lambda}{k^2 + k'^2 \cos^2\lambda}$$

$$= \frac{\pi(-)^n}{2(2n+1)^2} \perp \frac{(-)^{n-1}}{(2n+1)^4}; \quad . \quad (64)$$

also

$$\int_0^1 k^2 k' dk \int_0^{\frac{\pi}{2}} \frac{F'(\lambda) \sin \lambda \tan^{-1}\left(\frac{k'}{k}\cos\lambda\right) d\lambda}{k^2 + k'^2 \cos^2\lambda} = \frac{\pi}{2} - 1 \, ; \quad (65)$$

and squaring (62) and integrating to k from 0 to 1,

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{k^3 k'^2 F'(\lambda) F'(\mu) \tan^{-1}\left(\frac{k'}{k}\cos\lambda\right) \tan^{-1}\left(\frac{k'}{k}\cos\mu\right) \sin\lambda \sin\mu \, dk \, d\lambda \, d\mu}{(k^2 + k'^2 \cos^2\lambda)(k^2 + k'^2 \cos^2\mu)}$$

$$= \frac{1}{4} \sum_0^\infty \frac{2}{2n+1}\left(\frac{(-)^n \pi^2}{2n+1} + \frac{2(-)^{n-1}}{(2n+1)^3}\right)^2$$

$$= \frac{\pi^4}{2}\left(1 - \frac{1}{2^3}\right)\sigma_3 + 2\left(1 - \frac{1}{2^7}\right)\sigma_7 - 2\pi^2\left(1 - \frac{1}{2^5}\right)\sigma_5$$

$$= \frac{7}{16}\pi^4 \sigma_3 + \frac{127}{64}\sigma_7 - \frac{31}{16}\sigma_5 \, . \quad . \quad . \quad . \quad . \quad (66)$$

It may be remarked that formulæ of these kinds can be successively obtained, and the expansion of the integral

$$\int_0^{\frac{\pi}{2}} F'(\lambda)\left[\tan^{-1}\left(\frac{k'}{k}\cos\lambda\right)\right]^n \frac{\sin\lambda \, d\lambda}{k^2 + k'^2 \cos^2\lambda}$$

in terms of Legendre's functions, when n is any positive integer, by taking a succession of Fourier series of the kinds already considered in the last two cases.

We may write (56) in a form involving only complete elliptic integrals outside, for

$$\int_0^\theta \frac{\phi \, d\phi}{\sqrt{\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}}} = \left[\phi \cdot 2K'\cos\frac{\phi}{2}\right]_0^\theta - 2\int_0^\theta K'\left(\cos\frac{\phi}{2}\right) d\phi$$

$$\therefore \quad \int_0^\theta \frac{\phi \, d\phi}{\sqrt{\cos^2\frac{\phi}{2} - \cos^2\frac{\theta}{2}}} = 4\left\{K' \cdot \cos^{-1}k - \frac{1}{2}\int_0^\theta K'\left(\cos\frac{\phi}{2}\right) d\phi\right\}$$

$\therefore$ by (56),

$$2K'\cos^{-1}k - \int_0^\theta K'\left(\cos\frac{\phi}{2}\right) d\phi = 2\pi \sum_0^\infty \frac{(-)^n}{2n+1} P_n(\mu) - 4\sum_0^\infty \frac{P_n(\mu)}{(2n+1)^2} \, ;$$

$\therefore$ by (9),

$$K' \cdot (2\pi - \cos^{-1}k) + \frac{1}{2}\int_0^\theta K'\left(\cos\frac{\phi}{2}\right) d\phi = 2\sum_0^\infty \frac{P_n(\mu)}{(2n+1)^2} . \quad (67)$$

By changing θ to $\pi - \theta$,

$$\mathrm{K}\left(\cos^{-1}k - \frac{\pi}{2}\right) - \frac{1}{2}\int_0^{\pi-\theta}\mathrm{K}'\left(\cos\frac{\phi}{2}\right)d\phi = 2\overset{\infty}{\underset{0}{\Sigma}}\frac{(-)^n}{(2n+1)^2}\mathrm{P}_n(\mu). \quad (68)$$

These with (8) and (9) give the curious results

$$\left\{2\pi - \theta + \frac{1}{2}\int_0^\theta \frac{\mathrm{K}'\left(\cos\dfrac{\phi}{2}\right)}{\mathrm{K}'\left(\cos\dfrac{\theta}{2}\right)}d\phi\right\}\overset{\infty}{\underset{0}{\Sigma}}\frac{(-)^n}{2n+1}\mathrm{P}_n(\cos\theta)$$

$$= 4\overset{\infty}{\underset{0}{\Sigma}}\frac{\mathrm{P}_n(\cos\theta)}{(2n+1)^2}, \quad . \quad . \quad (69)$$

and

$$\left\{\theta - \frac{\pi}{2} + \frac{1}{2}\int_\theta^\pi \frac{\mathrm{K}\left(\cos\dfrac{\phi}{2}\right)}{\mathrm{K}\left(\cos\dfrac{\theta}{2}\right)}d\phi\right\}\overset{\infty}{\underset{0}{\Sigma}}\frac{\mathrm{P}_n(\cos\theta)}{2n+1},$$

$$= 4\overset{\infty}{\underset{0}{\Sigma}}\frac{(-)^n\mathrm{P}_n(\cos\theta)}{(2n+1)^2}, \quad (70)$$

which hold from $\theta = 0$ to 2π except at $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$.

If

$$\mathrm{I}_1 = \int_0^\theta \frac{\mathrm{K}'\left(\cos\dfrac{\phi}{2}\right)}{\mathrm{K}'\left(\cos\dfrac{\theta}{2}\right)}d\phi, \qquad \mathrm{I}_2 = \int_\theta^\pi \frac{\mathrm{K}\left(\cos\dfrac{\phi}{2}\right)}{\mathrm{K}\left(\cos\dfrac{\theta}{2}\right)}\,.\,d\phi.$$

We may at once deduce certain integrals as follows:

$$\int_0^1 k(4\pi - 2\cos^{-1}k + \mathrm{I}_1)\,.\,\mathrm{K}'\,.\,dk = 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

$$\int_0^1 k(\mathrm{I}_2 - \pi + 2\cos^{-1}k)\,.\,\mathrm{K}\,.\,dk = 2 \quad . \quad . \quad . \quad . \quad . \quad (72)$$

$$\int_0^1 k(4\pi - 2\cos^{-1}k + \mathrm{T}_1)\,.\,\mathrm{K}'\,.\,\mathrm{P}_n(2k^2-1)\,.\,dk = \frac{7}{4}\sigma_3 \quad . \quad (73)$$

$$\int_0^1 k(\mathrm{I}_2 - \pi + 2\cos^{-1}k)\,.\,\mathrm{K}\,.\,\mathrm{P}_n(2k^2-1)\,.\,dk = \frac{\pi^3}{16} \quad . \quad . \quad (74)$$

$$\int_0^1 k(4\pi - 2\cos^{-1}k + \mathrm{I}_1)^2\mathrm{K}'^2 dk = \frac{31}{4}\sigma_5 \quad . \quad . \quad . \quad . \quad (75)$$

$$\int_0^1 k(\mathrm{I}_2 - \pi + 2\cos^{-1}k)^2\,.\,\mathrm{K}^2\,.\,dk = \frac{31}{4}\sigma_5\,. \quad . \quad . \quad . \quad (76)$$

From (67) and (7),

$$\int_0^1 (4\pi - 2\cos^{-1}k + I_1) \cdot K' \cdot dk = 4\left(1 - \frac{1}{3^3} + \frac{1}{5^3}\cdots\right)$$

$$= \frac{\pi^3}{8};$$

∴ by (21),

$$\int_0^1 K' \cdot (I_1 - 2\cos^{-1}k) \cdot dk = \frac{7}{8}\pi^3 . \quad . \quad . \quad . \quad (77)$$

By (68) and (7),

$$\int_0^1 (2\cos^{-1}k - \pi + I_2) \cdot K \cdot dk = \frac{7}{2}\sigma_3. \quad . \quad . \quad (78)$$

Integrating all the above by parts, a large number of integrals involving E and K, E′ and K′ may be deduced.

By combinations of (67) and (68) with previous series of Legendre functions, we may deduce a large number of definite integrals of functions of k from 0 to 1. The following are examples :

(1) $k\mathrm{PQ}\left\{\cos^{-1}k + \frac{1}{2}\left(I_1, I_2\right)\right\}$ where P=K, K′, E, E′

(2) $k\mathrm{PQ}\left\{\sin^{-1}k + \frac{1}{2}\left(I_1, I_2\right)\right\}$,, Q=K, K′, E, E′

and these integrals, when found, will on integration by parts give rise to great numbers of elliptic integrals, integrated with respect to their modulus.

It may here be noted, in connexion with the result (62) and similar results, that by a similar investigation with series of the type

$$\frac{\pi y}{4} = \sin y - \frac{1}{3^2}\sin 3y + \frac{1}{5^2}\sin 5y\ldots$$

when y is between $\pm\frac{\pi}{2}$, we may in all cases expand, in a series of Legendre functions, the integral

$$\int_0^{\frac{\pi}{2}} \mathrm{F}(\lambda) \frac{\left(\tan^{-1}\left(\frac{k'}{k}\sin\lambda\right)\right)^n}{k'^2 + k^2\sin^2\lambda}\cos\lambda\, d\lambda,$$

where n is a positive integer, and $F(\lambda)$ is the incomplete elliptic integral

$$\int_0^\lambda \frac{d\lambda}{\sqrt{1-k^2 \sin^2 \lambda}}.$$

Considering now the equation (62), which may be written

$$\int_0^\theta \frac{\phi^2 d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} = 8\pi^2 K' - 32\sum_0^\infty \frac{(-)^n}{(2n+1)^3} P_n(\mu),$$

we have

$$\int_0^\theta \frac{\phi^2 d\phi}{\sqrt{\cos^2 \frac{\phi}{2} - \cos^2 \frac{\theta}{2}}} = \left[2\phi^2 K'\left(\cos\frac{\phi}{2}\right) \right]_0^\theta - 4\int_0^\theta \phi K'\left(\cos\frac{\phi}{2}\right) d\phi.$$

In this and above, (66)–(70), $K'\left(\cos\dfrac{\phi}{2}\right)$ denotes an incomplete integral equal to K' when $\phi = \theta$.

$$\therefore \ (\pi - \cos^{-1} k)(\pi + \cos^{-1} k) K' + \int_0^\theta \phi K'\left(\cos\frac{\phi}{2}\right) d\phi = 4\sum_0^\infty \frac{(-)^n}{(2n+1)^3}(P_n\mu), \quad (79)$$

and putting $\pi - \theta$ for θ in the above,

$$\therefore \ \left[2\phi^2 K'\left(\cos\frac{\phi}{2}\right) \right]_0^{\pi-\theta} - 4\int_0^{\pi-\theta} \phi K'\left(\cos\frac{\phi}{2}\right) d\phi = 8\pi^2 K - 32\sum_0^\infty \frac{P_n(\mu)}{(2n+1)^3}$$

$$\therefore \ \sin^{-1} k\{\pi + \sin^{-1} k\} K + \int_0^{\pi-\theta} \phi K'\left(\cos\frac{\phi}{2}\right) d\phi = 4\sum_0^\infty \frac{P_n(\mu)}{(2n+1)^3} \quad . \quad . \quad . \quad (80)$$

Proceeding in this manner, it follows that

$$\sum_0^\infty \frac{P_n(\mu)}{(2n+1)^r}, \quad \sum_0^\infty \frac{(-)^n P_n(\mu)}{(2n+1)^r},$$

where r is a positive integer, can be expressed in a similar manner to the above in terms of Jacobian elliptic functions, and their integrals with respect to a function of their amplitude.

The sum of all such series can therefore be concisely expressed.

XXVII. *Notices respecting New Books.*

Die Heterogenen Gleichgewichte vom Standpunkte der Phasenlehre.
Von Dr. H. W. BAKHUIS ROOZEBOOM, *Professor an der Universität Amsterdam.* Zweites Heft. Systeme aus zwei Komponenten. Erster Teil. Mit 149 eingedruckten Abbildungen und zwei Tafeln. Braunschweig: F. Vieweg und Sohn. 1904. Pp. xii + 468.

AN interval of three years has elapsed since the publication of Vol. I. of this monumental work, and we now welcome the appearance of Part 1 of Vol. II., which deals in a very exhaustive manner with the equilibrium between the solid, liquid, and gaseous phases of such binary systems as contain only the two components in the solid phase, the case of compounds or mixed crystals being reserved for future treatment in the second part of Vol. II.

After a brief general introduction, the author begins with the study of the equilibrium between a liquid and a gaseous phase. A characteristic feature of the author's method of treatment is the free use made of graphical representations and models in three dimensions. In such a solid construction, the three coordinate axes are taken to represent the percentage composition of a variable phase, the pressure, and the temperature. The use of the solid model is carefully explained in the section specially devoted to this subject. Then follows a detailed study of various special cases, including the equilibrium, under constant pressure, of one component having solid and liquid phases, and of two solid components with or without a liquid phase. The various experimental methods for determining the solidification curves are explained, and numerous examples are given in illustration of the general theory. The next section deals with phase complexes under variable pressure, and in the concluding section are considered binary systems in which the components exhibit polymorphism.

The compilation of the present volume must have involved an immense amount of care and thought ; it embodies the views of one of the most eminent authorities on the subject, and must be regarded as one of the most valuable recent contributions to the rapidly-growing literature of physical chemistry.

We have received the following Catalogues :—

International Catalogue of Scientific Literature. Second Annual Issue.—B. Mechanics. Pp. viii + 132. C. Physics. Pp. viii + 328. E. Astronomy. Pp. viii + 224. London : Harrison and Sons. 1903.

Halbmonatliches Literaturverzeichnis der ' Fortschritte der Physik.' Dargestellt von der Deutschen Physikalischen Gesellschaft, redigiert von KARL SCHEEL und RICHARD ASSMANN. 2 Jahrg., Nos. 22–24 ; 3 Jahrg., Nos. 1–7. Braunschweig : F. Vieweg und Sohn. 1903 & 1904.

XXVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from vol. viii. p. 668.]

November 9th, 1904.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

THE following communications were read :—

1. 'Notes on Upper Jurassic Ammonites, with Special Reference to Specimens in the University Museum, Oxford : II.' By Miss Maud Healey.

2. 'Sarsen-Stones in a Clay-Pit.' By the Rev. E. C. Spicer, M.A., F.G.S.

Near to Bradenham, midway between High Wycombe and Prince's Risborough, certain clay-pits yield a clay for brick-making, in which are embedded large angular sarsen-stones, white saccharoidal sandstones with a siliceous cement. This clay is quite flintless : it rests upon, and is 'contained' by, Clay-with-Flints based upon Chalk, and is covered with roughly-mingled material containing horizontal bands with worn flint-pebbles and drifted sarsens of smaller size. Each patch of the clay fits roughly into a funnel-shaped depression lined with Clay-with-Flints. The depressions are probably swallow-holes, formed by underground solution of the Chalk, which was covered by wet clay and that by sands of the Bagshot Series. As the clay oozed out into the hollows the sarsens of these sands broke away, and became involved in the clay, much as blocks of the Malmstone at Atherfield become involved in the mud-glaciers which descend from the underlying Gault in the sea-cliffs. The overlying formation is due to much later, and probably Pleistocene, action after the flints of the Upper Chalk had been exposed by denudation.

3. " On the Occurrence of *Elephas meridionalis* at Dewlish (Dorset). Second communication : Human agency suggested.' By the Rev. Osmond Fisher, M.A., F.G.S.

November 23rd.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

The following communications were read :—

1. 'On an Ossiferous Cavern of Pleistocene Age at Hoe-Grange Quarry, Longcliffe, near Brassington (Derbyshire).' By Henry Howe Arnold-Bemrose, M.A., F.G.S., and Edwin Tulley Newton, F.R.S., V.P.G.S.

2. 'The Superficial Deposits and pre-Glacial Valleys of the Northumberland and Durham Coalfield.' By David Woolacott, D.Sc., F.G.S.

Six volumes, published by the North-of-England Institute of

Mining & Mechanical Engineers, contain a large number of borings made in the northern coalfield. A considerable proportion of these are most valuable in showing the nature and distribution of the superficial deposits. From them and from field-mapping it is possible to form a fairly-accurate conception of the pre-Glacial floor of the district and its drainage, and also of the relative changes of level before, during, and after the Glacial Period. The chief superficial deposits include the following :—(1) the stony Boulder-Clay ; (2) the Upper Clay, including (*a*) the prismatic clay and (*b*) the leafy clay ; and (3) deposits of Sand and Gravel occurring below, in, or upon the Clay. The last include Raised Beaches, which decline in height when traced northward and southward from Cleadon and Fulwell, with a gradient of from 3 to 6 feet per mile.

The surface-deposits lie arranged in the valleys formed before the Glacial Period, and often reach a considerable depth. They fit upon and level up the Glacial inequalities of the country, and are not found above 1000 feet. The base of the Drift is in some cases as much as 140 feet below sea-level, and in one spot the thickness bored through is 233 feet. By inserting all available borings in maps, it is possible to gain some idea of the direction and depth of the valleys. In this way the following pre-Glacial valleys have been located :—The Tyne and its tributaries, corresponding in general trend with the present Tyne, which is superimposed upon it ; the ' Wash,' which joins the present course of the Wear with the Tyne, and has the Team superimposed on its northern part ; the upper Wear, which received all the waters from the west of Durham county, took up several large tributaries between Bishop Auckland and Durham city, and passed through the ' Wash ' into the Tyne ; the Sleekburn Valley ; and the Druridge-Bay depression, into which the pre-Glacial Coquet probably ran. Thus in pre-Glacial times the Tyne and the Tees were the major rivers, and the other streams were tributary to them ; the main changes have taken place in the lower, and not in the higher, parts of the rivers. The post-Glacial Wear has breached the Permian escarpment, and many gorge-like valleys have been cut since the uplift which produced the Raised Beaches. The dependence of the contour of the country on its geological structure was much more pronounced in pre-Glacial times than present.

<div style="text-align:center">

December 7th.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

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The following communication was read :—

' The Chemical and Mineralogical Evidence as to the Origin of the Dolomites of Southern Tyrol.' By Prof. Ernest Willington Skeats, D.Sc., F.G.S.

Recent work on modern coral-reefs has shown that these limestones contain very little, if any, insoluble residue. The study of

the relative proportions of the organisms composing these reefs, and the alterations that they undergo, has further shown that corals play a subordinate part in them, and that calcareous algæ, foraminifera, and other organisms form the bulk of the rocks of the reefs. The author has applied this information in the examination of collections from the much debated area of the Dolomites of Southern Tyrol. The chemical examination of numerous specimens from the Schlern dolomites of the Schlern, the Langkofl, the Marmolata, the Sella, the St. Cassian district, the Richthofen Reef, and numerous other localities, is described, so far as relates to the proportions of lime and magnesia and of insoluble residue. These results are compared with similar analyses of limestones from lower and higher horizons. Many of the dolomites are devoid of insoluble residue, and where this is present it can generally be attributed, as in the raised reefs of Fiji, etc., to the association with contemporaneous volcanic rocks. The action of solution during the slow accumulation of deep-sea limestones, and the presence of terrigenous material in organic deposits forming near the shore, give to these limestones a much larger proportion of insoluble residue. It is therefore probable that the Schlern dolomites represent a Triassic ' coral-reef,' using the term in the modern, more extended, sense. The author proceeds, on this hypothesis, to suggest the conditions under which the dolomite may have been deposited in the different parts of the area in question. He next proceeds to the microscopical examination of specimens, with a view to the study of the organisms contained in them. While the limestones have undergone mineralogical and chemical changes similar to those of recent coral-reefs, they contain a suite of organisms comparable with those of the latter, when due allowance is made for the loss of structure due to more complete dolomitization. 45 per cent. of the sections yielded no trace of organic structure. In the others, calcareous algæ are the commonest fossils, echinoderm-spines and lamellibranch-shells are not uncommon, while foraminifera are occasionally found; undoubted corals are extremely rare. But reef-forming corals, gasteropoda, and some of the calcareous algæ are built of aragonite, and hence are likely to disappear under the operation of mineralogical changes; while even the more stable organisms with calcite-skeletons begin to lose structure, owing to the invasion of crystals of dolomite.

With regard to the origin of the dolomite-masses, it is shown that the conditions favourable to their formation were :—(*a*) shallow water between 0 and 150 feet, and corresponding to a pressure of 1 to 5 atmospheres ; (*b*) the presence of carbon-dioxide in comparative abundance causing partial solution of the limestone, and the possibility of chemical change with the magnesian salts in sea-water ; (*c*) porosity of the limestones facilitating the movement of magnesian solutions ; and (*d*) sufficiently-slow subsidence or elevation to render the change from calcite to dolomite complete. Locally, dolomite is deposited directly from solution in confined areas or cavities in the rock ; while the Raibl Dolomite, associated with gypsum, was formed by the concentration of sea-water in landlocked areas.

in Small

amp-Oil in Small Drum.

200 c.c.

150 c.c.

260 c.c.

100 c.c.

310 c.c.
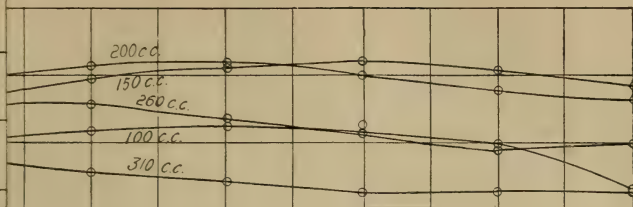
Fig. 1.—Castor-Oil in Small Drum.

Fig. 2.—Lamp-Oil in Small Drum.
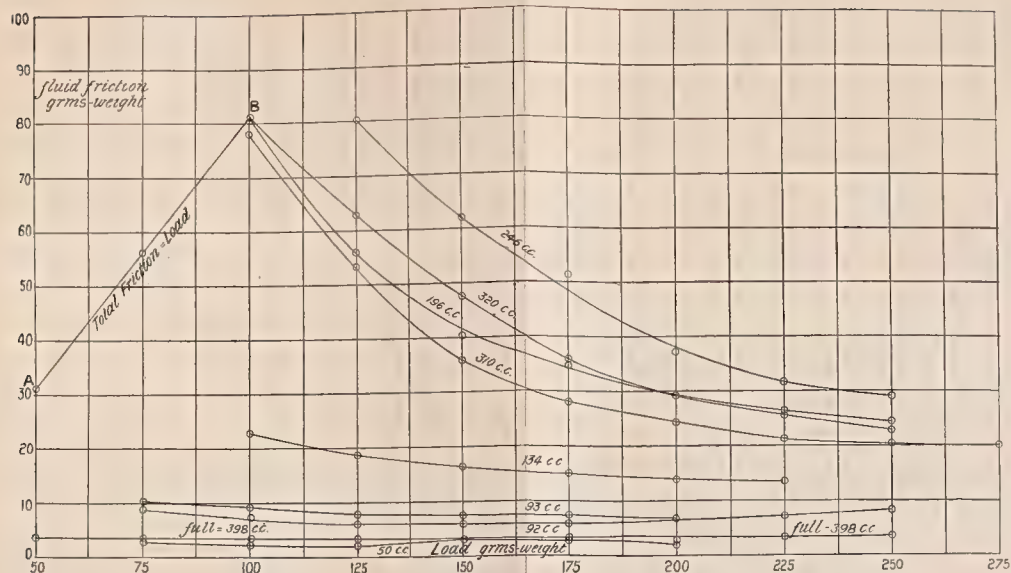
Fig. 3.—Water in Small Drum.

Fig. 4.—Lamp-Oil and Water in Large Drum.
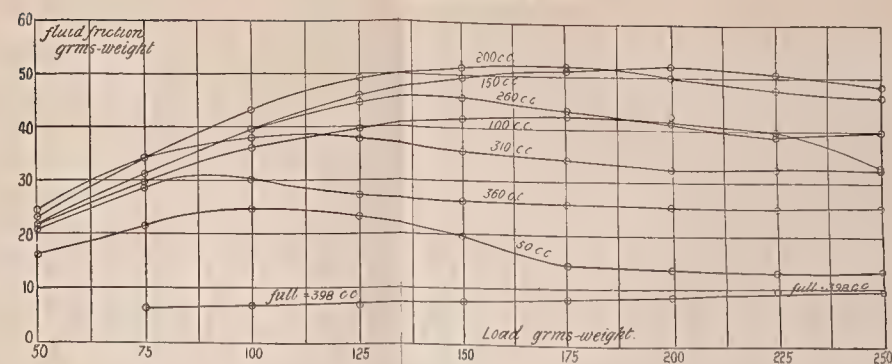
Fig. 5.—Castor-Oil, Lamp-Oil, and Water in Large Drum.
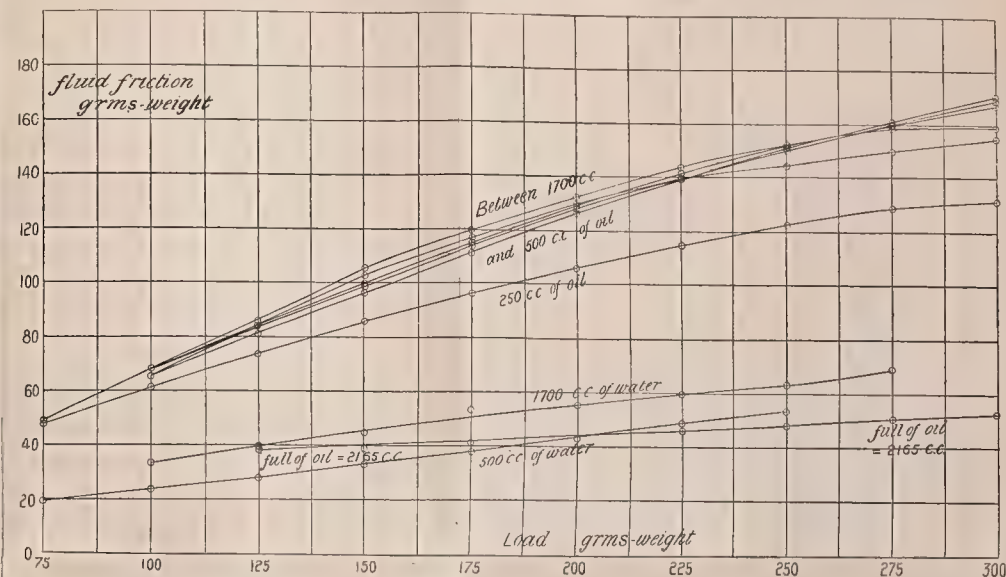
Fig. 6.—Castor-Oil, Lamp-Oil, and Water in Small Drum.
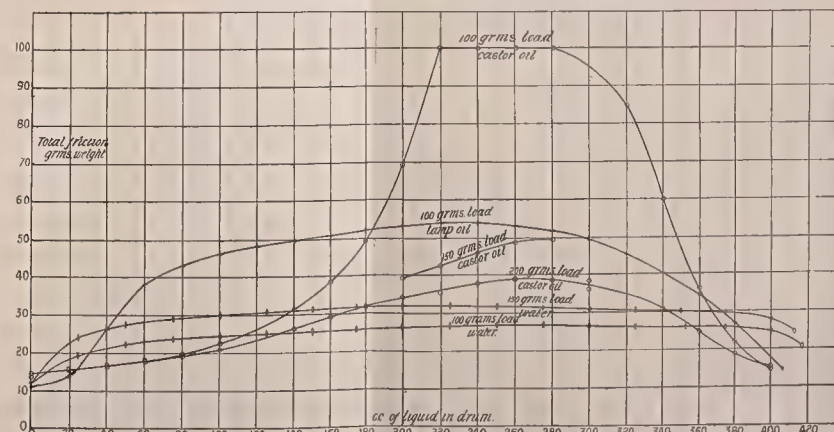
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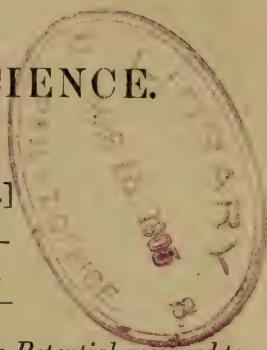
XXIX. *A Theory of the Variation of the Potential required to maintain a Current in a Gas.* By JOHN S. TOWNSEND, *M.A., Wykeham Professor of Physics, Oxford* *.

1. THE researches † which have been made on the ionization of a gas by the motion of positive and negative ions, show that a satisfactory explanation of the sparking potentials between two parallel plate-electrodes is obtained on the supposition that all the ions are produced by collisions from the molecules of the gas. The calculations which lead to the various conclusions are very simple, as the theory rests upon the results of experiments made with small currents between the plates so that the uniformity of the electric field is not disturbed.

The results of the experiments may also be applied to explain a large number of the variations of the electric field which ensue when a steady current is maintained in the gas.

Although the experimental results show that these fields of force are widely different under different circumstances, still the number of hypotheses and theories which have been proposed in order to explain them appear to be superfluous. It is to be expected that there is a close connexion between many of these phenomena; and it is desirable, if possible, to obtain a simple theory which will afford a satisfactory explanation of the various predominating features and at the same time show how they are related to one another.

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* Communicated by the Author.

† J. S. Townsend, Phil. Mag. November 1903; J. S. Townsend & H. E. Hurst, Phil. Mag. December 1904.

The present investigation deals with the field of force which is required to maintain a continuous current in a gas. From the results of the experiments that have been made on the effects of the collisions between ions and molecules of a gas, it is easy to show that the sparking potential is greater than the potential required to maintain a discharge, that the latter potential diminishes as the current increases, and that a fall of potential in a certain layer of gas near the cathode approaches the value of the minimum sparking potential[*]. The same analysis which shows that the potential required to maintain a discharge is less than the sparking potential also shows that the potential required to produce a discharge from an electrified point is smaller when the point is negatively charged than when it is positively charged.

The effects produced by various disturbing influences such as the heating of the gas and the recombination of the ions would be very small, and may be neglected when dealing with small currents which are not accompanied by the normal cathode-fall of potential.

When the ordinary cathode-fall of potential is developed, the effect of heating would introduce a complication in the calculations when the currents are large; but in these cases a method of treating the problem may be used which with the aid of some simple experimental results will lead to an explanation of some of the phenomena which accompany the ordinary vacuum-tube discharge. In addition, the radiation, from the electrodes and luminous portions of the discharge, produces ionization to a small extent in the conducting gas. Thus a small amount of ionization may be produced near the positive electrode due to Röntgen rays, but the effects would most probably be inappreciable except at very low pressures, to which this investigation is not intended to apply.

2. It is evident that a continuous steady current would be maintained if the positive and negative ions produced ions in sufficient quantities by their motion through the gas. . It is possible to find the exact extent to which ionization must be produced in the gas when the field of force is not uniform, and it is possible to decide whether the currents which are obtained in practice could be maintained by the process of ionization by collision.

Let the current pass between two parallel plate-electrodes at a distance (a) apart, and let x be the distance of any point in the gas from the negative electrode. Let u be the velocity of the negative ions, α the number of molecules

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[*] A brief account of the results thus obtained is given in a note on this subject: Phil. Mag. December 1904.

ionized by a negative ion in moving through a centimetre of the gas, n_1 the number of negative ions per cubic centimetre, v, β, and n_2 similar quantities for positive ions. Let X be the electric force at any point in the gas, and V the difference of potential between the electrodes. The current i per unit area of the electrodes will be constant when the current is steady. The quantities α, β, u, and v (as well as n_1 and n_2) depend on the distance x since the electric force between the plates is not in general the same at different points.

If e is the charge on an ion, then $n_1 u + n_2 v = \dfrac{i}{e}$ is constant. Let c denote the value of this quantity.

When the current is steady there are no variations with respect to the time in any part of the gas, so that

$$\frac{dn_1}{dt} = -\frac{d}{dx}(n_1 u) + \alpha n_1 u + \beta n_2 v = 0,$$

and

$$\frac{dn_2}{dt} = \frac{d}{dx}(n_2 v) + \alpha n_1 u + \beta n_2 v = 0.$$

Hence

$$\frac{d}{dx}(n_1 u) = \alpha n_1 u + \beta n_2 v$$
$$= (\alpha - \beta)n_1 u + \beta c.$$

So that

$$n_1 u = A\epsilon^{\int (\alpha-\beta)dx} + \epsilon^{\int(\alpha-\beta)dx}\int \beta \times c \times \epsilon^{-\int(\alpha-\beta)dx}dx$$

where A is a constant.

Let $Z(x)$ denote the quantity $\epsilon^{\int_0^x (\alpha-\beta)dx}$ so that $n_1 u$ may be expressed

$$n_1 u = AZ(x) + cZ(x)\int_0^x [Z(x)]^{-1}\beta dx.$$

Similarly

$$n_2 v = BZ(x) - cZ(x)\int_0^x [Z(x)]^{-1}\alpha dx.$$

Since all the ions are produced in the gas, the following conditions hold at the electrodes :—

At the negative electrode $(x=0)$, $n_1 = 0$ and $n_2 v = c$.

At the positive electrode $(x=a)$, $n_2 = 0$ and $n_1 u = c$.

The first two conditions give

$$A = 0 \quad \text{and} \quad B = c,$$

U 2

and the second give

$$1 = Z(a) \int_0^a \beta [Zx]^{-1} \, dx \quad \text{and} \quad 1 = \int_0^a \alpha \left[Z(x) \right]^{-1} dx.$$

Since

$$\int_0^x (\beta - \alpha) \left[Z(x) \right]^{-1} dx = \left[Z(x) \right]^{-1} - 1,$$

the two latter equations reduce to the same condition

$$1 = \int_0^a \alpha \times \epsilon^{\int_0^x (\beta - \alpha) dx} \times \, dx, \quad . \quad . \quad . \quad . \quad (1)$$

and the values of n_1 and n_2 become

$$n_1 u = c \times Z(x) \times \int_0^x \left[Z(x) \right]^{-1} \times \beta \times dx, \quad . \quad . \quad (2).$$

$$n_2 v = c \times Z(x) \times \int_x^a \left[Z(x) \right]^{-1} \times \alpha \times dx. \quad . \quad . \quad (3)$$

Equation (1) represents the condition which must be satisfied by the values of α and β along the path from $x = 0$ to $x = a$ in order that a continuous current should be maintained by collisions. Since the values of α and β may be found in terms of the pressure and electric force by independent investigations, it would be possible to decide whether any given field of force would maintain a steady current. If this condition is satisfied by a field of force determined experimentally, while a current is passing through the gas, it is evident that the ionization must be produced principally by collisions.

The field of force obviously depends on the current, the connexion between these quantities being obtained from the equation

$$4\pi (n_2 - n_1) e = \frac{dX}{dx},$$

which gives

$$\frac{dX}{dx} = 4\pi i Z(x) \left[\frac{\int_x^a \left[Z(x) \right]^{-1} \times \alpha \times dx}{v} - \frac{\int_0^x \left[Z(x) \right]^{-1} \times \beta \times dx}{u} \right]. \quad (4)$$

The rate of change of the force should also satisfy this equation, but since the velocities u and v are not known for the large forces and small pressures which are used in the experiments, no very satisfactory verification of the theory could be deduced from this condition. Equation (4) might be used to determine the velocities at different parts of the discharge.

3. Some interesting results may be obtained from equation (1), as it may be put into forms which are suggested by the experimental investigations.

Let the current be very small so that the field of force is uniform. The quantities α and β are then independent of the distance x, so that the equation

$$1 = \int_0^a \epsilon^{a \int_0^x (\beta - \alpha) dx} \times \alpha \times dx \quad \text{reduces to}$$

$$1 = \frac{\alpha}{\beta - \alpha} \left(\epsilon^{(\beta - \alpha)a} - 1 \right),$$

or $\alpha - \beta \epsilon^{(\alpha - \beta)a} = 0.$

This is the condition which, as has been already shown, determines the sparking potential. Hence the sparking potential may be defined as the potential which is required to maintain a very small current in the gas.

This result has also been established experimentally. When the sparking potential is obtained by increasing the number of cells in a battery connected to the electrodes through a high resistance and a galvanometer, it was found that the potential difference at the electrodes did not drop appreciably below the sparking potential during the passage of a very small current which was accompanied by a faint glow in the gas.

4. The difference of potential between the electrodes required to maintain a current diminishes as the current increases when the pressure of the gas is greater than that which corresponds to the minimum sparking potential. It is easy to show that this result is in accordance with the theory.

When new ions are generated in the gas by the collisions of ions with molecules, the negative ions which are formed are of much smaller mass than the positive ions. The latter would therefore move more slowly so that a resultant positive * charge would accumulate in the gas, and the electric force near the negative electrode would in consequence become greater than the force near the positive electrode. Since all the positive ions must pass through the gas in the neighbourhood of the negative electrode, the positive charge would be greatest near that electrode, unless the velocity of the ions becomes much greater near the cathode than at other parts of the discharge owing to the increase of

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* This explanation of the positive charge near the negative electrode was first suggested by Professor Schuster (Bakerian Lecture, Proc. Royal Soc. xlvii. 1890, p. 541).

the electric force. The effect of the positive charge in the gas on the potential required to maintain a discharge can easily be illustrated by finding the potential required to maintain a current in a field made up of two portions in each of which the force is constant, the value of the constant being greater near the negative electrode. It is interesting at first to investigate cases in which the discrepancy between the forces in the two parts of the field is small compared with the variations which are found when the ordinary cathode-fall of potential is developed. In general, let α and β have the constant values α_1 and β_1 from $x=0$ to $x=b$; and let the values be α_2 and β_2 from $x=b$ to $x=a$.

Equation (1) may be expressed in general in the following form:

$$\epsilon^{\int_0^b (\alpha-\beta)dx} \times \left(1 - \int_0^b \alpha \times \epsilon^{\int_0^x (\beta-\alpha)dx} dx \right) + \epsilon^{\int_b^a (\beta-\alpha)dx} \times \left(1 - \int_b^a \beta \times \epsilon^{\int_a^x (\beta-\alpha)dx} dx \right) = 1.$$

With the above constant values of α and β for the two parts of the field, this equation reduces to

$$\frac{\alpha_1 - \beta_1 \epsilon^{(\alpha_1-\beta_1)b}}{\alpha_1 - \beta_1} + \frac{\beta_2 - \alpha_2 \epsilon^{(\beta_2-\alpha_2)(a-b)}}{\beta_2 - \alpha_2} = 1.$$

As an example, let the gas between the electrodes be hydrogen at 10 millimetres pressure, and let $b=2$ millimetres. Let the force be 80 volts per millimetre in the two millimetres of gas near the negative electrode, and 50 volts per millimetre in the rest of the field. The values of $\frac{X}{p}$ will then be 80 and 50 in the two parts of the field, and the values of α_1, β_1, α_2, and β_2 are as follows:[*]

$$\alpha_1 = 10 \cdot 0, \qquad \alpha_2 = 3 \cdot 6,$$
$$\beta_1 = \cdot 081, \quad \beta_2 = \cdot 021.$$

The value of a may be found by substituting these values in the foregoing equation. The distance between the electrodes requisite for the maintenance of the discharge is thus found to be 10·0 millimetres. Hence, if the electrodes are a centimetre apart, a continuous current would be maintained in the gas if the force is 80 volts per millimetre in a layer 2 millimetres thick near the cathode, and 50 volts per millimetre in the remaining 8 millimetres of the gas. The total fall of potential between the electrodes would be 560 volts, which is about twenty volts less than the sparking

[*] J. S. Townsend, Phil. Mag. June 1902 and November 1903.

potential in hydrogen at 10 millimetres pressure between
plates a centimetre apart.

Larger differences between the sparking potential and the
potential required to maintain a discharge are obtained when
the force near the cathode is again increased. If the force
be 200 volts per millimetre in the first millimetre near the
cathode and 40 volts per millimetre for the rest of the
distance, then the values of α_1, β_1, α_2, and β_2 are

$$\alpha_1 = 28, \qquad \alpha_2 = 2\cdot0,$$
$$\beta_1 = \cdot9, \qquad \beta_2 = 0, \text{ q.p.}$$

In the same manner as before the value of a may be found.
In this case $a = 4\cdot25$ millimetres, and the fall of potential
between the electrodes is $200 + 130 = 330$ volts, whereas the
sparking potential for the distance $4\cdot25$ millimetres between
the plates is 385 volts.

The same analysis shows that if the force is 200 volts per
millimetre in the first millimetre and 30 volts per millimetre
in the rest of the path of the discharge, the current would be
maintained if the distance between the plates is 9 millimetres
so that the total fall of potential between the electrodes would
be 440 volts, whereas the sparking potential for the same
distance and pressure of gas is 545 volts.

Similar results may be obtained by using the values of α
and β which have been found for air.

Thus it is evident that the more the force near the cathode
exceeds the force in the other parts of the discharge, the less
will be the potential required to maintain the discharge.

5. The first example which has been chosen shows that a
potential of 560 volts is sufficient to maintain a discharge of
1 centimetre long in hydrogen at 10 millimetres pressure
when the force is uniform and equal to 80 volts per milli-
metre in the layer two millimetres thick near the cathode,
and 50 volts per millimetre in the remaining 8 millimetres
of the path of the discharge. It is easy to show that the
same field of force reversed would not suffice to maintain a
discharge, the larger force being then confined to the layer
2 millimetres thick in contact with the positive electrode.
This may be seen by substituting the following values of
α_1, β_1, α_2, β_2, a, and b :—

$$\alpha_1 = 3\cdot6, \qquad \alpha_2 = 10\cdot0, \qquad b = \cdot8,$$
$$\beta_1 = \cdot021, \quad \beta_2 = \cdot081, \quad a = 1\cdot0,$$

in the expression

$$\frac{\alpha_1 - \beta_1 \epsilon^{(\alpha_1 - \beta_1)b}}{\alpha_1 - \beta_1} + \frac{\beta_2 - \alpha_2 \epsilon^{(\beta_2 - \alpha_2)(a-b)}}{\beta_2 - \alpha_2} = \Upsilon.$$

When the values of α and β are very small, the potential between the electrodes does not suffice to maintain a discharge, and the two fractions in the above value of Y are each unity, so that Y = 2. As the values of α and β increase the two fractions diminish, and their sum becomes equal to unity when the forces are sufficiently great to maintain a discharge. Hence, if Y has a value between 1 and 2 for any system of values of α_1, β_1, α_2, β_2, a, and b, the forces between the electrodes will not be sufficiently great to maintain a discharge. When the values of α and β corresponding to the case in which the large force is near the positive electrode are taken, Y becomes 1·033, so that the forces are not sufficiently great to maintain the discharge. (Before reversal Y = 1·000.)

In the other two numerical examples which have been chosen, where there are greater differences in the two parts of the field, it may also be shown that Y exceeds unity when the field is reversed. This may be seen at a glance, as β_1 vanishes in these cases, so that one of the fractions is unity and the other has a small positive value.

Perhaps the most familiar experiment which can be explained by these results is the difference between the potentials required to cause a discharge from a point, depending on the sign of the electrification of the point.

When the point is negative, the intensity of the force is greatest near the negative electrode, so that the total potential difference between the point and the surrounding conductors required to produce a discharge is less than when the point is positively electrified.

[There is really no reason to expect that the potential should be the same in the two cases. The condition that a discharge should start along a path S from one conductor A to another B is

$$1 = \int_A^B \alpha \times \epsilon^{\int_A^S (\beta - \alpha) dS} \times dS,$$

when the conductor A is the negative electrode, and

$$1 = \int_A^B \beta \times \epsilon^{\int_A^S (\alpha - \beta) dS} \times dS,$$

when A is the positive electrode.]

6. The condition connecting the quantities α and β which must hold along the line of a discharge may be examined from another point of view, and it may be shown that the fall of potential in a certain layer of gas near the negative

electrode cannot exceed a certain fixed value which is equal to the minimum sparking potential. The condition which has been given may be expressed in the form

$$1 - \int_0^b \alpha \epsilon^{\int_0^x (\beta-a)dx} dx = \int_b^a \alpha \epsilon^{\int_0^x (\beta-a)dx} dx.$$

If the value of α is very small in the column of gas of thickness $a-b$ in contact with the positive electrode, then the quantity on the right of the above equation becomes very small. This implies that nearly all the ionization takes place in the layer of thickness b near the negative electrode. Also since the quantity

$$1 - \int_0^b \alpha \epsilon^{\int_0^x (\beta-a)dx} dx$$

becomes very small in this case, the potential fall along the distance from $x=0$ to $x=b$ is practically the same as would maintain the same current between electrodes at a distance b apart, the pressure of the gas being unchanged.

The theory shows that the current could be maintained in the gas even when the value of α vanishes in the positive column, and in this case the potential fall in the layer near the negative electrode would be determined accurately by the condition

$$1 - \int_0^b \alpha \epsilon^{\int_0^x (\beta-a)dx} dx = 0.$$

It is possible to show the connexion between the cathode-fall of potential and the minimum sparking potential by using the above investigation in connexion with the following experimental results.

The sparking potential for parallel plate-electrodes has a minimum value corresponding to a certain amount of gas between the plates. Let m be the product of the distance between the plates and the pressure p which corresponds to the minimum sparking potential. When the product exceeds the value m the potential required to maintain a discharge is less than the sparking potential, but when the product approaches the value m the potential required to maintain a discharge does not differ much from the sparking potential. For hydrogen $m=1\cdot1$, the distance between the plates being measured in centimetres and the pressure in millimetres of mercury. A set of experiments with the plates ·44 centi-

metre apart and the gas at 2·55 millimetres pressure gave the following results :—

Sparking potential, 273 volts.
Potential required to maintain a current of 2 10^{-5} ampere, 272 volts.

| „ | „ | „ | 2 10^{-4} | „ | 255 | „ |
| „ | „ | „ | 4·6 10^{-3} | „ | 250 | „ |

Let these potentials be represented by u, so that within an error of about 4 per cent. the value of u may be taken as 260 volts for hydrogen for any current.

The cases in which the product $p \times a$ is greater than m may now be considered.   Let b be determined so that $p \times b = m$.

When the value of α becomes very small in the column of gas of thickness $(a-b)$ near the positive electrode, the following condition is satisfied approximately,

$$1 = \int_0^b \alpha \times \epsilon^{\int_0^x (\beta - \alpha) dx} \, dx.$$

This is the condition that the fall of potential from $x=0$ to $x=b$, should maintain the same current between two plates separated by the distance b, so that this fall of potential must be equal to u. Consequently the fall of potential in the layer of gas m near the negative electrode will in general attain a value nearly equal to the minimum sparking potential, and will remain constant for the larger currents.

Experiments on the cathode-fall of potential show that this phenomenon connected with the discharge has the general properties which have been found theoretically for the fall of potential across the layer m near the cathode.   The cathode-fall of potential is independent of the pressure, the distance between the plates, and the current, and in addition it has been shown [*] that the cathode-fall of potential is practically the same as the minimum sparking potential. Also when the normal cathode-fall of potential is developed, the electric force in those parts of the gas which are remote from the cathode is comparatively small.

The cathode-fall of potential as determined experimentally is the fall of potential between the negative electrode and a point at a certain short distance C from the electrode.   The electric force is very high near the electrode, and at a short distance (C) from the electrode the force becomes very small, so that the fall of potential across the layer of thickness $b = \dfrac{m}{p}$ will also be equal to the cathode-fall of potential,

* Hon. R. J. Strutt, Phil. Trans. vol. cxciii. p. 377 (1900).

provided that b is small and greater than C. The distance b satisfies this condition, as the thickness of the layer across which

the cathode-fall of potential acts is less than $\dfrac{m}{p}$.

It is thus seen that the theory leads to a consistent explanation of many of the phenomena which accompany a continuous current through a gas.

There are several other points on which further comparisons might be made between the theory and experimental results. For example, the field of force between the electrodes at a distance a apart, when a current is passing through the gas at pressure p, ought to be double the intensity at corresponding points when the same current is passing between electrodes at a distance $2a$ apart, and the pressure $\frac{1}{2}p$. Experimental investigations are at present being made with a view to obtaining further results which may be easily compared with the theory. The most important of these consists in tracing the changes in the field which occur as the current rises. Some unexpected difficulties arose in making these determinations, so that the progress of the researches has been somewhat delayed.

XXX. *A High-Frequency Alternator.* By W. DUDDELL [*].

[Plates V. & VI.]

IN view of the increasing interest taken in high-frequency alternating currents, especially in connexion with wireless telegraphy, a description of a small alternator designed by the writer in 1900 and of the difficulties encountered in its construction may prove of use to other experimenters on this subject.

The alternator was made for some experiments on the resistance of the electric arc [†]. As the experiments progressed, and it was found that with each increase of frequency the solid arc behaved more and more like an ordinary resistance, the alternator was altered until finally the very high frequency of 120,000 $\sim$ per second was attained. The inductor type of alternator was chosen ; the final design is shown in plan and elevation in Pl. V. figs. 1 and 2.

The inductor, 6 cms. in diameter, was built up of 53 disks cut out of ferrotype plate, the iron being about 0·007 inch thick, enclosed between two plates 0·019 inch thick, the

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[*] Communicated by the Physical Society : read December 9, 1904.
[†] Phil. Trans. A. vol. cciii. p. 305.

whole clamped together on a tool-steel spindle. The inductor alone weighed about 200 grammes. As at first constructed, the edge of the inductor had 30 V-shaped notches cut in it so as to have 30 flat-topped teeth.

Surrounding the inductor is a laminated soft-iron ring having two inwardly projecting poles; the clearance between these and the teeth of the inductor, which formed the air-gap of the machine, being less than 0·1 mm. The ring itself is wound so that a direct current flowing round the winding tends to produce lines of force from one pole to the other through the inductor. The number of lines of force varies from a maximum when the teeth on the inductor are opposite the poles to a minimum when the poles are in front of spaces. This variation in the total number of lines of force produces alternating E.M.F.'s in any winding either on the pole-pieces or on the ring itself ; the movement of the inductor through the distance between two consecutive teeth producing one complete period. One winding on the ring could serve to carry both the direct field-current which magnetizes the ring and the alternating current produced. In practice, owing to the great importance of any self-induction in the alternate-current circuit, it was found advisable at high frequencies to use separate windings for the two currents. The ring was, therefore, wound in three sections as follows :—Two sections, each consisting of a single layer of 113 turns of No. 33 double cotton-covered wire wound next the core on each side of the ring which could be connected in series or parallel, were used as the armature. The third section consisting of 4 layers wound on top of the above, having 430 turns on each side of the ring, the whole being permanently connected in series, was used as the field-coil. At a later date coils on the pole-tips themselves were added.

In whatever way the windings are arranged or used, it is necessary to prevent the induced E.M.F.'s from sending alternate currents round the exciting circuit, outside the alternator, as these currents would tend to annul the changes in lines of force through the coils and so prevent any useful alternating current from being produced. To prevent this the exciting circuit contained a choking-coil.

As no steam turbine or other high-speed motor was available, the alternator was at first driven by means of a figure-of-8 belt-drive, illustrated in fig. 3, from a $\frac{1}{2}$ H.P. direct-current motor D fixed, with its shaft vertical, to a heavy wooden framework. Two bicycle-wheels $24\frac{1}{2}$ inches actual diameter, No. 14 tangent-spokes, and "Palmer's" jointless hollow rims, were employed: the one fixed direct to

the motor-shaft as the driving-wheel; the other, which was carried in an ordinary front-fork and hub with ball-bearings, acting as a tension-pulley to balance the pull on the alternator-spindle.

The pulleys on the alternator-spindle were of steel, having a diameter of only 10 mms. at the bottom of the V-grooves in them. The ratio of the belt-drive, that is the number of turns of the alternator-spindle for each revolution of the motor, was found experimentally to be 42·5.

To measure the frequency, a Young's speed-indicator was used geared down from the inductor-spindle by means of worm and wheel, having a ratio of 20 : 1. Although the spindle was constantly run at speeds of between 30,000 and 40,000 revolutions per minute, no trouble was experienced with this gear.

CURVES I.—Characteristics of High-Frequency Alternator on Open Circuit. Inductor with 30 teeth.
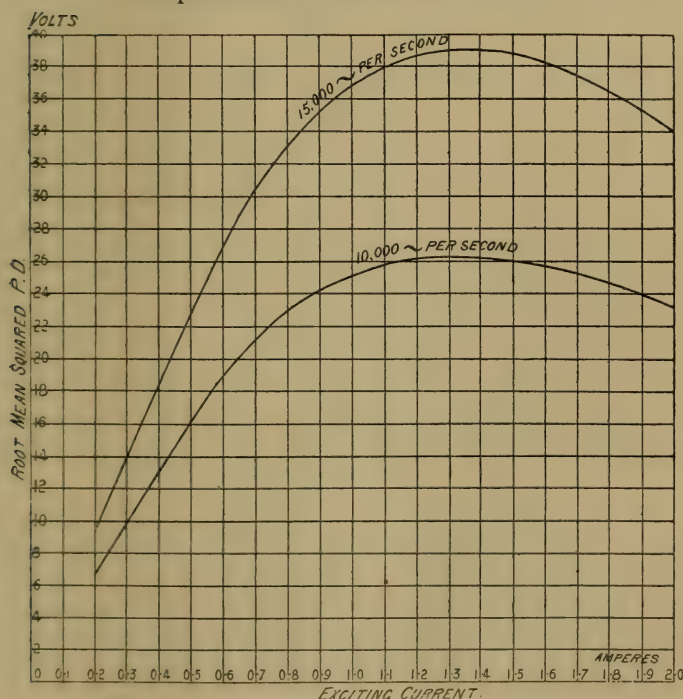
With this method of driving, frequencies up to 18,000 ∼ per second could be obtained. Curves I. are characteristic

curves for the alternator; and Curves II. are between R.M.S. current and P.D. with constant exciting current at frequencies of 15,000 and 10,000 ∼ per second.

CURVES II.—Between P.D. and Current.   High-Frequency Alternator.
Constant Exciting Current=1 ampere.   Inductor with 30 teeth.
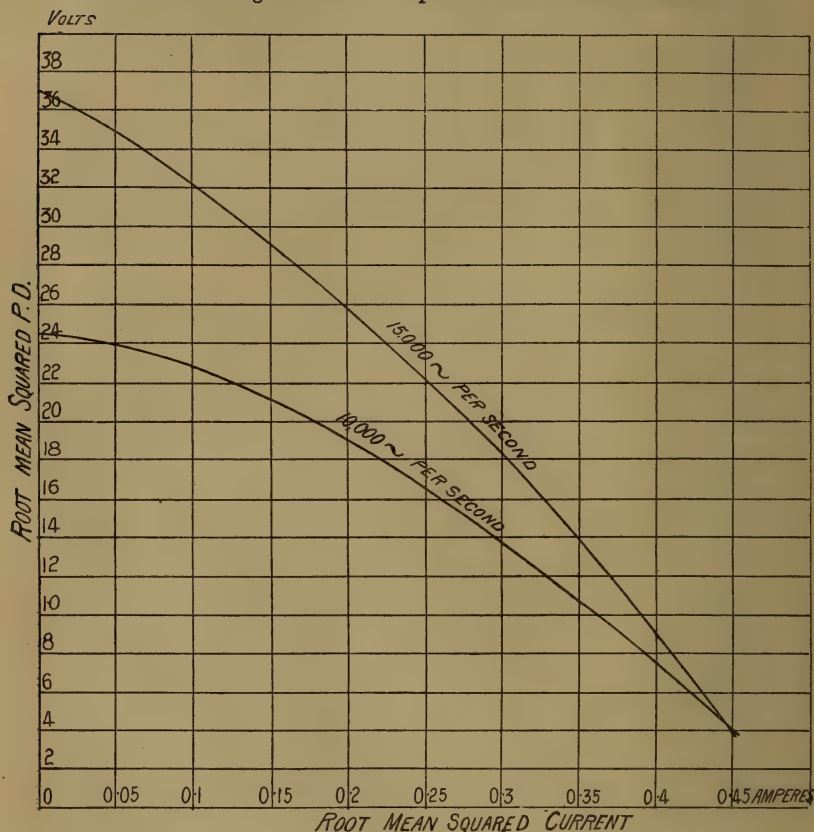
Subsequently the inductor having 30 teeth with which the above curves were obtained was replaced by inductors having 60 and 90 teeth, with which frequencies up to 50,000 ∼ per second were obtained. This high frequency being still hardly sufficient to settle the question of the resistance of the solid arc, some means for still further increasing it had to be devised.

As the highest speed of the inductor was only 600 revolutions per second, whereas the limit at which it was considered safe to run the inductor, owing to considerations of bursting, was

1000 revolutions per second, attempts were made to increase its speed; but these failed owing to the large power required seriously overloading the motor.

In order to find out why such a large increase of power was necessary to drive the motor and alternator at higher speeds, the power supplied to the motor-armature was measured. At a motor speed of 800 revolutions per minute, which would correspond, if the alternator were connected, with a speed of its spindle of 567 revolutions per second, the power required to drive each part was as follows,

TABLE I.

					Watts.
Motor alone, no bicycle-wheels or alternator					63
,, with one bicycle-wheel only					111
,, ,, both bicycle-wheels bolted together...					162
,, ,, ,, ,, and alternator....					532

CURVES III.—Between Speed and Power required to drive:—Motor alone; Motor and one Bicycle-wheel; Motor and both Bicycle-wheels.
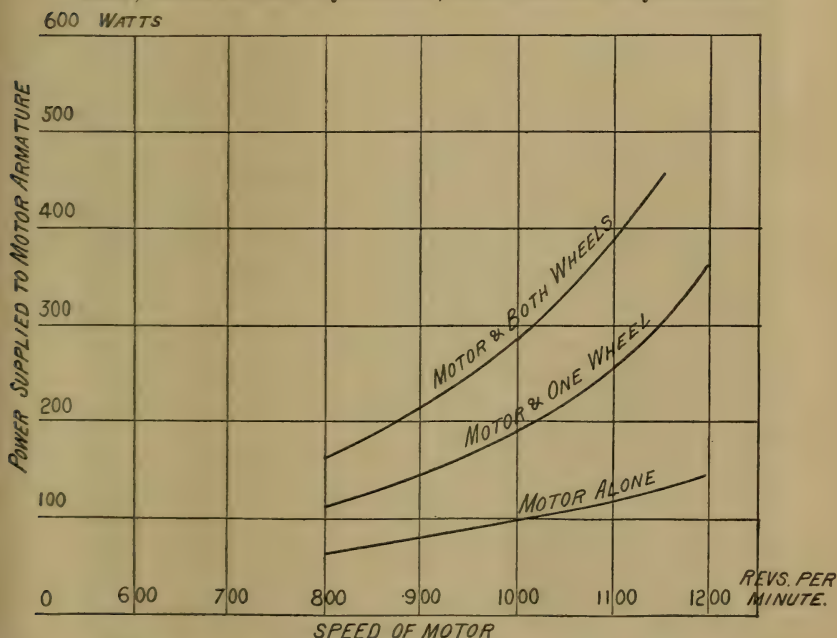
The power required to drive the motor alone and with one or two wheels at higher speeds is given in Curves III., which shows how very rapidly the power required to drive the bicycle-wheels increases with the speed even without the

alternator. Practically, the whole of the difference between the power required to drive the motor and one wheel and that required to drive the motor alone, was expended in overcoming air-friction on the wheel, as the increase of the bearing friction due to the extra weight of the wheel cannot be large, the armature being many times heavier than the wheel; and the extra C^2R loss due to the extra armature current is only 22 watts at a speed of 1200 revolutions per minute.

The air-friction on this single bicycle-wheel running at 1200 revolutions per minute, or at a rim-velocity of about $85\frac{1}{2}$ miles per hour, required an expenditure of energy at the rate of about 200 watts; so that for a cyclist to attain this speed he would have to develop over $\frac{1}{2}$ H.P. to overcome the air-friction on his two wheels alone.

The motor and bicycle-wheels were therefore replaced by two phosphor-bronze disks driven from an 8 K.W. motor by means of $\frac{1}{4}$-inch gut bands. This arrangement is illustrated by the photograph Pl. VI. fig. 4, and a working drawing of one of the disks is given in Pl. V. fig. 5.

In the figure-of-8 drive with only one wheel acting as driver, the spindle cannot be relieved of all pressure due to the belt, if the tension-wheel takes any considerable power to drive it. Both disks were therefore made drivers, each being separately belted to the motor.

Before taking these disks into use, each one was tested by twice running it up to about 6500 revolutions per minute for a few minutes to see if there was any risk of its bursting. From this test it was concluded that, as far as safety was concerned, any speed up to 5000 revolutions per minute, or a rim-velocity of 327 feet per second, might be employed; but owing to the excessive vibration, due to the disks not being perfectly balanced, 4000 revolutions per minute, or a rim-velocity of 262 feet per second, was the highest speed at which they could be continually run.

This rim-velocity, which is practically the same as the linear belt-velocity, being so much higher than that attained with the bicycle-wheels, the 10 mm. steel pulleys on the alternator spindle were removed and replaced by larger ones of red fibre, as it was thought that this change would reduce the tension in the belt necessary to drive the alternator. It had, however, quite the opposite effect, owing to the inertia forces lifting the belt out of contact with the pulleys at the higher belt speeds necessary to obtain the same speed of the alternator-spindle. The extra tension required to make the belt wrap round and drive the pulleys was much larger than

the saving in tangential pull on the pulleys, owing to their larger radius. It is evident that there is in any given case a best diameter of pulley which will give the least strain on the belt; this diameter was found by experiment in the present case to be the size of pulleys shown on the spindle in Pl. V. fig. 1. The object of making the new pulleys of red fibre was to prevent the heat developed by any slip of the belt being conducted to the spindle, and so expanding it and causing it to seize in the bearings.

At these high speeds the centrifugal force on the belt introduced a serious difficulty, as it tended to lift the belt right out of contact with the small pulleys, so that the slip became excessive. If, on the other hand, the belt was tightened up sufficiently to prevent this, then the strain on it became too great, and it broke after a short run. The joint in the belt was also a source of trouble owing to the inertia forces acting on it. Many different kinds of belts and ways of joining them were tried, such as : gut with connectors or splices; leather both round and flat with different kinds of wire and sewn joints; hempen cord with various splices; common cotton-tape twisted up and sewn together at the ends; and $\frac{3}{16}$-inch diameter cotton-cord joined with a long splice. Of all these, the last gave the best results. In order to get sufficient friction between the belt and the pulleys, a small quantity of thick solution of resin in turpentine was applied to it; and if it was then found to be too sticky, a trace of oil was put on the belt with the finger whilst running. The life of one of these belts was not very long, being only about four hours continuous use ; but owing to their cheapness, the ease with which a new one could be made, and to the fact that they did no damage when they broke, this was not considered a very serious disadvantage.

In spite of every care, all attempts to run the inductor at 1000 revolutions per second failed, owing to the heating of the bearings ; but on removing the inductor from the spindle, this latter could be run at 1000 revolutions per second with comparative ease. This was evidently due to the axis of inertia of the inductor not coinciding with its mechanical axis; all attempts to adjust this, such as making new inductors, &c., failed.

To give some idea of the importance of the two axes coinciding, suppose that they are parallel and 1 mm. apart; then at 1000 revolutions per second the pressure on the two bearings works out to about 0·8 metric ton ; so that an error of only $\frac{1}{100}$ mm. would mean a prohibitive pressure on such small bearings.

TABLE II.—List of High-Frequency Alternators.

Alternator belonging to	Made by	Type of Alternator.	Method of driving.	Speed of Alternator-shaft. Revs. per min.	Peripheral Speed. ft. per sec.	Frequency per sec.	Output on N.I. Load. Watts.	P.D. Volts.	Current. Amperes.
Prof. Lodge, Birmingham University.	Messrs. C. A. Parsons & Co.	Revolving field with radial poles. The armature-coils are wound on teeth projecting inwards from a laminated ring.	Electrical. Alternator on the same shaft as motor.	6000	250	3000	360	112	3·2
Prof. Elihu Thomson. The General Electric Co., Lynn, Mass, U.S.A.	Elihu Thomson, in mechanical laboratory of Thomson-Houston Electric Co.	Inductor, of Thomson design. Two stationary field-coils in iron frame surrounding ends of revolving inductor. Inductor-poles central. 50 poles=50 cycles per revolution.	Belt and pulley.	5000	150	4165	1000	1000	1·0
N. Tesla, New York, U.S.A.	....	Revolving armature. Zig-zag winding on edge 384 field-poles(1)	Belt.	1600	209	5100	.....	100?	10
M. Wien, Physikal Institut der Technische Hochschule, Aachen.	....	Inductor (2).	Direct, driven by a motor.	2040	185	8500(2)	4	20	0·2
Sir D. Salomons.	Messrs. Pyke & Harris.	Field & armature both revolve in opposite directions. 174 pairs of poles on each (3).	Motor forming part of alternator.	1500	79	8700	200 (3)	200	1

			Induction motor or belt.	3750	310	10,000	2000	100	20
C. P. Steinmetz. The General Electric Co., Schenectady, U.S.A.	The General Electric Co.	Revolving armature single phase.	Induction motor or belt.	3750	310	10,000	2000	100	20
M. Leblanc.	The Westinghouse Co.	Inductor, 200 polar projections (4).	Direct, coupled to motor.	3000	327	10,000	2000		
	Messrs. Siemens & Halske.	Inductor (5).	Ditto.	...	...	10,000			
	Thury.	„ (6).	Ditto.	3000	...	10,000	1600	200	8
Prof. Ewing. Cambridge University.	Messrs. C. A. Parsons & Co.	Special Disk field-magnet (7).	Steam turbine.	12,000	600	14,000	500	100	5
N. Tesla. New York, U.S.A.	....	Revolving armature Ferranti type, 480 polar projections.	Belt.	...	...	15,000			
Central Technical College, London.	The author and students at the Central Technical College.	Inductor with 30 teeth. Inductor with 204 teeth	Belt. {	30,000	310	15,000	5·5	22	0·25
				35,400	366	120,000	0·20	2·0	0·1
M. Tureettini,(8) Geneva, Switzerland.									

X 2

(1) The 'Electrical Engineer' of New York, 1889, vol. xi. p. 338. Data as to output are wanting.

(2) Wiedemann's *Annalen*, 1901, vol. iv. p. 425. By resonating the second harmonic a frequency of 17,000 is said to have been obtained.

(3) The Journal of the Institution of Electrical Engineers, 1892, vol. xxi. p. 709. There seems a slight uncertainty as to whether this output was ever actually obtained.

(4) The American Institute of Electrical Engineers, 1904, pp. 405, 416.

(5) Exhibited St. Louis, 1904.

(6) *L'Eclairage Electrique*, 1904, vol. xli. p. 65.

(7) 'The Electrician,' 1892, vol. xxx. p. 65.

(8) Said to possess a high frequency alternator; no data available.

Parallel bearings working in two loose sleeves were tried, and also combinations of conical and parallel bearing surfaces; but the simple conical bearings shown in Pl. V. fig. 1 were found to give the best results, probably owing to their shorter length allowing the inductor more freedom to spin on its axis of gyration rather than forcing it to spin on its mechanical axis. An ideal method would have been to allow the inductor to spin like a top and choose its own axis of rotation; but this could not be done with the actual alternator, as the clearance between the inductor and the pole-tips was less than 0·1 mm.

Experiments were next started to see whether the efficiency of the alternator could not be increased so as to allow the small amount of power required for the experiments to be obtained with an inductor having more than 90 teeth. An inductor having 204 teeth was therefore constructed.

By placing search-coils on different parts of the field-ring it was found that the E.M.F.'s induced were much larger in those coils placed near the poles than in those placed on any other parts of the ring, from which it seemed that the changes in the number of lines of force produced by the movement of the inductor were confined to the neighbourhood of the pole-tips, probably due to the magnetic leakage and to the eddy-currents set up in the laminations. Coils were therefore wound right on the pole-tips themselves. Using the coil on one pole-tip only, which consisted of 10 turns of No. 30 double cotton-covered wire, it was found that the output of the alternator was 10 times as great as could be obtained with the original armature-winding on the ring itself.

With this inductor and the coils on the pole-tips as shown in Pl. V. fig. 1, it was finally possible to obtain a R.M.S. current of 0·1 ampere at frequencies up to 120,000 ∼ per second. The maximum P.D. at 100,000 ∼ per second with 0·1 ampere flowing was 2 volts, and on open circuit 3·6 volts. As this output, though small, was ample for the purposes of the research, no attempt was made to further increase it. The wave-form of the current, when this inductor was driven so slowly that the frequency was only 2670 ∼ per second, is given in Pl. VI. fig. 6.

An illustration will perhaps convey some idea of how high a frequency of 120,000 ∼ per second really is. In plotting curves for ordinary frequencies of 50 to 100∼ per second, a scale often adopted is 10 inches for 100∼. If it were attempted to plot a curve up to 120,000∼ per second to this scale the curve paper would require to be 12,000 inches, or nearly *one-fifth of a mile long*.

At the time of constructing the present alternator the

owners of various high-frequency machines kindly furnished the author with data of their alternators. These data are collected in Table II. (pp. 306–7) together with three more recent alternators. The writer hopes that a description of other machines may be forthcoming in the discussion on the present paper.

In conclusion the author would like to point out that he has described this machine not with a view of suggesting it as the best, or even as a satisfactory, design, but rather to indicate some of the difficulties which are encountered in attempting to construct an alternator for high frequencies. The machine served its purpose and allowed the research for which it was designed to be completed, and no further work has been done on it, though there are several very obvious improvements which would greatly increase its output and utility. Firstly, the greater part of the difficulties were encountered with the belt-drive and the bearings, these could obviously be got over by mounting the inductor direct on the shaft of a steam turbine. A larger inductor, say 10 to 15 cm. diameter, could also probably be used with 200 teeth and a speed of 30,000 r.p.m. to give 100,000∼ per second. Secondly, the lamination of the iron must be carried to a very high degree to prevent eddy-current losses, and the armature-coils must be fixed as near as possible to the very tips of the poles. Thirdly, with accurately-spaced teeth on the inductor a large number of armature poles could be used.

All the experiments described in this communication were carried out at the Central Technical College. The author wishes to express his indebtedness to Prof. Ayrton and Mr. Mather for much valuable assistance and advice.

XXXI. *Alternators in Parallel.*
By Captain E. Henderson, *Royal Marine Artillery* [*].

THE recent investigation by Professor B. Hopkinson on the conditions causing " Hunting in alternators " have re-directed interest to the older analytical treatment by Dr. John Hopkinson of the coupling of alternators in parallel. In Perry's ' Calculus for Engineers ' this problem is treated in a very concise manner, owing to his symbolical modification of the complicated mathematics once thought necessary.

It may be advisable to point out that there are slight algebraical omissions in the results given by Professor Perry on page 261 of the ' Calculus for Engineers '; the work

below is a continuation of his investigation, the same nota-
tion being used throughout. The subject-matter dealt with
by Professor Perry opens out a large number of problems,
but it will be sufficient for the present if we consider the
simplest case only, that of two alternators coupled up in
parallel to a non-inductive resistance, and if we deal with the
following points :—

(*a*) To prove that the law for alternators in parallel is the
same as that for cells in parallel.

(*b*) To deduce the relation between self-induction, fre-
quency, and resistance, for maximum control under given
conditions.

The problem is complicated further if we take into con-
sideration the condenser effect of a long cable, in addition
to the self-induction due to transformers in the external
circuit.

When two alternators are running together in parallel,
they give out equal power when in phase, but directly they
are out of step the leading machine gives out more power,
and so may be said to drive the lagging machine, which gives
out less, as a motor.

This extra power, together with that supplied by the
engine, tends to bring up the lagging machine and retard
the leading machine, and the two come thus into phase
again.

It is requisite, then, that for as small a difference in phase
as possible, the difference in power given by the two alter-
nators should be as large as possible.

But the total power given by the two varies as the dif-
ference in phase increases.

Therefore, for any particular difference in phase the ratio
of the difference to the sum of the powers would be a measure
of the controlling force between the two.

When the two machines are in phase this ratio is zero,
and the ratio should increase as the angle of phase increases,
being a maximum when the lagging machine is doing no
work, but is running altogether as a motor.

The problem to calculate P_1 and P_2, the powers of two
alternators having electromotive force

$$e_1 = \text{E} \sin (nt + \alpha) \text{ and } e_2 = \text{E} \sin (nt - \alpha),$$

internal resistance r, and self-induction l, coupled up in
parallel to a non-inductive resistance R, can be easily
deduced.

Employing the data given in Perry's 'Calculus for

Engineers,' paragraph 189, we find that

$$P_1 + P_2 = E^2 \frac{(l^2 n^2 + r^2)(2R + r) + R(2Rr + r^2 - l^2 n^2)\cos 2\alpha}{(l^2 n^2 + r^2)\{(2R + r)^2 + l^2 n^2\}}. \quad (1)$$

$$= \tfrac{1}{2} E^2 \frac{R + \tfrac{1}{2} r}{(R + \tfrac{1}{2} r)^2 + (\tfrac{1}{2} l)^2 n^2} \quad \text{when } \alpha = 0, \quad . \quad . \quad . \quad (2)$$

$$\frac{P_1 - P_2}{P_1 + P_2} = \frac{2R(R + r)\sin 2\alpha}{\dfrac{(2Rr + r^2)(R + r - R\cos 2\alpha)}{ln} + (R + r + R\cos 2\alpha)ln} \quad .(3)$$

The expression in (2) for $P_1 + P_2$, when the alternators are in phase, is similar to the power given by a single alternator

$$P = \tfrac{1}{2} E^2 \frac{R + r}{(R + r)^2 + l^2 n^2}$$

substituting $\dfrac{r}{2}$ and $\dfrac{l}{2}$ for r and l, so that the total power of two alternators in parallel is the same as that of a single alternator having half the internal resistance and half the self-induction.

It can be proved similarly that in the case of three alternators in parallel, the total power is the same as that of one alternator having $\frac{1}{3}$ the internal resistance and $\frac{1}{3}$ the self-induction.

It is reasonable to suppose, by induction, that the law holds good generally, so that the law for alternators in parallel is the same as that for cells in parallel.

Another point of interest is, that if any number m of alternators are working in parallel, then when

$$l^2 n^2 = (m - 1)(mRr + r^2)$$

the total power equals $\dfrac{\tfrac{1}{2} E^2}{R + r}$, and the current follows Ohm's law.

Examining the expression in (1) for $P_1 + P_2$, it is seen that as the difference in phase α increases, and $\cos 2\alpha$ diminishes, the total power increases if $l^2 n^2$ is smaller than $2Rr + r^2$, and *vice versa*.

In the former case, part of the extra work done by the leading machine is used to bring up the lagging machine into step again, and the remainder goes into the external circuit.

In the latter case, the controlling power does not all come directly from the leading machine, but the lagging machine partly borrows it from the external circuit.

When $l^2 n^2 = 2\mathrm{R}r + r^2$ the total power remains the same, whatever the phase difference, the leading machine supplying the whole of the controlling power, and there is no variation of power $+$ or $-$ in the external circuit.

Thus it would seem desirable, so as to have no fluctuations of current in the external circuit, to make $l^2 n^2 = 2\mathrm{R}r + r^2$.

Considering the expression for $\dfrac{\mathrm{P}_1 - \mathrm{P}_2}{\mathrm{P}_1 + \mathrm{P}_2}$ in equation (3),

we find that it is a maximum for variation of ln, when

$$l^2 n^2 = \frac{(2\mathrm{R}r + r^2)(\mathrm{R} + r - \mathrm{R}\cos 2\alpha)}{\mathrm{R} + r + \mathrm{R}\cos 2\alpha}, \quad . \quad . \quad . \quad (4)$$

so that when α is very small, $ln = r$.

This is the same result that Hopkinson arrives at (see Fleming's 'Alternating Current Transformer,' vol. ii. p. 356), and ln may not be reduced beyond this minimum value without serious loss of control.

Again $\dfrac{\mathrm{P}_1 - \mathrm{P}_2}{\mathrm{P}_1 + \mathrm{P}_2}$ is a maximum for a given value of ln by variation of α when

$$\cos 2\alpha = \frac{\mathrm{R}(2\mathrm{R}r + \mathrm{R}^2 - l^2 n^2)}{(\mathrm{R} + r)(2\mathrm{R}r + \mathrm{R}^2 + l^2 n^2)}, \quad . \quad . \quad . \quad (5)$$

so that if ln is very small the ratio $\dfrac{\mathrm{P}_1 - \mathrm{P}_2}{\mathrm{P}_1 + \mathrm{P}_2}$ rapidly increases to a maximum for a very small difference in phase.

At first sight this might appear to be an argument in favour of having the self-induction as small as possible, but on consideration it will be seen that the alternators would then be very " stiff," in the same sense as a sailing boat is " stiff" as opposed to " crank," and that the least alteration in the working of the engine would not only cause large currents to surge backwards and forwards between the two alternators, but in addition would affect considerably the current in the external circuit.

This rigid control is decidedly to be avoided, for it is impossible so to regulate two engines that they will work exactly at the same number of revolutions, and some fluctuation of the engines is to be expected in speed.

Some elasticity then is required in the system, whereby a small alteration in speed of the engine should

    (1) Not affect the power in the external circuit,
    (2) Damp down the surging currents,
    (3) Give maximum control.

The first is effected by putting $l^2n^2 = 2\mathrm{R}r + r^2$, and the second by making the self-induction a maximum. We shall now see how the third can be effected.

The difference in phase when $\dfrac{\mathrm{P}_1 - \mathrm{P}_2}{\mathrm{P}_1 + \mathrm{P}_2}$ is a maximum is given by equation (5).

Replacing $\sin 2\alpha$ and $\cos 2\alpha$ by their corresponding values, we get

$$\frac{\mathrm{P}_1 - \mathrm{P}_2}{\mathrm{P}_1 + \mathrm{P}_2} = \frac{2\mathrm{R}ln(\mathrm{R}+r)}{\sqrt{(2\mathrm{R}r + r^2)\{4\mathrm{R}^2 l^2 n^2 + (2\mathrm{R}r + r^2 + l^2 n^2)^2\}}}$$

$$= \frac{2\mathrm{R}(\mathrm{R}+r)}{\sqrt{2\mathrm{R}r + r^2}\sqrt{(2\mathrm{R}+r)^2 + r^2 + \left(\dfrac{2\mathrm{R}r + r^2}{ln} - ln\right)^2}}. \quad (6)$$

This is a maximum when $l^2 n^2 = 2\mathrm{R}r + r^2$, that is when the difference in phase is 90°; and it will be found that the self-induction can be altered to a considerable extent above or below this value, without affecting the control materially.

Taking into consideration, then, the fact that the self-induction is not constant, but varies to a certain extent with the current, and that in any case it is only possible to measure it approximately, it seems advisable to adjust it in such a manner as to make $l^2 n^2$ as nearly as possible equal to $2\mathrm{R}r + r^2$.

This allows a large margin either way without any appreciable loss of control, damps down the surging currents, and finally avoids all fluctuation in the external circuit by keeping constant the total power given out by the two alternators.

I have to thank Professor Greenhill for suggestions and for verification of the formulas involved.

XXXII. *Radiation and Electromagnetic Theory.*
By R. HARGREAVES[*].

REFLEXION at a moving surface has certain distinct features : the angles of incidence and reflexion are unequal, the periods of the incident and reflected waves are different. At the same time energy is converted to mechanical work by the action of pressure on a moving surface.

* Communicated by the Author.

The special feature of the transformation is the relation of the formula for Doppler's effect to the mode in which allowance is made for this conversion of energy.

The translation in the first section is not general, but it will appear in § 7 that when reflexion only is in question it is not necessary to consider other components.

§ 1. First take the case of reflexion at a perfectly reflecting plane surface, which is retreating directly from the free æther space with uniform velocity w. If $p\{t-(lx+my+nz)/\mathrm{V}\}$ is the argument of an original, and $p'\{t-(l'x+m'y-n'z)/\mathrm{V}\}$ that of the reflected wave, the kinematical part of the law of reflexion consists in the identification of arguments at the moving surface, *i. e.*, for a value $z=z_0+wt$: the identification to hold for all values of t, x, and y. The conditions are

$$pl=p'l', \quad pm=p'm', \quad p(1-wn/\mathrm{V})=p'(1+wn'/\mathrm{V}). \quad (1)$$

The former pair may be replaced by $p^2(1-n^2)=p'^2(1-n'^2)$, and the law of reflexion is

$$(1-n^2)/(1-wn/\mathrm{V})^2=(1-n'^2)/(1+wn'/\mathrm{V})^2, \quad . \quad (2)$$

with a common value ϕ for an azimuthal angle involved in

$$l=\sqrt{1-n^2}\cos\phi, \quad m=\sqrt{1-n^2}\sin\phi, \quad l'=\sqrt{1-n'^2}\cos\phi,$$
$$m'=\sqrt{1-n'^2}\sin\phi.$$

The differential of (2) gives

$$(\mathrm{V}n-w)dn/(1-wn/\mathrm{V})^3=(\mathrm{V}n'+w)dn'/(1+wn'/\mathrm{V})^3,$$

or $\qquad p^3(\mathrm{V}n-w)dn=p'^3(\mathrm{V}n'+w)dn',$

and multiplication by the last of (1) gives

$$p^4(\mathrm{V}n-w)(1-wn/\mathrm{V})dn=p'^4(\mathrm{V}n'+w)(1+wn'/\mathrm{V})dn', \quad (3)$$

a formula of fundamental importance. The law of reflexion may be put in linear form, viz. with $r=w/\mathrm{V}$,

$$\frac{\mathrm{V}n-w}{\mathrm{V}-wn}=\frac{\mathrm{V}n'+w}{\mathrm{V}+wn'}, \text{ or } \frac{n-r}{1-rn}=\frac{n'+r}{1+rn'}, \quad . \quad . \quad (4)$$

which follows from (2) by using

$$1-\left(\frac{n-r}{1-rn}\right)^2=\frac{(1-r^2)(1-n^2)}{(1-rn)^2}.$$

We may note also the relations

$$p(\mathrm{V}n-w)=p'(\mathrm{V}n'+w), \text{ and } p^2dn=p'^2dn'. \quad . \quad . \quad (5)$$

It is proposed to obtain a relation between mean values of p^4 and p'^4 independent of the angle of incidence by integrating the two sides of (3) with regard to n and n' respectively, while leaving the mean values p^4 and p'^4 outside the sign of integration. We may dispense with a separate notation, and describe the resulting relation between p and p' as a collective relation: this is regarded as appropriate to two streams of radiation incident and reflected, the former evenly distributed as regards direction. The proper limits for n are r and 1, for n' they are $-r$ and 1; it is evident that a wave for which $n < r$ fails to reach the surface, and $n = r$ corresponds to $n' = -r$ by (4). The result of the integration is

$$p^4(1-r)^3\left(1+\frac{r}{3}\right) = p'^4(1+r)^3\left(1-\frac{r}{3}\right). \quad . \quad . \quad . \quad (6)$$

Both in (3) and in (6) p^4 and p'^4 may be replaced by $p^3 dp$ and $p'^3 dp'$, and for (5) may be written $p\,dp\,dn = p'dp'dn'$. Since the wave-length $\lambda = 2\pi V/p$, λ^{-1} and λ'^{-1} may take the places of p and p', or $d\lambda/\lambda^5$ and $d\lambda'/\lambda'^5$ those of $p^3 dp$ and $p'^3 dp'$.

§ 2. Analytically the process is that of transformation of a double integral with subsequent integration with regard to one variable on each side. A continuous range of values for p and p' is assumed, and if the range is limited the relation is not applicable within a certain margin near the limit.

The form of (3) points to its interpretation as an equation for the transfer of energy by reflexion, with an allowance for the work done by pressure and for the filling of the new space opened out by the motion, with energy-content. In Wien's theory energy-content is proportional to $d\lambda/\lambda^5$ multiplied by a function of $\lambda\theta$, or to $p^3 dp$ multiplied by a function of p/θ, when p is variable. If in the two streams $p/\theta = p'/\theta'$ the energies are as $p^3 dp : p'^3 dp'$ or as $p^4 : p'^4$; and the above equation (6) connects the energy-content in a stream of incident radiation with that in the reflected stream. The possibility of writing $p/\theta = p'/\theta'$ depends on the invariable collective relation between p and p'. On the other hand, the notion of temperature as a general energy parameter which is implied in Wien's function, is not admissible for the individual wave, and its application seems to demand a preliminary treatment of a collective character.

If χ is used for the energy-content belonging to radiation proceeding in one sense (directions covering a hemisphere), and is evenly distributed, the section between the cones n and $n + dn$ is χdn. We take it that χ and χ' may replace

p^4 and p'^4 in the integrated form (6), and consider the interpretation. With odd and even powers of r separated, the exact formula is

$$(\chi-\chi')\left(1+2r^2-\frac{r^4}{3}\right)=\frac{8r}{3}(\chi+\chi');$$

and if r is small,

$$\frac{V}{2}(\chi-\chi')=w(\chi+\chi')+\frac{w}{3}(\chi+\chi'), \quad . \quad . \quad . \quad (7)$$

which gives a rough idea of the meaning. At a fixed plane parallel to the moving reflector, the left-hand member gives the excess of the rate of passage of the outgoing radiation above the returning radiation (radiation formula $\chi\int_0^1 Vn\,dn$) ; the first term on the right is the rate at which energy-content is supplied to the new space, and the second term is the rate of working of pressure (formula $\int_0^1 n.wn.\chi\,dn$). When r is small, (7) gives the ratio $\chi:\chi'$ correctly to the second order ; but when r approaches 1 (or w approaches V) the difference is important, and the exact formula makes χ' vanish with $1-r$.

Electromagnetic theory gives the more detailed interpretation through the form

$$\chi\int Vn\,dn=\chi\int(Vn-w)(1-rn)\,dn+w\chi\int(1-rn)\,dn+w\chi\int n^2dn. \quad (8)$$

The radiation across a fixed plane provides (1) the exchange value, or flux at a moving surface, with (2) the energy-content proper to a moving standpoint for the space opened out, and (3) the work done by pressure.

The formula for pressure in reflexion is

$$\chi\int_r^1 n^2dn+\chi'\int_{-r}^1 n'^2dn' \quad \text{or} \quad \frac{\chi}{3}(1-r^3)+\frac{\chi'}{3}(1+r^3),$$

or

$$\frac{1}{3}(\chi+\chi')-\frac{r^3}{3}(\chi-\chi');$$

and the exact division into parts with the meaning assigned in (7) is

$$\frac{V}{2}(\chi-\chi')=w\left\{\chi+\chi'-r\left(1-\frac{r^2}{2}\right)(\chi-\chi')\right\}$$
$$+\frac{w}{3}\left\{\chi+\chi'-r^3(\chi-\chi')\right\}.$$

§ 3. The collective formula presupposes a continuous range

of values of p; if the range is limited, then a margin of values of p' will be imperfectly represented, or the relation between p and p' ceases to have the uniform character given above within a margin at either extreme. The extent of such margin depends on r or w/V, and when r is small is inappreciable; but is not so if r is finite. The conclusions drawn as to the fourth power of temperature require the *range of p to be made sufficiently wide to ensure this penumbral effect being of no importance*, but need not be infinite unless $w=V$.

In connexion with this question of the fourth power, if we look back to the construction of (3) the use of

$$p^3(Vn-w)dn=p'^3(Vn'+w)dn'$$

or $\qquad p^2(Vn-w)\ dp\ dn=p'^2(Vn'+w)\ dp'\ dn'$

seems to be essential, the further subdivision into

$$pdpdn=p'dp'dn' \quad \text{and} \quad p(Vn-w)=p'(Vn+w)$$

not being possible in refraction. As a matter of analysis, a more general collective relation, involving transit with some loss or modification, would be got by multiplying the above by an arbitrary function of

$$p(1-qn) \qquad \text{or} \qquad p'(1+qn').$$

When the object is to allow for work done by pressure it is *primâ facie* reasonable to make the work depend directly on wn the component of translation along the wave-normal, and use the linear factor. In this way the theory is brought into relation with the laws of Stefan and Wien; but the point will be mentioned later.

In reflexion two of the factors p in p^4 are connected with the angular correspondence in the original and reflected streams, the other two are connected with the relative velocities $Vn-w$ and $V-wn$, the former a velocity of transit, the latter connected with change of period. When we pass to the collective relation this is true as to the source of the power in θ^4. In effect radiation being a question of exchange, its law depends on the degree of freedom for exchange: for the analogues in two dimensions and in one dimension the appropriate powers are the cube and the square.

For two dimensions (2) is correct, but (3) suggests a wrong angular element: the element is $d\theta$ (not $\sin\theta d\theta$, *i. e.* $\sin\theta d\theta d\phi$ with $d\phi$ discarded), and

$$dn : dn' = d\theta\ \sqrt{1-n^2} : d\theta'\ \sqrt{1-n'^2} = pd\theta : p'd\theta';$$

so that for (3) should be written

$$p^3 \int (V\cos\theta-w)(1-r\cos\theta)d\theta = p'^3 \int (V\cos\theta'+w)(1+r\cos\theta')d\theta'.$$

The integrated relation is

$$p^3\left[r'(1+r^2)-\frac{r}{2}\Big(rr'+3\text{ arc sin }r'\Big)\right]$$
$$=p'^3\left[r'(1+r^2)+\frac{r}{2}\Big(\overline{3\pi-\text{arc sin }r'}-rr'\Big)\right],$$

where $r'=\sqrt{1-r^2}$. In the limit when r' is small or w nearly $=\text{V}$, p'^3/p^3 varies as r'^5 or $(1-r^2)^{5/2}$. The approximate form analogous to (7) is

$$\frac{2\text{V}}{\pi}(\chi-\chi')=w(\chi+\chi')+\frac{w}{2}(\chi+\chi');$$

and the exact estimate of pressure is

$$\frac{\chi}{\pi}(\text{arc sin }r'+rr')+\frac{\chi'}{\pi}(\pi-\text{arc sin }r'+rr').$$

In one dimension we have the single relation

$$p(1-w/\text{V})=p'(1+w/\text{V}),$$

which must be duplicated in the form

$$p^2(\text{V}-w)(1-w/\text{V})=p'^2(\text{V}+w)(1+w/\text{V}),$$

to show the change of period and velocity of transit. No integration occurs, χ and χ' are written for p^2 and p'^2, and the pressure is $\chi+\chi'$.

The special features in the comparison of the cases for different dimensions are :— for radiation formulæ

$$\frac{\text{V}\chi}{2}, \quad \frac{2\text{V}\chi}{\pi}, \quad \text{and} \quad \text{V}\chi;$$

for the limits when $w=\text{V}$ or $r=1$, p' varies as $(1-r)^{3/4}$, $(1-r)^{5/6}$, $1-r$; and χ' as $(1-r)^3$, $(1-r)^{5/2}$, $(1-r)^2$.

§ 4. When refraction accompanies reflexion there are two channels into which energy is directed, and the two transformations concerned are only part of the problem : the partition of energy requires distinctive optical theory. The special features affecting the transformation for refraction are (1) a modified velocity of propagation in the dielectric, and (2) the necessity for interpretation with reference to the moving standpoint of the dielectric.

At first we use an argument referred to the same coordinates as the original wave, take p'' for period number, and suppose Ω the velocity of propagation to depend on

$l''u + m''v + n''w$ or U''. The comparison of arguments gives

$$\frac{pl}{V} = \frac{p''l''}{\Omega}, \quad \frac{pm}{V} = \frac{p''m''}{\Omega}, \quad p\left(1 - \frac{wn}{V}\right) = p''\left(1 - \frac{wn''}{\Omega}\right); \left.\right\} (9)$$

and then

$$p\left(1 - \frac{U}{V}\right) = p''\left(1 - \frac{U''}{\Omega}\right),$$

follows by combining the other ratios. The correspondence of p and p'' is the same whether deduced from agreement at a moving point or surface. If the ratio $p : p''$ is eliminated,

$$(1-n^2)/(V-wn)^2 = (1-n''^2)/(\Omega-wn'')^2,$$

or

$$(1-n^2)/(V-U)^2 = (1-n''^2)/(\Omega-U'')^2. \left.\right\} (10)$$

An azimuthal angle ϕ ($l = \sqrt{1-n^2}\cos\phi$) is the same for both waves, this fact and (10) together constituting the law of refraction. Since ϕ is the same for both waves it may be ignored in forming the differentials of (10), and we take account of the dependence of l'', m'' on n'' by writing

$$\frac{dl''}{l''} = \frac{-n''dn''}{1-n''^2} = \frac{dm''}{m''}.$$

Hence

$$d\Omega = \Sigma\,\frac{d\Omega}{dl''}dl'' = \frac{d\Omega}{dU''}\Sigma u\,dl'',$$

or

$$(1-n''^2)d\Omega = \frac{d\Omega}{dU''}(w-n''U'')dn'',$$

and the differential of (10) yields

$$\frac{(Vn-w)dn}{(V-U)^3} = \frac{\left\{n''\left(\Omega-U''\frac{d\Omega}{dU''}\right) - w\left(1-\frac{d\Omega}{dU''}\right)\right\}dn''}{(\Omega-U'')^3}.$$

Either form of (10) may be used, and the denominators here may be replaced by $(V-wn)^3$ and $(\Omega-wn'')^3$. The numerators contain a differential and a velocity of transit, and $d\phi$ may be appended to each side. [If Ω is defined as a function of $l''m''n''$ without reference to U'', the transit-form is

$$n''\left(\Omega-\Sigma l''\frac{d\Omega}{dl''}\right) + \frac{d\Omega}{dn''} - w.]$$

With Fresnel's modification, which is used throughout,

$$\Omega = \frac{V}{\mu} + U''\left(1 - \frac{1}{\mu^2}\right),$$

and the above takes the simpler form

$$\frac{(\mathrm{V}n-w)dn}{(\mathrm{V}-\mathrm{U})^3} = \frac{\left(\dfrac{\mathrm{V}n''}{\mu} - \dfrac{w}{\mu^2}\right)dn''}{\left(\dfrac{\mathrm{V}}{\mu} - \dfrac{\mathrm{U}''}{\mu^2}\right)^3}; \quad \cdot \quad \cdot \quad \cdot \quad (11)$$

the transit-velocity, or component of ray-velocity, depending only on n''. Introducing the fourth powers of p and p'',

$$p^4\mathrm{V}^{-4}(\mathrm{V}n-w)(\mathrm{V}-\mathrm{U})dn = p''^4\Omega^{-4}\left(\frac{\mathrm{V}n''}{\mu} - \frac{w}{\mu^2}\right)\left(\frac{\mathrm{V}}{\mu} - \frac{\mathrm{U}''}{\mu^2}\right)dn'' . (12)$$

With the wave-length as variable, since $p''/\Omega = 2\pi/\lambda''$,

$$\lambda : \lambda'' = l : l'' = m : m'' = \mathrm{V} - wn : \Omega - wn''$$

$$= \mathrm{V} - \mathrm{U} : \Omega - \mathrm{U}'' = \mathrm{V} - \mathrm{U} : \frac{\mathrm{V}}{\mu} - \frac{\mathrm{U}''}{\mu^2}, \quad (13)$$

and

$$\lambda^{-4}(\mathrm{V}n-w)(\mathrm{V}-\mathrm{U})dn = \lambda''^{-4}\left(\frac{\mathrm{V}n''}{\mu} - \frac{w}{\mu^2}\right)\left(\frac{\mathrm{V}}{\mu} - \frac{\mathrm{U}''}{\mu^2}\right)dn''. (14)$$

We now give a transformation of a different kind, in which the ray-cosines define direction, and a period-number is used which has reference to a moving standpoint. For free æther this is $\varpi = p(1 - \mathrm{U}/\mathrm{V})$, and for the dielectric $\varpi'' = p''(1 - \mathrm{U}''/\Omega)$, and by (9) $\varpi = \varpi''$. With (11) the transfer from p to ϖ or ϖ'' is given by

$$p^4(\mathrm{V}n-w)(1 - \mathrm{U}/\mathrm{V})dn/\mathrm{V}^3 = \varpi^4(\mathrm{V}n-w)dn/(\mathrm{V}-\mathrm{U})^3$$

$$= \varpi''^4\left(\frac{\mathrm{V}n''}{\mu} - \frac{w}{\mu^2}\right)dn'' \Big/ \left(\frac{\mathrm{V}}{\mu} - \frac{\mathrm{U}''}{\mu^2}\right)^3. \quad (15)$$

Ray-cosines (l_r, m_r, n_r) are connected with the wave-normal cosines by the scheme

$$\left.\begin{array}{l} \mathrm{V}_r l_r = \mathrm{V}l - u, \quad \mathrm{V}_r m_r = \mathrm{V}m - v, \quad \mathrm{V}_r n_r = \mathrm{V}n - w, \\ \mathrm{V}_r^2 = \mathrm{V}^2 - 2\mathrm{V}\mathrm{U} + \Sigma u^2, \quad \mathrm{V}_r^2 + 2\mathrm{V}_r\mathrm{U}_r = \mathrm{V}^2 - \Sigma u^2, \\ \mathrm{V}_r\Sigma l_r l = \mathrm{V} - \mathrm{U}, \quad \mathrm{V}\Sigma l_r l = \mathrm{V}_r + \mathrm{U}_r = \sqrt{\mathrm{V}^2 - \Sigma u^2 + \mathrm{U}_r^2}, \end{array}\right\} , (16)$$

V_r being a ray-velocity and U_r standing for $\Sigma l_r u$. In a dielectric $\dfrac{\mathrm{V}}{\mu}$, $\dfrac{u}{\mu^2}$, ... take the places of V, u, ... The radical is the wave-velocity in the direction of the ray. Comparing $\sqrt{\mathrm{V}^2 - \Sigma u^2 + \mathrm{U}_r^2} - \mathrm{U}_r$ and $\sqrt{\mathrm{V}^2 - \Sigma u^2 + \mathrm{U}_r^2} + \mathrm{U}_r$, which are the numerical values of V_r, with $\mathrm{V} - \mathrm{U}$ and $\mathrm{V} + \mathrm{U}$ belonging to translation with or against the wave, the radical

plays a part corresponding to that of V. For the transfor-
mation use the variables $(n_r \, \phi_r)$, $(n \, \phi)$, where $\tan \phi_r = m_r/l_r$,
$\tan \phi = m/l$; the Jacobian after some reduction gives

$$\mathrm{V}_r^3 \left(\frac{dn_r}{dn} \frac{d\phi_r}{d\phi} - \frac{dn_r}{d\phi} \frac{d\phi_r}{dn} \right) = \mathrm{V}^2(\mathrm{V} - \mathrm{U}).$$

Hence

$$\varpi^4(\mathrm{V}n - w) dn \, d\phi/(\mathrm{V} - \mathrm{U})^3 = \varpi^4 \, \mathrm{V}_r \, n_r \, dn \, d\phi/(\mathrm{V} - \mathrm{U})^3$$
$$= \mathrm{V}^2 \varpi^4 \, n_r \, dn_r \, d\phi_r/\{\mathrm{V}^2 - \Sigma u^2 + \mathrm{U}_r^2\}^2,$$

since by (16) $\quad \mathrm{V}(\mathrm{V} - \mathrm{U}) = \mathrm{V}_r \sqrt{\mathrm{V}^2 - \Sigma u^2 + \mathrm{U}_r^2}.$

Apply to the third member of (15) with $d\phi$ appended a
like transformation, and we get

$$p^4(n - w/\mathrm{V})(1 - \mathrm{U}/\mathrm{V}) \, dn \, d\phi = \varpi^4 \, n_r \, dn_r \, d\phi_r/\{1 - \Sigma u^2/\mathrm{V}^2 + \mathrm{U}_r^2/\mathrm{V}^2\}^2$$
$$= \mu^2 \varpi''^4 \, n_r'' \, dn_r'' \, d\phi_r''/\{1 - \Sigma u^2/\mu^2 \mathrm{V}^2 + \mathrm{U}_r''^2/\mu^2 \mathrm{V}^2\}^2. \quad (17)$$

§ 5. Consider first the *statical* case, in which ϖ coalesces
with p. It is then indifferent whether p'' or λ'' is used as
variable, and the form of equations (12) and (14) suggests
for the ratio of energy-contents in the two media

$$\chi : \chi'' = \mathrm{V}/\lambda^4 : \mathrm{V}''/\lambda''^4 = p^4/\mathrm{V}^3 : p''^4/\mathrm{V}''^3, \quad . \quad . \quad (18)$$

with V'' for V/μ: since $p = p''$ or $\mu\lambda'' = \lambda$, $\chi'' = \mu^3\chi$. We
may then write $p/\theta = p''/\theta''$ or its equivalent $\lambda\theta = \mu\lambda''\theta''$.
This is consistent with the use of Wien's formula in the shape

$$\mathrm{V}''^{-3} p''^3 \, dp'' \, f(p''/\theta'') \quad \text{or} \quad \mathrm{V}''\lambda''^{-5} \, d\lambda'' \, f(\mathrm{V}''/\lambda''\theta''),$$

for the result of integration through an infinite range of
p'' or λ'' is proportional to $\theta''^4/\mathrm{V}''^3$, and the condition of
statical exchange $\mathrm{V}n(\theta^4/\mathrm{V}^3) dn = \mathrm{V}''n''(\theta''^4/\mathrm{V}''^3) dn''$ makes
$\theta = \theta''$ follow from $ndn/\mathrm{V}^2 = n''dn''/\mathrm{V}''^2$.

When there is motion the use of (p, p') or (λ, λ') in *reflexion*
is indifferent. But there is this real difference between the
original and reflected waves on the one hand, and the re-
fracted wave on the other, that for the former the moving
standpoint represents a momentary phase, while for the latter
it is permanent, and moreover is that with reference to which
we naturally seek to interpret matters. Now an argument
$\Sigma lx' - \mathrm{V}t$ becomes $\Sigma lx - (\mathrm{V} - \mathrm{U})t$, when by the use of
$x' = x + ut, \ldots$ a change is made to a moving standpoint; the
change alters the period not the wave-length, and this is
equally true if the variable velocity Ω is used instead of V.

Thus if the formula in λ'' is used it is clear that

$$\left(\frac{V}{\mu} - \frac{U''}{\mu^z}\right)\lambda''^{-4}\,dn'' \quad \text{not} \quad \frac{V}{\mu}\lambda''^{-4}\,dn''$$

must be proportional to energy-content, otherwise there would be nothing to mark the moving standpoint. This agrees with a formula used above (8) for the energy-content in space opened out by the motion. The formula uses wave variables for direction.

If energy-content were taken proportional to $\dfrac{V}{\mu}\lambda''^{-4}\,dn''$,

then the term in U'' would necessarily be interpreted as on the other side of the equation, and would imply a back-pressure on the æther caused by the refracted ray (*v. infra*).

The formula in ϖ has no term which could be interpreted in this manner, and a special feature is that $(u\,v\,w)$ only occur in second order terms. The use of ϖ and ray-cosines both imply a new standpoint; the fact that ϖ and ϖ'' are equal makes the change of standpoint rather than that of medium the important fact in (17).

§ 6. The collective relations when reflexion and refraction are combined must take account of the proportion in which the radiation is divided into two streams, a proportion varying with the angle of incidence and with the plane of polarization. For the present we denote by A^z and B^2 the fractions (sum unity) of the original energy which undergo the respective transformations; the value of A^2 is given by electromagnetic theory, and will be found to depend on $(u\,v\,w)$ as well as on the angular variables. It is proposed to weight the equations of transformation with A^2 for the reflected and B^2 for the refracted section ; and then to integrate them in a manner which will clear the results of all reference to particular waves. The relations involve n and ϕ (through U), and A^2 also involves an angle defining the plane of polarization. But if we suppose A^2 to stand for the mean with regard to this last angle, the plane of polarization taking all possible positions for each wave-argument, then only n and ϕ need appear, *i. e.*

$$p^4 \iint A^2(Vn-w)(1-U/V)\,dn\,d\phi$$
$$= p'^4 \iint A^2(Vn'+w)(1-U'/V)\,dn'\,d\phi$$
$$p^4 \iint B^2(Vn-w)(1-U/V)\,dn\,d\phi$$
$$= \varpi^4 \iint B^2 Vn_r\,dn_r\,d\phi_r/(1-\Sigma u^2/V^2 + U_r^2/V^2)^2$$

$$\left.\right\} \quad (19)$$

and the sum

$$p^4 \iint (Vn-w)(1-U/V)\,dn\,d\phi$$
$$= p'^4 \iint A^2(Vn'+w)(1-U'/V)\,dn'\,d\phi + \varpi^4 \iint B^2 Vn_r \ldots$$

is the equation for flux of energy. [The formula for ϖ or ϖ'' may be used, *v. infra.*] The terms representing a rate of mechanical working are

$$p^4 \iint n\mathrm{U} \, dn \, d\phi - p'^4 \iint \mathrm{A}^2 n' \mathrm{U}' \, dn' \, d\phi,$$

and the pressure depends on the term multiplying w contained in U and U' or $l'u + m'v - n'w$. If we are only concerned with its main term, that independent of $u\,v\,w$, we may write $n' = n$, $p' = p$, use statical limits for n and a statical value of A^2, and then with χ for the energy-content of the incident radiation, the pressure is

$$\frac{\chi}{2\pi} \cdot \iint (1 + \mathrm{A}^2) n^2 \, dn \, d\phi. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (20)$$

The terms containing the other section of U, viz. $lu + mv$ will vanish in the integration with regard to ϕ, if A^2 is a rational integral function of $(lu + mv)^2$ and of $(lv - mu)$, *i. e.* of $(u \cos \phi + v \sin \phi)^2$ and of $(v \cos \phi - u \sin \phi)$. This is readily seen by taking $v = 0$, *i. e.* by measuring ϕ from the component of translation in the reflecting surface, in which case the quantities named vary as $\cos^2 \phi$ and $\sin \phi$. The component which must only enter through its square is for each wave-front the component in the reflecting surface along its line of intersection with the wave-front. On the present electromagnetic scheme this will be shown to be true in an exact manner, when A^2 stands as above for the mean of all polarizations taken for that particular wave-front. Thus there is no work expended in connexion with tangential forces (in the collective result) ; a further consequence is that in (19) nw may be written for U and $-n'w$ for U', and that all first-order results are independent of u and v.

§ 7. For perfect reflexion $\mathrm{A}^2 = 1$, and the vanishing of the above integrals is obvious, but in that case u and v do not intervene at all, as the variables of the reflected wave are determined independently of u and v. In fact

$$l : l' = m : m' = \mathrm{V} - \mathrm{U} : \mathrm{V} - \mathrm{U}'$$

is equivalent to

$$l : l' = \mathrm{V} - wn : \mathrm{V} + wn'$$

or to $\qquad \sqrt{1 - n^2} : \sqrt{1 - n'^2} = \mathrm{V} - wn : \mathrm{V} + wn' ;$

i. e. n' is determined in terms of n precisely as in (4) with only w present. The omission of u and v in the account of perfect reflexion is therefore justified, and that not merely as a preliminary course to secure simplicity in the opening statement.

In a general formula it may be convenient to measure n' in the same direction as n; with this arrangement, which may be marked by the use of suffixes, the group of relations for reflexion is

$$l_1 : l_2 = m_1 : m_2 = V - wn_1 : V - wn_2$$
$$= -(Vn_1 - w) : (Vn_2 - w) = V - U_1 : V - U_2, \Bigg\} \quad (21)$$

and we may deduce

$$l_1 l_2 + m_1 m_2 - n_1 n_2 = 1 - (n_1 + n_2)w/V,$$

which will be of service later.

Thus
$$\frac{1}{l_1}(l_1 l_2 + m_1 m_2 - n_1 n_2) = -\frac{n_1 n_2}{l_1} + \frac{1}{l_2}(1 - n_2^2)$$
$$= \frac{1}{l_2} - n_2\Big(\frac{n_1}{l_1} + \frac{n_2}{l_2}\Big) = \frac{1}{l_2} - \frac{wn_2}{V}\Big(\frac{1}{l_1} + \frac{1}{l_2}\Big),$$

using
$$l_2(Vn_1 - w) + l_1(Vn_2 - w) = 0,$$
$$= \frac{1}{l_1}\Big\{1 - \frac{w}{V}(n_1 + n_2)\Big\},$$

using
$$\frac{1}{l_2}\Big(1 - \frac{wn_2}{V}\Big) = \frac{1}{l_1}\Big(1 - \frac{wn_1}{V}\Big).$$

When V/μ is written for V and w/μ^2 for w, the results are applicable to the reflexion of a wave, moving with modified velocity in a dielectric, at a surface sharing the motion of the dielectric. In refraction when u and v exist, ϕ always intervenes in the determination of n'' from n, viz., from

$$l : l'' = m : m'' = V - U : \frac{V}{\mu} - \frac{U''}{\mu^2}$$

follows

$$\frac{1 - wn/V}{\sqrt{1 - n^2}} - \frac{1 - wn''/\mu V}{\mu\sqrt{1 - n''^2}} = \Big(1 - \frac{1}{\mu^2}\Big)(u\cos\phi + v\sin\phi)/V.$$

The first-order relation between ray-cosines may be noticed. We have $l/(1 - U/V) = \mu l''(1 - U''/\mu V)$ exactly, and therefore to the first order

$$(l - u/V)/(1 - U/V) = \mu(l'' - u/\mu V)/(1 - U''/\mu V);$$

but to this order

$$l_r = (Vl - u)/(V - U) \text{ and } l_r'' = \Big(\frac{V}{\mu}l'' - \frac{u}{\mu^2}\Big)\Big/\Big(\frac{V}{\mu} - \frac{U''}{\mu^2}\Big),$$

and so $l_r = \mu l_r''$, $m_r = \mu m_r''$.

§ 8. It was noted above, in regard to the formulæ for ϖ and ϖ'', that the change to a moving standpoint was the

main fact indicated, and the second and third members of (17) differ only geometrically. When the middle term is used we are in effect interpreting from a moving standpoint radiation which is proceeding in free æther, without reference to reflexion or refraction. If we form a collective relation as before and take first the simpler case $u = v = 0$, we have

$$p^4 \int (n - w/V)(1 - wn/V)\, dn = \varpi^4 \int_0^1 n_r dn_r / \{1 - w^2(1 - n_r^2)/V^2\}^2$$

$$\text{or} \qquad\qquad = \varpi^4 \int_0^1 \nu d\nu / (1 - w^2/V^2) \,;$$

the use of the kinematical cosine $\nu = (n - w/V)/(1 - wn/V)$ being easier. The result is

$$p^4 \left(1 - \frac{w}{V}\right)^3 \left(1 + \frac{w}{3V}\right) = \varpi^4 \Big/ \left(1 - \frac{w^2}{V^2}\right).$$

In the general case a double integration is needed, viz. :—

$$p^4 \iint \left(n - \frac{w}{V}\right)\left(1 - \frac{U}{V}\right) dn d\phi = \varpi^4 \iint n_r dn_r d\phi_r \Big/ \left[1 - \frac{\Sigma u^2}{V^2} + \frac{U_r^2}{V^2}\right]^2 ;$$

but as each of the variables $n_r \phi_r$ has full range, *i.e.* 0 to 1, and 0 to 2π, the integral on the right is $\varpi^4 \left(1 - \frac{\Sigma u^2}{V^2}\right)$, that on the left being unchanged. This gives a mean alteration of period due to Doppler's effect and a diminished rate of approach of energy. If we are dealing with refraction B^2 is used under the sign of integration, because the transformation applies only to a part of the energy, and the integral is more complicated. But the ϖ thus obtained is also the ϖ'' of the refracted ray ; there is no further change.

Through the factor belonging to Doppler's effect a proper contribution to mechanical work is levied on the section of the incident energy which suffers refraction. Again, since

$$p(1 - U_i/V) = p'(1 - U'/V) = \varpi,$$

the process of reflexion can be divided into two, the first a change from p to ϖ, the second a change from ϖ to p', each accompanied by a levy to mechanical work. Thus in effect a levy is made on the whole incident energy, and a further levy on the energy of the reflected wave, and each is associated with a change of standpoint.

The pressure, as pointed out above, can be found (in its main term) by the use of a statical value of A^2 and B^2 ; the collective values of p' and of ϖ and ϖ'' require a knowledge of the way in

which the components of translation enter into A^2 and B^2, and on that ground the calculations are deferred to a later stage.

Electromagnetic Scheme.

In the section of Electromagnetic theory presented here we are concerned almost entirely with a moving standpoint. Such a standpoint seems to be demanded for the consideration of reflexion at a moving surface, and of refraction into a moving dielectric. But it has further applications. When radiation is viewed from a moving standpoint the energy ceases to have purely the character of radiant energy; a part of it assumes a mechanical character. That is, a part of the energy is then radiant energy as interpreted from a moving standpoint, and another part is kinetic energy; and this latter is given by the scalar product of the translation and a momentum belonging to the passage of radiation.

For a dielectric there is a question of the coefficient in the motional term of the energy, and two values claim consideration. One gives a constant velocity of propagation for electromagnetic waves, the other a modified velocity. Both are regarded as actual, the former referring to waves originating in the dielectric, the latter to waves produced by the impact of waves from outside on a moving dielectric. The motion of the dielectric is a motion relative to the standpoint in respect to which the originating wave shows a constant velocity.

The modification is that of Fresnel's formula, and the formula is here exact, not an approximation. This simplifies the character of the equation for transit of energy in reflexion and refraction. Lorentz's equations, when treated exactly, involve a double refraction appertaining to the translation; *i. e.* the propagation of a plane wave is only possible for two modes of polarization, defined by the directions of the translation and of the wave-normal. This introduces great complication into the question of reflexion and refraction, especially as neither velocity of propagation is so easy to deal with as that of Fresnel's formula.

§ 9. For the propagation of electromagnetic waves, the scheme of equations referred to the moving standpoint is taken to be

$$\frac{\mathrm{K}}{\mathrm{V}}\frac{d\mathrm{X}}{dt} = \frac{d\gamma'}{dy} - \frac{d\beta'}{dz}, \quad \frac{\mathrm{M}}{\mathrm{V}}\frac{d\alpha}{dt} = \frac{d\mathrm{Y}'}{dz} - \frac{d\mathrm{Z}'}{dy}, \cdots \quad \text{(I.)}$$

$$\mathrm{X}' - \mathrm{X} = \epsilon\mathrm{M}(v\gamma - w\beta)/\mathrm{V}, \quad \alpha' - \alpha = \epsilon\mathrm{K}(w\mathrm{Y} - v\mathrm{Z})/\mathrm{V}, \cdots \text{(II.)}$$

with $\mathrm{X}' \mathrm{Y}' \alpha' \beta'$ continuous at a surface of separation $z =$ constant.

$\mathrm{KX}, \ldots \mathrm{M}\alpha, \ldots$ are the components of electric and magnetic induction, $\mathrm{X}' \ldots \alpha' \ldots$ are the electric and magnetic

forces modified by the translation $(u\,v\,w)$. The index of refraction will be denoted by μ, *i. e.* $\mu^2 = \mathrm{K}\,\mathrm{M}$, the product of the electric and magnetic constants of the medium. The use of the constant ϵ to cover two cases which are held to be distinct, will be pointed out after a short account of the propagation of a plane wave.

Write

$$\mathrm{X} = \mathrm{X}_0 f(lx + my + nz - \Omega t), \quad \alpha = \alpha_0 f'(lx + my + nz - \Omega t), \dots$$

with the same argument and function throughout ; then

$$\frac{d\mathrm{X}}{dt} = -\Omega \mathrm{X}_0 f', \text{ and } \frac{d\mathrm{X}}{dx} = l \mathrm{X}_0 f', \dots .$$

Thus (I.) is replaced by the algebraical equations

$$\mathrm{K}\Omega\mathrm{X} = \mathrm{V}(n\beta' - m\gamma'), \quad \mathrm{M}\Omega\alpha = \mathrm{V}(m\mathrm{Z}' - n\mathrm{Y}').$$

From these follow

$$\Sigma l\mathrm{X} = 0, \quad \Sigma l\alpha = 0 \quad . \quad . \quad . \quad . \quad . \quad (22)$$

corresponding to

$$\Sigma \frac{d\mathrm{X}}{dx} = 0, \quad \Sigma \frac{d\alpha}{dx} = 0.$$

With the help of (II.),

$$\mathrm{K}\Omega\mathrm{X} = \mathrm{V}(n\beta - m\gamma) + \epsilon\mathrm{K}\{n(u\mathrm{Z} - w\mathrm{X}) - m(v\mathrm{X} - u\mathrm{Y})\}$$
$$= \mathrm{V}(n\beta - m\gamma) - \epsilon\mathrm{K}\mathrm{U}\mathrm{X},$$

writing U for Σlu, and quoting (22). Thus

$$\left.\begin{aligned} \mathrm{K}\mathrm{X}(\Omega + \epsilon\mathrm{U}) &= \mathrm{V}(n\beta - m\gamma) ; \\ \mathrm{M}\alpha(\Omega + \epsilon\mathrm{U}) &= \mathrm{V}(m\mathrm{Z} - n\mathrm{Y}) \end{aligned}\right\} \quad . \quad (2\,3)$$

and follows by similar work. Therefore

$$\mu^2\mathrm{X}(\Omega + \epsilon\mathrm{U})^2 = \mathrm{V}\mathrm{M}(\Omega + \epsilon\mathrm{U})(n\beta - m\gamma)$$
$$= \mathrm{V}^2\{n(n\mathrm{X} - l\mathrm{Z}) - m(l\mathrm{Y} - m\mathrm{X})\} = \mathrm{V}^2\mathrm{X}$$
$$\text{or} \qquad \Omega + \epsilon\mathrm{U} = \pm\,\mathrm{V}/\mu. \quad . \quad . \quad . \quad . \quad (24)$$

The two values correspond to waves travelling in the sense of the translation and opposite to it, both velocities measured in the sense of the translation : the positive sign will be taken for the standard case.

If we write $x' = x + ut, \dots$ the differential equations (I.) become

$$\frac{\mathrm{K}}{\mathrm{V}}\frac{\mathrm{D}\mathrm{X}}{\mathrm{D}t} = \frac{d\gamma'}{dy'} - \frac{d\beta'}{dz'}, \quad \frac{\mathrm{M}}{\mathrm{V}}\frac{\mathrm{D}\alpha}{\mathrm{D}t} = \frac{d\mathrm{Y}'}{dz'} - \frac{d\mathrm{Z}'}{dy'}, \dots \quad (\mathrm{I.}\,b)$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{d}{dt} + u\frac{d}{dx'} + v\frac{d}{dy'} + w\frac{d}{dz'}.$$

The argument is then $lx' + my' + nz' - (\Omega + U)t$, and

$$\Omega + U = \frac{V}{\mu} + U(1 - \epsilon).$$

(i.) Thus if $\epsilon = 1$, the wave travels with constant velocity V/μ in relation to $(x'y'z')$, and appears to have the variable velocity $\frac{V}{\mu} - U$ in relation to (xyz).

(ii.) But if $\epsilon = \frac{1}{\mu^2}$,

$$\Omega + U = \frac{V}{\mu} + U\left(1 - \frac{1}{\mu^2}\right), \quad \Omega = \frac{V}{\mu} - \frac{U}{\mu^2};$$

both are variable unless $\mu = 1$, when there is no distinction between (i.) and (ii.).

For a dielectric (i.) is held to apply when the disturbance originates in the dielectric, and the translation in question is a motion relative to the dielectric. Case (ii.) is applicable to waves produced in a dielectric by waves in free æther; the moving standpoint is that of the dielectric itself, and the fixed standpoint is that in respect to which the originating wave travels with constant velocity.

The use of the moving standpoint simplifies the transition from (i.) to (ii.).

§ 10. We proceed to certain relations between the two types of variables for a plane wave, and they will be written for case (ii.) for which they are less obvious, because neither standpoint shows a contant velocity of propagation. Applying (24) to (23), we have

$$\left. \begin{array}{ll} \text{KX} = \mu(n\beta - m\gamma), & \text{M}\alpha = \mu(m\text{Z} - n\text{Y}) \\ \text{or} \\ \mu\text{X} = \text{M}(n\beta - m\gamma), & \mu\alpha = \text{K}(m\text{Z} - n\text{Y}) \end{array} \right\} ; \quad (25)$$

i. e., the relations between the inductions or inducing forces are the same as for the statical case, though in (ii.) the velocity is variable. An immediate consequence of (25) is

$$\text{K}\Sigma\text{X}^2 = \text{M}\Sigma(n\beta - m\gamma)^2 = \text{M}\Sigma\alpha^2 = \frac{\text{K}}{2}\Sigma\text{X}^2 + \frac{\text{M}}{2}\Sigma\alpha^2 \equiv \text{E}, \quad (26)$$

the last giving a definition of E. We now express α' in terms of α; thus

$$\alpha' = \alpha + \text{K}(w\text{Y} - v\text{Z})/\mu^2\text{V} = \alpha + \{w(l\gamma - n\alpha) - v(m\alpha - l\beta)\}/\mu\text{V}$$

$$= \alpha\left(1 - \frac{\text{U}}{\mu\text{V}}\right) + \frac{l}{\mu\text{V}}\Sigma u\alpha;$$

and similarly

$$\text{X}' = \text{X}\left(1 - \frac{\text{U}}{\mu\text{V}}\right) + \frac{l}{\mu\text{V}}\Sigma u\text{X}.$$

$\left. \vphantom{\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}} \right\} (27)$

Also

$$\alpha\left(1-\frac{U}{\mu V}\right)=\alpha'-\frac{l}{\mu V}\Sigma u\alpha', \text{ and } \Sigma\alpha'\left(l-\frac{u}{\mu V}\right)=0 \ . \ (28)$$

with similar properties for X, since $\Sigma u\alpha=\Sigma u\alpha'$ and $\Sigma uX=\Sigma uX'$. Thus the letters with dash denote quantities perpendicular to the ray, not to the wave-normal; also since

$$\Sigma\alpha'X'=\Sigma(\alpha'-\alpha)(X'-X)$$

they are not mutually perpendicular, except for first-order work in (uvw). From (27),

$$M\Sigma\alpha\alpha'=M\left(1-\frac{U}{\mu V}\right)\Sigma\alpha^2=K\left(1-\frac{U}{\mu V}\right)\Sigma X^2=K\Sigma XX'$$

$$=\frac{K}{2}\Sigma XX'+\frac{M}{2}\Sigma\alpha\alpha'\equiv S=E\left(1-\frac{U}{\mu V}\right); \ . \ . \ (29)$$

this giving a definition of S.

Again we may express α' in terms of the electric induction; thus

$$V\alpha' = V\alpha + \frac{K}{\mu^2}\left(wY-vZ\right)=\frac{KV}{\mu}\left(mZ-nY\right)+\frac{K}{\mu^2}\left(wY-vZ\right)$$

$$= K\left\{\left(\frac{Vm}{\mu}-\frac{v}{\mu^2}\right)Z-\left(\frac{Vn}{\mu}-\frac{w}{\mu^2}\right)Y\right\};$$

and similarly

$$VX'=M\left\{\left(\frac{Vn}{\mu}-\frac{w}{\mu^2}\right)\beta-\left(\frac{Vm}{\mu}-\frac{v}{\mu^2}\right)\gamma\right\},$$

$\left.\right\}(30)$

relations which involve components of ray-velocity. From (27) and (30),

$$V(X'\beta'-Y'\alpha')$$

$$= M\left\{\beta\left(\frac{Vn}{\mu}-\frac{w}{\mu^2}\right)-\gamma\left(\frac{Vm}{\mu}-\frac{v}{\mu^2}\right)\right\}\left\{\beta\left(1-\frac{U}{\mu V}\right)+\frac{m}{\mu V}\Sigma u\alpha\right\}$$

$$-M\left\{\gamma\left(\frac{Vl}{\mu}-\frac{u}{\mu^2}\right)-\alpha\left(\frac{Vn}{\mu}-\frac{w}{\mu^2}\right)\right\}\left\{\alpha\left(1-\frac{U}{\mu V}\right)+\frac{l}{\mu V}\Sigma u\alpha\right\}$$

$$= \left(\frac{Vn}{\mu}-\frac{w}{\mu^2}\right)\left(1-\frac{U}{\mu V}\right)M\Sigma\alpha^2$$

$$= \left(\frac{Vn}{\mu}-\frac{w}{\mu^2}\right)\left(1-\frac{U}{\mu V}\right)E=\left(\frac{Vn}{\mu}-\frac{w}{\mu^2}\right)S \ . \ . \ . \ . \ . \ (31)$$

From (25), $\qquad V(X\beta-Y\alpha)=\frac{Vn}{\mu}E. \ . \ . \ . \ . \ . \ (32)$

This completes the more important algebraical connexions between the two types of variables for a plane wave which are useful for the interpretation of formulæ, and in reflexion and refraction. We may note also that by (II.),

$$K\Sigma X(X'-X) = V^{-1}\Sigma X(v\gamma - w\beta)$$
$$= V^{-1}\Sigma\alpha(wY - vZ) = M\Sigma(\alpha'-\alpha)\alpha,$$

so that

$$K\Sigma XX' = M\Sigma\alpha\alpha' \quad \text{whenever} \quad K\Sigma X^2 = M\Sigma\alpha^2,$$

as for plane waves.

§ 11. We proceed to the equations for the movement of energy, first within the medium, and then across a surface of separation. We have

$$K\Sigma X\frac{d}{dt}\Big(X'-X\Big) = V^{-1}\Sigma X\Big(v\frac{d\gamma}{dt} - w\frac{d\beta}{dt}\Big)$$
$$= V^{-1}\Sigma\frac{d\alpha}{dt}\Big(wY - vZ\Big) = M\Sigma\frac{d\alpha}{dt}\Big(\alpha'-\alpha\Big),$$

and again

$$M\Sigma\alpha\frac{d}{dt}\Big(\alpha'-\alpha\Big) = K\Sigma\frac{dX}{dt}\Big(X'-X\Big).$$

Hence, by addition,

$$K\Sigma X\frac{dX'}{dt} + M\Sigma\alpha\frac{d\alpha'}{dt} = K\Sigma X'\frac{dX}{dt} + M\Sigma\alpha'\frac{d\alpha}{dt},$$

and therefore

$$\frac{d}{dt}\Big(\frac{K}{2}\Sigma XX' + \frac{M}{2}\Sigma\alpha\alpha'\Big) = \Sigma K\frac{dX}{dt}X' + \Sigma M\frac{d\alpha}{dt}\alpha'$$
$$= V\Sigma\Big(\frac{d\gamma'}{dy} - \frac{d\beta'}{dz}\Big)X' + V\Sigma\Big(\frac{dY'}{dz} - \frac{dZ'}{dy}\Big)\alpha'$$
$$= -V\Sigma\frac{d}{dz}\Big(X'\beta' - Y'\alpha\Big)$$

or

$$\frac{dS}{dt} + V\Sigma\frac{d}{dz}\Big(X'\beta' - Y'\alpha'\Big) = 0; \quad . \quad . \quad . \quad (33)$$

i. e., change in S is due to the flux of which $V(X'\beta' - Y'\alpha')$ is the *z*-component. For a plane wave this is ζS by (31), ζ being *z*-component of ray-velocity, and then (33) is expressed by

$$\frac{dS}{dt} + \xi\frac{dS}{dx} + \eta\frac{dS}{dy} + \zeta\frac{dS}{dz} = 0.$$

The flux across the *z*-face contains precisely those quantities

X′Y′α′β′ which the fundamental equations require to be continuous at a surface of separation. For the transfer of energy in reflexion and refraction something more than the continuity of $V(X'\beta' - Y'\alpha')$ is wanted, viz. the independence of the flux formulæ for the original and reflected waves expressed by

$$V(X'\beta' - Y'\alpha)_1 + V(X'\beta' - Y'\alpha')_2 = V(X'\beta' - Y'\alpha')_3 \ . \ (34)$$

the suffixes attaching to the original reflected and refracted waves. If we set out from $X_3' = X_1' + X_2'$, and similar equations for $Y'\alpha'\beta'$, the proof requires

$$X_1'\beta_2' - Y_1'\alpha_2' + X_2'\beta_1' - Y_2'\alpha_1' = 0,$$

or the vanishing of cross-products. Take the first medium for convenience to be æther and apply (27) and (30) exactly as in establishing (31). The quantity which should vanish is

$$\Sigma\alpha_1\alpha_2\{(Vn_1 - w)(1 - U_2/V) + (Vn_2 - w)(1 - U_1/V)\}$$
$$- \{(V - U_1)\gamma_2\Sigma l_2\alpha_1 + (V - U_2)\gamma_1\Sigma l_1\alpha_2\}$$
$$+ (1 - \Sigma l_1 l_2)(\gamma_2\Sigma u\alpha_1 + \gamma_1\Sigma u\alpha_2)$$
$$+ \frac{1}{V}\left\{(Vn_2 - w)\Sigma u\alpha_1\Sigma l_1\alpha_2 + (Vn_1 - w)\Sigma u\alpha_2\Sigma l_2\alpha_1\right\} ;$$

the work so far involving no special relations. The first line vanishes by (21), which also makes

$$(V - U_1)\Sigma l_2\alpha_1 = (V - U_2)(l_1\alpha_1 + m_1\beta_1) + (V - U_1)n_2\gamma_1$$
$$= \gamma_1\{(V - U_1)n_2 - (V - U_2)n_1\},$$

and the second line vanishes when the other part is similarly treated. Again using (21),

$$(Vn_2 - w)\Sigma l_1\alpha_2 = \gamma_2\{n_1(Vn_2 - w) + n_2(Vn_1 - w)\}.$$

and

$$1 - \Sigma l_1 l_2 = -2n_1 n_2 + \frac{w}{V}(n_1 + n_2)$$
$$= -\frac{1}{V}\{n_1(Vn_2 - w) + n_2(Vn_1 - w)\},$$

so that the residue vanishes.

The cross-products do therefore vanish in virtue of $\Sigma lX = 0$, $\Sigma l\alpha = 0$, and the geometrical laws of reflexion.

Equation (34) written with the formula of interpretation (32) for waves incident in æther on a moving dielectric is

$$(Vn_1 - w)S_1 + (Vn_2 - w)S_2 = \left(\frac{Vn_3}{\mu} - \frac{w}{\mu^2}\right)S_3, \ . \ (35)$$

an equation for the continuity in the rates of arrival and

departure of S at the surface, in interpreting which we must remember that n_2 is measured in the same sense as n_1, so that for the statical case $n_2 = -n_1$. The moving standpoint, that of the dielectric, is essential for interpretation on the right hand, *i. e.* S_3 is to be taken as a substantive quantity. The waves in free æther should be interpreted with reference to the fixed standpoint for which E_1 and E_2 are the substantive expressions. Hence, the proper form is

$$(Vn_1 - w)(1 - U_1/V)E_1 + (Vn_2 - w)(1 - U_2/V)E_2 = \left(\frac{Vn_3}{\mu} - \frac{w}{\mu^2}\right)S_3 \quad (36)$$

and E_1 experiences a transformation at the surface. The interpretation in detail is got by applying

$$Vn_1 E_1 = (Vn_1 - w)S_1 + n_1 U_1 E_1 + wS_1$$

to the two waves in æther, and is:—The rate of progression of energy in the original wave across a fixed plane provides the rate of transit of energy $\left(\dfrac{Vn_2}{\mu} - \dfrac{w}{\mu^2}\right) S_3$ in the dielectric, the rate of transit of energy $-Vn_2 E_2$ for the reflected wave, the work shown by $n_1 U_1 E_1 + n_2 U_2 E_2$, and the energy-content $S_1 + S_2$ for the space opened out by the motion. To express the same facts without use of wave-parameters (n, U), (II.) must be applied to the forms for free æther, giving

$$V(X\beta - Y\alpha) = V(X'\beta' - Y'\alpha') + wS + (wE - Z\Sigma uX - \gamma\Sigma u\alpha) \Big\} \quad (37)$$
$$= V(X'\beta' - Y'\alpha') + wS + (uX_z + vY_z + wZ_z),$$

where

$$X_z = -(XZ + \alpha\gamma), \quad Z_z = \tfrac{1}{2}(X^2 + Y^2 - Z^2 + \alpha^2 + \beta^2 - \gamma^2),$$

are the components of a stress.

In (36) the fraction of incident energy which experiences the transformation for reflexion is

$$A^2 = -\frac{(Vn_2 - w)(1 - U_2/V)E_2}{(Vn_1 - w)(1 - U_1/V)E_1} = \frac{\lambda_2^2 E_2}{\lambda_1^2 E_1} = \frac{p_2^{-2} E_2}{p_1^{-2} E_1}.$$

This quantity was required in the previous section, and will be evaluated in solving the problem of refraction. [Or the ratio of the terms in $E_1 E_2$ of (36) may be compared directly with the last of (19).]

§ 12. We may now consider more fully the expression S,

$$S = \frac{K}{2}\Sigma XX' + \frac{M}{2}\Sigma\alpha\alpha' \quad . \quad . \quad . \quad . \quad . \quad (a) \left.\begin{array}{c} \\ \\ \end{array}\right|$$
$$= \frac{K}{2}\Sigma X^2 + \frac{M}{2}\Sigma a^2 - \frac{\epsilon KM}{V}\Sigma u(Y\gamma - Z\beta) \quad (b) \left.\begin{array}{c} \\ \\ \end{array}\right| \quad (38)$$

The use of $X'\ldots \alpha'\ldots$ as multipliers in forming the energy-equation suggests that in a dynamical scheme they have the character of velocities, confirmed to some extent by their appearance in external conditions; while $KX\ldots M\alpha\ldots$ have the character of momenta which accords with the use of $K\dfrac{dX}{dt}\ldots$ in the fundamental equations. Thus in (38), (a) is a mixed, and (b) a momental expression for S; and if we start from (b), X' may be defined as

$$X' \equiv \frac{dS}{d(KX)} = X + \epsilon M(v\gamma - w\beta)/V,$$

and α' as

$$\equiv \frac{dS}{d(M\alpha)} = \alpha + \epsilon K(wY - vZ)/V,$$

which are the relations II. To express S in terms of $X'\ldots \alpha'\ldots$, the equations II. must be inverted, giving

$$
\left.
\begin{aligned}
&X(1-\epsilon^2\mu^2\Sigma u^2/V^2) = X' - \epsilon^2\mu^2 V^{-2}u\Sigma uX' - \epsilon MV^{-1}(v\gamma' - w\beta') \\
&\alpha(1-\epsilon^2\mu^2\Sigma u^2/V^2) = \alpha' - \epsilon^2\mu^2 V^{-2}u\Sigma u\alpha' - \epsilon KV^{-1}(wY' - vZ') \\
&\text{and} \\
&S'(1-\epsilon^2\mu^2\Sigma u^2/V^2) = \frac{K}{2}\Sigma X'^2 + \frac{M}{2}\Sigma\alpha'^2 + \epsilon\mu^2 V^{-1}\Sigma u(Y'\gamma' - Z'\beta') \\
&\qquad\qquad\qquad\qquad\quad - \frac{K\epsilon^2\mu^2}{2V^2}(\Sigma uX')^2 - \frac{M\epsilon^2\mu^2}{2V^2}(\Sigma u\alpha')^2
\end{aligned}
\right\} (39)
$$

using S' for the expression in terms of letters with dash. The result of differentiating with regard to $X'\ldots \alpha'\ldots$ is

$$\frac{dS'}{dX'} = KX, \quad \frac{dS'}{d\alpha'} = M\alpha ;$$

as we should expect. It will also be found that

$$\frac{dS'}{dw} = \frac{\epsilon KM}{V}(X\beta - Y\alpha) = -\frac{dS}{dw}. \quad . \quad . \quad (40)$$

The vector of which the z-component is $\dfrac{KM}{V}(X\beta - Y\alpha)$ will be denoted by (PQR), and we have for $\epsilon = 1$,

$$S = E - \Sigma Pu, \text{ or } E = S + \Sigma Pu \equiv S + T ; \quad . \quad . \quad (41)$$

but for the aberrational case, $\epsilon = 1/\mu^2$,

$$S = E - \Sigma\frac{Pu}{\mu^2}, \text{ or } E = S + \Sigma\frac{Pu}{\mu^2} = S + \Sigma\frac{w(\beta X - \alpha Y)}{V} . \quad (42)$$

Attending first to the normal case (41), E appears as the

sum of two terms, the first of which is radiant energy as
viewed from a moving standpoint. The second is kinetic in
character, *i. e.* it is a composite term of kinetic energy, in
which the components of translation are associated with those
of a momentum belonging to radiation. If we have an aber-
rational argument $\Sigma lx - \left(\dfrac{V}{\mu} - \dfrac{U}{\mu^2}\right)t$, and look at the wave
from a standpoint in motion $(u'v'w')$ relative to the *dielectric,*
i. e. put $x = \xi + u't, \ldots$ the argument is

$$\Sigma l\xi - \left(\frac{V}{\mu} - \frac{U}{\mu^2} - U'\right)t.$$

The corresponding motional term in the energy is

$$\Sigma \frac{KM}{V}\left(X\beta - Y\alpha\right)\left(w' + \frac{w}{\mu^2}\right).$$

We take the expression for momentum to retain its normal
value, while a reduced effective value of the velocity appears
in the aberrational case. Thus (40) gives the momentum of
radiation in the dielectric when the differentiation is with
regard to a velocity relative to the dielectric (w').

The case as regards aberration is briefly as follows :—For
free æther the electric and magnetic sections of energy have
expressions $\frac{1}{2}\Sigma X^2$ and $\frac{1}{2}\Sigma\alpha^2$; when the radiation is viewed
from a moving standpoint only a part of this has the character
of radiant energy, and there is a kinetic term $\Sigma \dfrac{w}{V}(X\beta - Y\alpha)$.

In a dielectric $\frac{1}{2}\Sigma X^2$ and $\frac{1}{2}\Sigma\alpha^2$ are weighted with the coeffi-
cients K and M respectively. Is the kinetic term to be
weighted or not ? If it is not weighted we have the
aberrational effect, *i. e.* the latter amounts to the retention of
the same expression for the part of the formula which denotes
kinetic energy. If it is weighted, with a coefficient KM,
the meaning corresponds to an alteration of standpoint in
viewing a wave which is proceeding with a velocity V/μ.
It is here supposed that this case is actual, that a wave
originating in a dielectric does travel with constant velocity,
and that the translation in the formula is a motion relative
to the dielectric.

There is I think a *primâ facie* case for a real difference
between the cases which are here separated; *i. e.* it is reason-
able to expect a difficulty in boarding, or gaining foothold in,
the moving medium ; and the fact of aberration shows that a
wave in such circumstances does not obtain a grip of the full
propagating power of the medium.

It is a consequence of this separation of cases that no

aberrational effect is to be got from a terrestrial source of
light without an entry into and passage through a medium
in motion relative to the earth.

§ 13. We proceed to the solution of the problem of
reflexion and refraction, which will be written for an
originating wave in free æther; and for the dielectric M is
taken $=1$, $K=\mu^2$, so that the connecting equations (25)
and (27) become

$$\alpha = \mu(mZ-nY), \quad \mu X = n\beta - m\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (25\,b)$$

$$\alpha' = \alpha\left(1-\frac{U}{\mu V}\right) + \frac{l}{\mu V}\Sigma u\alpha, \quad X' = X\left(1-\frac{U}{\mu V}\right) + \frac{l}{\mu V}\Sigma uX \quad . \quad (27)$$

The solution is simplest when $u=v=0$, so that $U=nw$, and
this case is taken first. For axis of x the line of intersection
of wave-front and reflecting surface is taken, and the cases
for which the magnetic and electric inductions are along
this axis are taken separately.

(i.) Magnetic induction along x, i.e. $\beta=\gamma=0$ and $X=0$.
The scheme is, with r for w/V,

$$\alpha = H, \quad \mu Y = -nH, \quad \mu Z = mH,$$
$$\left.\alpha' = H\left(1-\frac{rn}{\mu}\right), \; \beta'=0, \; X'=0, \; Y'=-\frac{H}{\mu}\left(n-\frac{r}{\mu}\right)\right\} . (43)$$

Writing these for the three waves with H, H' and H'', and
equating $\alpha'Y'$ at the surface,

$$\left.\begin{aligned}
&H(1-rn) + H'(1+rn') = H''\left(1-\frac{rn''}{\mu}\right), \\
&H(Vn-w) - H'(Vn'+w) = H''\left(\frac{Vn''}{\mu} - \frac{w}{\mu^2}\right);
\end{aligned}\right\} (44)$$

and

$$(Vn-w)(1-rn)H^2 - (Vn'+w)(1+rn')H'^2 = \left(\frac{Vn''}{\mu} - \frac{w}{\mu^2}\right)\left(1-\frac{rn''}{\mu}\right)H''^2$$

is the equation for transit of energy, the product HH'
vanishing by (21). These equations may be altered in form
by introducing the ratios

$$\lambda : \lambda' : \lambda'' = V - wn : V + wn' : \frac{V}{\mu} - \frac{wn''}{\mu^2},$$

and by using the kinematical cosines

$$\frac{n-r}{1-rn} = \nu = \frac{n'+r}{1+rn'}, \quad \nu'' = \frac{n''-r/\mu}{1-rn''/\mu}.$$

The exact relation between ν and ν'' is

$$(1-\nu^2)/(1-r^2)=\mu^2(1-\nu''^2)/(1-r^2/\mu^2),$$

and ν has the range 0 to 1.   The equations are then

$$H\lambda + H'\lambda' = \mu H''\lambda'', \quad (H\lambda - H'\lambda')\nu = H''\lambda''\nu'',$$

the solution is

$$H\lambda : H'\lambda' : H''\lambda'' = \mu\nu + \nu'' : \mu\nu - \nu'' : 2\nu \qquad \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \ . \ (45)$$

and

$$A^2 = (H'\lambda')^2 : (H\lambda)^2 = (\mu\nu - \nu'')^2 : (\mu\nu + \nu'')^2$$

The solution in terms of $n\,n'\,n''$ is of course readily found : the object of using (45) is to present A^2 (the fraction determining the partition of energy, *cf.* § 6 and end of § 11) in simple shape.

(ii.) Electric induction along x, *i. e.* $Y = Z = 0$, $\alpha = 0$. The scheme is

$$\beta = nH, \quad \gamma = -mH, \quad X = H/\mu,$$

$$\alpha' = 0, \quad \beta' = H\left(n - \frac{r}{\mu}\right), \quad X' = \frac{H}{\mu}\left(1 - \frac{rn}{\mu}\right), \quad Y' = 0 \right\} (46)$$

For continuity we require,

$$H(1-rn) + H'(1+rn') = \frac{H''}{\mu}\left(1 - \frac{rn''}{\mu}\right),$$

$$H(Vn - w) - H'(Vn' + w) = \mu H''\left(\frac{Vn''}{\mu} - \frac{w}{\mu^2}\right)$$

or with λ,

$$H\lambda + H'\lambda' = H''\lambda'', \quad (H\lambda - H'\lambda')\nu = H''\lambda'' . \mu\nu''$$

$$\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} \ . \ (47)$$

and $\therefore$

$$H\lambda : H'\lambda' : H''\lambda'' = \nu + \mu\nu'' : \nu - \mu\nu'' : 2\nu$$

and

$$A^2 = (\nu - \mu\nu'')^2 : (\nu + \mu\nu'')^2.$$

The quantity A^2 is invariant for the successive actions which take place when a wave enters a dielectric plate with parallel faces and is repeatedly reflected; for perpendicular incidence it is $(\mu - 1)^2/(\mu + 1)^2$.

(iii.) For the general case when u and v exist, the polarizations may be taken in conjunction.   Case (i.) is altered to

$$\alpha' = H_1\left(1 - \frac{U}{\mu V}\right), \quad \beta' = \frac{mu H_1}{\mu V}$$

$$X' = 0, \qquad\qquad VY' = -H_1\left(\frac{Vn}{\mu} - \frac{w}{\mu^2}\right),$$

and case (ii.) to

$$\alpha'=0, \quad \beta'=H_2\left(n-\frac{w}{\mu V}\right),$$

$$VX'=H_2\left(\frac{Vn}{\mu}-\frac{w}{\mu^2}\right), \quad VY'=\frac{muH_2}{\mu^2}.$$

The special feature is the appearance of β' in the first and Y' in the second, in consequence of which a wave with the polarization of (i.) gives rise to a component with the polarization of (ii.), of which the amplitude varies as u. The continuity of $\alpha'X'Y'\beta'$ (in order) in the combined scheme is expressed by

$$\left. \begin{aligned} &H_1(1-U/V)+H_1'(1-U'/V)=H_1''(1-U''/\mu V), \\ &H_2(1-U/V)+H_2'(1-U'/V)=\frac{H_2''}{\mu}(1-U''/\mu V), \\ &H_1\left(n-\frac{w}{V}\right)-H_1'\left(n'+\frac{w}{V}\right)-\frac{muH_2}{V}-\frac{m'uH_2'}{V} \\ &\qquad\qquad =\frac{H_1''}{\mu}\left(n''-\frac{w}{\mu V}\right)-\frac{m''uH_2''}{\mu^2 V}, \\ &H_2\left(n-\frac{w}{V}\right)-H_2'\left(n'+\frac{w}{V}\right)+\frac{muH_1}{V}+\frac{m'uH_1'}{\mu} \\ &\qquad\qquad =H_2''\left(n''-\frac{w}{\mu V}\right)+\frac{m''uH_1''}{\mu V} \end{aligned} \right\} \quad (48)$$

For transit of energy form the product $V(\beta'X'-\alpha'Y')$, and the result is

$$(Vn-w)(1-U/V)(H_1^2+H_2^2)-(Vn'+w)(1-U'/V)(H_1'^2+H_2'^2)$$
$$=\left(\frac{Vn''}{\mu}-\frac{w}{\mu^2}\right)\left(1-\frac{U''}{\mu V}\right)\left(H_1''^2+H_2''^2\right),$$

all products disappearing in virtue of the geometrical equations of reflexion.

With the object of getting at the ratios $H\lambda : H'\lambda' : H''\lambda''$ and at the value of A^2, λ is introduced as before with

$$\lambda : \lambda' : \lambda''=m : m' : m''=V-U : V-U' : \frac{V}{\mu}-\frac{U''}{\mu^2};$$

also we write

$$N=(Vn-w)/(V-U), \quad N''=\left(\frac{Vn'}{\mu}-\frac{w}{\mu^2}\right)\Big/\left(\frac{V}{\mu}-\frac{U''}{\mu^2}\right),$$

$M=m/(1-U/V)$, with $N'=N$ and $M'=M$, and k for $1-\frac{1}{\mu^2}$.

The scheme is then

$$H_1\lambda + H_1'\lambda' = \mu H_1''\lambda'', \qquad H_2\lambda + H_2'\lambda' = H_2''\lambda'',$$
$$(H_1\lambda - H_1'\lambda')N = H_1''\lambda''N'' + kuMH_2''\lambda'',$$
$$(H_2\lambda - H_2'\lambda')N = H_2''\lambda''\mu N'' - \mu kuMH_1''\lambda'',$$

$$\left.\begin{array}{c}\end{array}\right\} \quad (49)$$

and the solution is

$$H_1'\lambda'\{\overline{N + \mu N''}\ \overline{\mu N + N''} + \mu k^2 u^2 M^2/V^2\}$$
$$= H_1\lambda\{\overline{\mu N - N''}\ \overline{N + \mu N''} - \mu k^2 u^2 M^2/V^2\} - \frac{2\mu ku MN}{V}\ H_2\lambda,$$

$$H_2'\lambda'\{\overline{N + \mu N''}\ \overline{\mu N + N''} + \mu k^2 u^2 M^2/V^2\}$$
$$= H_2\lambda\{\overline{N - \mu N''}\ \overline{\mu N + N''} - \mu k^2 u^2 M^2/V^2\} + \frac{2\mu ku MN}{V}\ H_1\lambda.$$

$$\left.\begin{array}{c}\end{array}\right\} \quad (50)$$

If $u = 0$, we note that the cases are separated, and the solutions differ from (i.), (ii.) in having N, N'' for ν, ν''. The general value of A^2 is

$$(H_1'^2 + H_2'^2)\lambda'^2 : (H_1^2 + H_2^2)\lambda^2.$$

Its dependence on the original polarization is shown by writing

$$H_1 = H\cos\psi, \qquad H_2 = H\sin\psi,$$

when A^2 has the form

$$\alpha\cos^2\psi + \beta\sin^2\psi + 2\gamma\sin\psi\cos\psi.$$

Here α and β contain u^2 explicitly, while γ shows the first power of u. When the mean of A^2 for all orientations with a given wave-front is taken, the term in γ vanishes and the mean is $\frac{1}{2}(\alpha + \beta)$, which contains u^2 explicitly. Now N N'' M contain v implicitly, but not u, since $l = 0$ in $U = lu + mv + nw$. Thus the component of translation which only appears through its square is that along the line of intersection of the wave-front and the reflecting surface. This is the condition stated at the end of § 6, that the tangential forces shall do no work in the aggregate, and it also secures a simplification of the general equations for the collective relations.

§ 14. In finding the pressure and the relations between $p\ p'$ and ϖ we shall be content with first-order work; and as the influence of u and v only appears in second-order terms, we shall suppose that w only exists. The use of the kinematical cosine ν is more convenient than n because A^2 is expressed in terms of $\nu\nu''$, and ν has the range 0 to 1, while to the first order $1 - \nu^2 = \mu^2(1 - \nu''^2)$. The relations are

$$p^4\int A^2(n - r)(1 - rn)dn = p'^4\int A^2(n' + r)(1 + rn')dn',$$
$$p^4\int B^2(n - r)(1 - rn)dn = \varpi^4\int B^2\nu d\nu.$$

Since $n - r = \nu(1 - rn)$, and $d\nu/(1 - r^2) = dn/(1 - rn)^2$,
$$(n - r)(1 - rn)dn = \nu d\nu(1 - rn)^4/(1 - r^2) = \nu d\nu(1 - 4r\nu)$$
to the order required. Thus

$$\left.\begin{aligned}
p^4 \int_0^1 \mathrm{A}^2 \nu d\nu(1 - 4r\nu) &= p'^4 \int_0^1 \mathrm{A}^2(1 + 4r\nu)\nu d\nu, \\
\text{and} \quad p^4 \int_0^1 \mathrm{B}^2 \nu d\nu(1 - 4r\nu) &= \varpi^4 \int_0^1 \mathrm{B}^2 \nu d\nu.
\end{aligned}\right\} \quad (51)$$

Since $\mathrm{A}^2 + \mathrm{B}^2 = 1$, only two independent integrals are wanted for each type of polarization, viz. $\int \mathrm{B}^2 \nu d\nu$ and $\int \mathrm{B}^2 \nu^2 d\nu$, say B_1 and B_2. Then
$$\mathrm{A}_1 + \mathrm{B}_1 = \tfrac{1}{2} \quad \text{and} \quad \mathrm{A}_2 + \mathrm{B}_2 = \tfrac{1}{3}.$$
In terms of these
$$p' = p(1 - 2r\mathrm{A}_2/\mathrm{A}_1) \quad \text{and} \quad \varpi = p(1 - r\mathrm{B}_2/\mathrm{B}_1) \;;$$
and the pressure is
$$\chi \int \nu^2 d\nu(1 + \mathrm{A}^2) = \chi(\tfrac{1}{3} + \mathrm{A}_2) = \chi(\tfrac{2}{3} - \mathrm{B}_2).$$
Also each of these integrals is a mean of that for the polarizations (i.) and (ii.) above, say $2\mathrm{B}_1 = \mathrm{B}_1' + \mathrm{B}_1''$. We may now replace $\nu\,\nu''$ in these formulæ by the statical $n\,n''$.

(i.) With $\mathrm{B}^2 = 4\mu n n''/(n + \mu n'')^2$, $\quad \mu n'' = \sqrt{n^2 + \mu^2 - 1}$,

and
$$\frac{1}{n + \mu n''} = \frac{\sqrt{n^2 + \mu^2 - 1} - n}{\mu^2 - 1},$$

$$\begin{aligned}
\frac{(\mu^2 - 1)^2}{4}\mathrm{B}_1' &= \int_0^1 n^2 dn \sqrt{n^2 + \mu^2 - 1}(\sqrt{n^2 + \mu^2 - 1} - n)^2 \\
&= \int_0^1 dn \{n^2(2n^2 + \mu^2 - 1)\sqrt{n^2 + \mu^2 - 1} - 2n^3(n^2 + \mu^2 - 1) \\
&= \left[\frac{n}{3}\{(n^2 + \mu^2 - 1)^{\frac{5}{2}} - (\mu^2 - 1)(n^2 + \mu^2 - 1)^{\frac{3}{2}}\} - \frac{\mu^2 - 1}{2}n^4 - \frac{n^6}{3}\right]_0^1 \\
&= \frac{\mu^3 - 1}{3} - \frac{\mu^2 - 1}{2}\;;
\end{aligned}$$

$$\begin{aligned}
\frac{(\mu^2 - 1)^2}{4}\mathrm{B}_2' &= \int_0^1 dn\{n^3(2n^2 + \mu^2 - 1)\sqrt{n^2 + \mu^2 - 1} - 2n^4(n^2 + \mu^2 - 1)\} \\
&= \left[\frac{2}{7}(n^2 + \mu^2 - 1)^{\frac{7}{2}} - \frac{3(\mu^2 - 1)}{5}(n^2 + \mu^2 - 1)^{\frac{5}{2}}\right. \\
&\qquad \left. + \frac{(\mu^2 - 1)^2}{3}(n^2 + \mu^2 - 1)^{\frac{3}{2}} - \frac{2n^7}{7} - \frac{2n^5(\mu^2 - 1)}{5}\right]_0^1 \\
&= \frac{2\mu^7}{105} - \frac{\mu^5}{15} + \frac{\mu^3}{3} - \frac{2}{105}(\mu^2 - 1)^{\frac{7}{2}} - \frac{2}{7} - \frac{2}{5}(\mu^2 - 1).
\end{aligned}$$

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(ii.) For $B^2 = 4\mu n n''/(\mu n + n'')^2$, the work is more troublesome.

Put $2ny = \sqrt{\mu^2-1}\,(1-y^2)$, $\quad 2\mu n'' y = \sqrt{\mu^2-1}\,(1+y^2)$,

and write

$$c^2 = \frac{\mu^2+1}{\mu^2-1}, \quad y_0{}^2 = \frac{\mu-1}{\mu+1};$$

then

$$B_1'' = \int_0^1 \frac{4\mu n^2 n'' dn}{(\mu n + n'')^2} = \frac{\mu^2}{\mu^2-1} \int_{y_0}^1 \frac{(1-y^4)^2 dy}{y^3(c^2-y^2)^2}$$

$$= \frac{\mu^2}{\mu^2-1} \int_{y_0}^1 dy \left\{ y + \frac{2}{c^6 y} + \frac{1}{c^4 y^3} - \frac{2(c^8-1)y}{c^6(c^2-y^2)} + \frac{(c^4-1)^2 y}{c^4(c^3-y^2)^2} \right\}$$

$$= \left[\frac{y^2}{2} + \frac{2}{c^6}\log y - \frac{1}{2c^4 y^2} + \frac{c^8-1}{c^6}\log(c^2-y^2) + \frac{(c^4-1)^2}{2c^4(c^2-y^2)} \right]_{y_0}^1$$

$$= \frac{\mu^2}{(\mu+1)^2(\mu-1)} + \frac{\mu^2(\mu+1)}{(\mu^2+1)^2} + \frac{4\mu^5}{(\mu-1)(\mu+1)^2(\mu^2+1)^2}$$

$$+ \frac{\mu^2(\mu^2-1)^2}{(\mu^2+1)^3}\log\frac{\mu+1}{\mu-1} - \frac{8\mu^4(\mu^4+1)}{(\mu^2-1)^2(\mu^2+1)^3}\log\mu.$$

$$B_2'' = \frac{\mu^2}{2\sqrt{\mu^2-1}} \int_{y_0}^1 \frac{(1-y^2)^3(1+y^2)^2 dy}{y^4(c^2-y^2)^2} = \frac{\mu^2}{2\sqrt{\mu^2-1}} \int_{y_0}^1 f'(y)\,dy,$$

where

$$f'(y) = -y^2 - (2c^2-1) - \frac{c^2-2}{c^6 y^2} + \frac{1}{c^4 y^4}$$

$$+ \frac{3c^4-2c^2-2-c^{-4}+2c^{-6}}{c^2-y^2} - \frac{(c^4-1)^2(c^2-1)}{c^4(c^2-y^2)^2},$$

$$f(y) = -\frac{y^3}{3} - (2c^2-1)y + \frac{c^2-2}{c^6 y} - \frac{1}{3c^4 y^3}$$

$$+ \frac{(c^2+1)(c^2-1)^2(5c^4+2c^2+5)}{4c^7}\log\frac{c+y}{c-y} - \frac{(c^4-1)^2(c^2-1)}{2c^6}\frac{y}{c^2-y^2},$$

giving

$$B_2'' = \frac{\mu^2}{2(\mu+1)}\left[\frac{\mu-1}{3(\mu+1)} + \frac{\mu^2+3}{\mu^2-1} + \frac{(\mu+1)^3(\mu-1)(\mu^2-3)}{(\mu^2+1)^3} \right.$$

$$\left. + \frac{(\mu+1)^4}{3(\mu^2+1)^2} + \frac{8\mu^3}{(\mu^2+1)^3(\mu^2-1)} \right] - \frac{4\mu^2(\mu^8+12\mu^4+2)}{3(\mu^2+1)^3(\mu^2-1)^{\frac{3}{2}}}$$

$$+ \frac{4\mu^4(3\mu^2+2)}{(\mu^2+1)^{\frac{7}{2}}(\mu^2-1)^2}\log\left[\mu(\mu^2+\sqrt{\mu^4-1})/\{\mu^2-\mu+1+(\mu-1)\sqrt{\mu^2+1}\}\right].$$

When μ is little greater than 1, each polarization gives approximately

$$B_1 = \frac{1}{2} - \frac{\mu^2 - 1}{12}, \quad B_2 = \frac{1}{3} - \frac{8}{105}(\mu^2 - 1)^{\frac{3}{2}},$$

i. e.
$$A_1 = \frac{\mu^2 - 1}{12}, \qquad A_2 = \frac{8}{105}(\mu^2 - 1)^{\frac{3}{2}}.$$

Hence the values of the ratios

$$B_2 : B_1 = \frac{2}{3}, \quad \text{and } A_2 : A_1 = \frac{32}{35}\sqrt{\mu^2 - 1}.$$

When μ is indefinitely great,

$$B_1' = \frac{4}{3\mu}, \quad B_2' = \frac{1}{\mu}, \quad B_1'' = \frac{4}{\mu}, \quad B_2'' = \frac{2}{\mu};$$

and for the means

$$B_1 = \frac{8}{3\mu}, \quad B_2 = \frac{3}{2\mu}.$$

Since $A_1 + B_1 = \frac{1}{2}$, $A_2 + B_2 = \frac{1}{3}$, the ratios are

$$B_2 : B_1 = \frac{9}{16} \quad \text{and } A_2 : A_1 = \frac{2}{3}.$$

The formula for pressure $\chi(\frac{2}{3} - B_2)$ gives for μ little greater than 1 the value

$$\chi\left\{ \frac{1}{3} + \frac{8}{105}(\mu^2 - 1)^{\frac{3}{2}} \ldots \right\}$$

and for μ very great

$$\chi\left\{ \frac{2}{3} - \frac{3}{2\mu} \ldots \right\}.$$

The notion of interpreting the two sides of (14) in a similar way so as to count a term $-\dfrac{V\lambda''^{-4}}{\mu^3} n'' U'' dn'' d\phi$ with the terms of the other side in the mechanical rate of working was rejected, *cf.* § 5. The consequences of such an interpretation may be noted. The main term in pressure when statical values are introduced, as for (20), is then

$$\frac{\chi}{2\pi} \iint \{ n(1 + A^2) - n'' B^2/\mu \} n \, dn \, d\phi,$$

and the new term means a back pressure from the dielectric

on the æther. The expression in brackets is for one principal polarization always negative for some part of the range of n; for the other principal polarization this is true if $\mu^2 < 2$. The integral for either or for a mean of both is positive in the upper range of μ, and when μ is infinite has the limit which belongs to perfect reflexion; but it is negative for a finite though short range of μ starting from $\mu = 1$. It seems difficult to accept an interpretation which gives a back pressure from the dielectric on the æther, or involves the attraction of a dielectric by impinging waves, if μ has a value below a certain limit (different for the two polarizations if taken separately), and for higher values a repulsive action. [Of course entry into an infinite dielectric is in question, not the resultant of actions on entry and emergence.]

§ 15. In the original argument it appeared that the result of differentiating the equations giving the laws of reflexion and refraction, was a connexion between cubes of periods, which was then altered by the use of the factors

$$p(1 - U/V) = p'(1 - U'/V) = \varpi\,;$$

and the possibility of using

$$f\{p(1 - U/V)\} = f\{p'(1 - U'/V)\} = f(\varpi)$$

in place of the linear factor was just mentioned, *cf.* § 3. We now examine the consequences of supposing this done, the method remaining in other respects as before. The relations between p, p', and ϖ, when only w exists, then become

$$\left.\begin{array}{l} p^3 \displaystyle\int_0^1 A^2 \nu\, d\nu (1 - 3r\nu) f\{p(1 - r\nu)\} = p'^3 \displaystyle\int_0^1 A^2 \nu\, d\nu (1 + 3r\nu) f\{p(1 + r\nu)\}, \\[3mm] p^3 \displaystyle\int_0^1 B^2 \nu\, d\nu (1 - 3r\nu) f\{p(1 - r\nu)\} = \varpi^3 f(\varpi) \displaystyle\int_0^1 B^2 \nu\, d\nu. \end{array}\right\} \quad (52)$$

With first-order work $f\{p(1 - r\nu)\} = f(p) - pr\nu f'(p)$; i. e.

$$p^3 \int_0^1 A^2 \nu\, d\nu \{f(p) - r\nu(3f(p) + pf'(p))\} = p'^3 \int_0^1 A^2 \nu\, d\nu \{f(p') + r\nu(3f(p') + p'f'(p'))\}$$

or

$$A_1 p^3 f(p) - r A_2 (3p^3 f(p) + p^4 f'(p)) = A_1 p'^3 f(p') + r A_2 (3p'^3 f(p') + p'^4 f'(p')).$$

If now $p' = p(1 + \epsilon')$,

$$p'^3 f(p') = p^3 f(p) + \epsilon'(3p^3 f(p) + p^4 f'(p));$$

and therefore

$$(3p^3 f(p) + p^4 f'(p))(2r A_2 + \epsilon' A_1) = 0\,;$$

i. e. $\epsilon' = -2r A_2/A_1$, whatever the function is, because the apparent case of exception $3p^3 f(p) + p^4 f'(p) = 0$ is one which makes the transformation nugatory. In a similar way if

$$\varpi = p(1 + \epsilon''), \text{ we get } \epsilon'' = -r B_2/B_1.$$

Again, if this function is used the estimate of the rate of conversion to mechanical work at the surface, instead of depending on rn in the factor $1 - rn$, must depend on the first-order term in $f\{p(1 - rn)\}$. The main term in pressure would then be proportional to

$$p^4 f'(p) \int n^2 (1 + A^2) dn \; ;$$

or, if we were dealing with $p^2 f(p) dp$ instead of $p^3 f(p)$, it would be proportional to

$$p^3 f'(p) dp \int n^2 (1 + A^2) dn.$$

Thus on the geometrical side we should have to deal with the same fraction $\frac{1}{3} + A_2$ or $\frac{2}{3} - B_2$; but this would be a fraction of $p^3 f'(p) dp$, the energy varying as $p^2 f(p) dp$; or the whole ratio of pressure to the energy-content of the incident wave would be

$$\frac{p f'(p)}{f(p)} (\tfrac{2}{3} - B_2).$$

When the fourth power is used

$$f(p) = p, \quad \text{or} \quad p f'(p) : f(p) = 1 \; ;$$

but in any other case the energy converted would be a different fraction of the energy-content in different parts of the periodic scale. Thus, so far as first-order work is concerned, when w/V is small, the choice of a different function does not alter the collective relations; but it does alter the way in which pressure is related to energy-content, *i. e.* it implies a difference of efficiency as regards the production of mechanical work by pressure, depending on the period of vibration.

§ 16. There are some quantities continuous at a surface of separation which have not received attention.

(i.) KZ and $M\gamma$ are continuous; *i. e.*, for reflexion and refraction $Z_1 + Z_2 = KZ_3$. The continuity follows from the differential equations; but in view of any doubt possibly arising in connexion with the forms $\dfrac{d}{dt}$ and $\dfrac{D}{Dt}$, an independent proof for the reflexion of a wave is given. It may be remarked

in this connexion that (I) is always to be regarded as the fundamental form of the equations.

$$KZ_3 = \mu(m_3\alpha_3 - l_3\beta_3) = \mu(m_3\alpha_3' - l_3\beta_3')/(1 - U_3/\mu V),$$

$$\text{since} \quad \alpha_2' = \alpha_3\left(1 - \frac{U_3}{\mu V}\right) + \frac{l_3}{\mu V}\Sigma u\alpha_3,$$

$$= \mu\{m_3(\alpha_1' + \alpha_2') - l_3(\beta_1' + \beta_2')\}/(1 - U_3/\mu V)$$

$$= (m_1\alpha_1' - l_1\beta_1')/(1 - U_1/V) + (m_2\alpha_2' - l_2\beta_2')/(1 - U_2/V),$$

$$\text{since} \quad \frac{m_1}{1 - U_1/V} = \frac{m_2}{1 - U_2/V} = \frac{\mu m_3}{1 - U_3/\mu V}$$

$$= m_1\alpha_1 - l_1\beta_1 + m_2\alpha_2 - l_2\beta_2 = Z_1 + Z_2.$$

(ii.) KZ, Mγ being continuous as well as $\alpha'\beta'X'Y'$, we expect $KZX' + M\gamma\alpha'$, and $KZY' + M\gamma\beta'$ to be continuous. Further, if we write

$$KZ_3X_3' = (Z_1 + Z_2)(X_1' + X_2') = Z_1X_1' + Z_2X_2' + (Z_1X_2' + Z_2X_1')$$

a question of the independence of the original and reflected waves in the first medium arises, *i. e.* the question whether

$$Z_1X_2' + Z_2X_1' + \gamma_1\alpha_2' + \gamma_2\alpha_1' = 0.$$

This proves to be the case when the method of § 11 is applied, and all the quantities are expressed in terms of (α, β, γ).
Now

$$KZX' + M\gamma\alpha' = KZX + M\gamma\alpha + \frac{1}{V}\{Z(v\gamma - w\beta) + \gamma(wY - vZ)\}$$

$$= KZX + M\gamma\alpha + \frac{w}{V}(Y\gamma - Z\beta),$$

when the aberrational forms

$$X' = X + \frac{M}{\mu^2}(v\gamma - w\beta), \quad \alpha' = \alpha + \frac{K}{\mu^2}(wY - vZ)$$

are used. Similarly,

$$KZY' + M\gamma\beta' = KZY + M\gamma\beta + \frac{w}{V}(Z\alpha - X\gamma).$$

When this is combined with the statement as to continuity there results

$$\left\{(ZX + \gamma\alpha) + \frac{w}{V}(Y\gamma - Z\beta)\right\}_1 + \left\{\quad\right\}_2 = \left\{KZX + M\gamma\alpha + \frac{w}{V}(Y\gamma - Z\beta)\right\}_3.$$

The tangential stress-component is not balanced until the motional terms are taken into account. In the above they are $w\mathrm{P}_1$, $w\mathrm{P}_2$ on the left, and $\dfrac{w\mathrm{P}_3}{\mu^2}$ on the right hand. For the other component Q takes the place of P.

§ 17. The modification of the equations to include current $(i_x\, i_y\, i_z)$ treated as a line-discontinuity is simple. In the main we are concerned with the case of free æther, but for the present K and M are retained, and the work is written for the non-aberrational form, *i. e.*, translation is relative to the dielectric. We have then, for the moving standpoint,

$$\left.\begin{array}{ll} \mathrm{K}\dfrac{d\mathrm{X}}{dt} + i_x = \mathrm{V}\left(\dfrac{d\gamma'}{dy} - \dfrac{d\beta'}{dz}\right), & \mathrm{M}\dfrac{d\alpha}{dt} = \mathrm{V}\left(\dfrac{d\mathrm{Y}'}{dz} - \dfrac{d\mathrm{Z}'}{dy}\right) \\[2mm] \mathrm{X}' = \mathrm{X} + \mathrm{M}(v\gamma - w\beta)/\mathrm{V}, & \alpha' = \alpha + \mathrm{K}(w\mathrm{Y} - v\mathrm{Z})/\mathrm{V} \end{array}\right\}, \quad (53)$$

with $\qquad \rho = \mathrm{K}\Sigma\dfrac{d\mathrm{X}}{dx}$, and $\dfrac{d\rho}{dt} + \Sigma\dfrac{di_x}{dx} = 0$.

In terms of letters without dash, we have, if $u\,v\,w$ are constant,

and $\qquad \dfrac{\mathrm{D}'}{\mathrm{D}t} = \dfrac{d}{dt} - u\dfrac{d}{dx} - v\dfrac{d}{dy} - w\dfrac{d}{dz}$,

$$\mathrm{K}\dfrac{\mathrm{D}'\mathrm{X}}{\mathrm{D}t} + \rho u + i_x = \mathrm{V}\left(\dfrac{d\gamma}{dy} - \dfrac{d\beta}{dz}\right), \quad \mathrm{M}\dfrac{\mathrm{D}\alpha}{\mathrm{D}t} = \mathrm{V}\left(\dfrac{d\mathrm{Y}}{dz} - \dfrac{d\mathrm{Z}}{dy}\right). \quad (54)$$

From the standpoint of the dielectric itself, these equations are

$$\mathrm{K}\dfrac{d\mathrm{X}}{dt} + \rho u + i_x = \mathrm{V}\left(\dfrac{d\gamma}{dy} - \dfrac{d\beta}{dz}\right), \quad \mathrm{M}\dfrac{d\alpha}{dt} = \mathrm{V}\left(\dfrac{d\mathrm{Y}}{dz} - \dfrac{d\mathrm{Z}}{dy}\right). \quad (55)$$

The energy equation attaching to (53) contains terms in $i_x\ldots$ in addition to those found in § 11; and we may also write those which result from supposing $u\,v\,w$ to depend on t. Thus

$$\dfrac{d\mathrm{S}}{dt} + \Sigma\mathrm{P}\dfrac{du}{dt} + \Sigma i_x\mathrm{X}' + \mathrm{V}\Sigma\dfrac{d}{dx}(\mathrm{Y}'\gamma' - \mathrm{Z}'\beta') = 0, \quad (56)$$

with a term for Joule's effect, and one combining the momentum of radiation with acceleration in translation.

But if we set out from (55), *i. e.* take the standpoint of the dielectric, and use E, we get with multipliers X … α,

$$\dfrac{d\mathrm{E}}{dt} + \rho\Sigma u\mathrm{X} + \Sigma i_x\mathrm{X} = -\mathrm{V}\Sigma\dfrac{d}{dx}(\mathrm{Y}\gamma - \mathrm{Z}\beta).$$

But $\Sigma u X = \Sigma u X'$,   and

$$\Sigma i_x X = \Sigma i_x X' - \frac{M}{V} \Sigma i_x (v\gamma - w\beta) = \Sigma i_x X' + \frac{M}{V} \Sigma u (i_y \gamma - i_z \beta); \quad (57)$$

and with these the above becomes

$$\frac{dE}{dt} + \Sigma i_x X' + \rho \Sigma u X' + \frac{M}{V} \Sigma u (i_y \gamma - i_z \beta) = -V \Sigma \frac{d}{dx} (Y\gamma - Z\beta). \quad (58)$$

The flux of energy to the unit-volume fixed in the dielectric accounts for (1) increase of E, (2) Joule's effect, and (3) the work done by electric and electromagnetic forces in virtue of the translation $(u\ v\ w)$. At first sight the use of multipliers X, α instead of X', α' threatens difficulties as regards Joule's effect: these are resolved by the difference showing the work done by electromagnetic force, that work not appearing in (56), because the standpoint implied the translation.

If $\rho u',\dots$ are written for $i_x\dots$, $(u'v'w')$ being a velocity additional to the general translation, Joule's effect is replaced by work done in virtue of the motion $(u'v'w')$, viz. $\rho \Sigma u' X'$; and the conservation of charge is then expressed by

$$\frac{d\rho}{dt} + \Sigma \frac{d}{dx} (\rho u') = 0.$$

The mechanical side of electrical action may be separately shown by forming the time-rate of the momentum of radiation, that rate taken from the standpoint of the dielectric ; that is, we start from (55) and form

$$\frac{dR}{dt} \quad \text{or} \quad \frac{d}{dt} \frac{KM}{V} (X\beta - Y\alpha).$$

In the course of the work the terms

$$M\gamma \Sigma \frac{d\alpha}{dx} \quad \text{and} \quad Z \left\{ K\Sigma \frac{dX}{dx} - \rho \right\}$$

are introduced, and the result is

$$-\left(\frac{dZ_x}{dx} + \frac{dZ_y}{dy} + \frac{dZ_z}{dz} \right) = \rho Z' + \frac{M}{V} (i\ \beta - i_y \alpha) + \frac{dR}{dt}. \quad (59)$$

That is, the rate of increase of (P Q R) together with forces on charge and current (where they exist), is expressible as the body-force of a stress. Here the coefficients K and M occur in the stress, *i. e.*,

$$Z_x = -(KZX + M\gamma\alpha), \quad Z_z = \frac{K}{2} (X^2 + Y^2 - Z^2) + \frac{M}{2} (\alpha^2 + \beta^2 - \gamma^2).$$

An energy equation on the mechanical side is obtained by multiplying these by $(u\,v\,w)$, viz.,

$$-\Sigma \frac{d}{dx}(u\mathrm{X}_x + v\mathrm{Y}_x + w\mathrm{Z}_x) = \rho\Sigma u\mathrm{X}' + \frac{\mathrm{M}}{\mathrm{V}}\Sigma u(i_y\gamma - i_z\beta) + \Sigma u\frac{d\mathrm{P}}{dt}. \quad (60)$$

This equation is the complement of (56), and the division of $\frac{d}{dt}\Sigma\mathrm{P}u$ into two parts is noticeable. Equation (60) may also be got by subtracting (56) from (58), and quoting (37); but $\frac{\mathrm{DS}}{\mathrm{D}t}$ must be used in (56) to transfer to the standpoint of (58).

§ 18. We now pass to the special problem of discontinuities moving in free æther under circumstances which give a steady condition from the moving standpoint.

The solution of (54) has the form

$$\alpha = \frac{d\mathrm{H}}{dy} - \frac{d\mathrm{G}}{dz}, \quad \mathrm{X} = -\frac{d\psi}{dx} - \frac{1}{\mathrm{V}}\frac{\mathrm{D}'\mathrm{F}}{\mathrm{D}t}$$

$$\alpha' = \alpha + \frac{1}{\mathrm{V}}\left(v\frac{d\psi}{dz} - w\frac{d\psi}{dy}\right) + \frac{1}{\mathrm{V}^2}\left(v\frac{\mathrm{D}'\mathrm{H}}{\mathrm{D}t} - w\frac{\mathrm{D}'\mathrm{G}}{\mathrm{D}t}\right),$$

$$\mathrm{X}' = -\frac{1}{\mathrm{V}}\frac{d\mathrm{F}}{dt} - \frac{d}{dx}\left(\psi - \frac{1}{\mathrm{V}}\Sigma\mathrm{F}u\right),$$

where $\psi\,\mathrm{F}\,\mathrm{G}\,\mathrm{H}$ are defined by

$$\nabla^2\psi - \frac{1}{\mathrm{V}^2}\frac{\mathrm{D}'^2\psi}{\mathrm{D}t^2} + \rho = 0, \quad \nabla^2\mathrm{F} - \frac{1}{\mathrm{V}^2}\frac{\mathrm{D}'^2\mathrm{F}}{\mathrm{D}t^2} + \frac{\rho u + i_x}{\mathrm{V}} = 0,$$

subject to

$$\frac{1}{\mathrm{V}}\frac{\mathrm{D}'\psi}{\mathrm{D}t} + \Sigma\frac{d\mathrm{F}}{dx} = 0.$$

In these we write

$$\frac{d}{dt} = 0, \quad \text{or} \quad \frac{\mathrm{D}'}{\mathrm{D}t} = -\left(u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz}\right).$$

Thus $\qquad \alpha = \dfrac{d\mathrm{H}}{dy} - \dfrac{d\mathrm{G}}{dz}, \quad \mathrm{X}' = -\dfrac{d}{dx}(\psi - \Sigma p\mathrm{F}), \qquad . \quad (61)$

where $(p,\,q,\,r) = (u,\,v,\,w)/\mathrm{V}$, and the other variables are given by $\alpha' = \alpha + r\mathrm{Y} - q\mathrm{Z}$, and $\mathrm{X}' = \mathrm{X} + q\gamma - r\beta$.

For the integration of $\int \mathrm{S}d\tau$, *i.e.* $\int(\frac{1}{2}\Sigma\mathrm{X}\mathrm{X}' + \frac{1}{2}\Sigma\alpha\alpha')d\tau$, the forms of $\mathrm{X}'\ldots$ and $\alpha\ldots$ as gradient and curl are essential. For a volume-distribution,

$$\int \mathrm{S}d\tau = \int \frac{1}{2}\rho(\psi - \Sigma p\mathrm{F})d\tau_i + \int \frac{1}{2\mathrm{V}}\Sigma i_x\mathrm{F}d\tau_i, \quad . \quad (62)$$

and for a typical current $i_x d\tau_i = \sigma i\, dx$, σ being a small cross-section. Now F depends on $u\rho + i_x$, and we may separate the discontinuities by writing $F = p\psi + F_0$, and F_0 then satisfies

$$\nabla^2 F_0 - \left(p\frac{d}{dx} + q\frac{d}{dy} + r\frac{d}{dz}\right)^2 F_0 + i_x/V = 0. \quad . \quad (63)$$

The condition $\dfrac{1}{V}\dfrac{D'\psi}{Dt} + \Sigma\dfrac{dF}{dx} = 0$ becomes, when $\dfrac{d}{dt} = 0$,

$$p\frac{d\psi}{dx} + q\frac{d\psi}{dy} + r\frac{d\psi}{dz} = \Sigma\frac{dF}{dx},$$

which gives $\Sigma\dfrac{dF_0}{dx} = 0$. We have, then,

$$\int S d\tau = \int \frac{1}{2}\rho\left[\psi(1 - \Sigma p^2) - \Sigma p F_0\right]d\tau_i + \int\frac{1}{2V}\Sigma i_x(F_0 + p\psi)d\tau_i$$

$$= \int\frac{1}{2}\rho\psi(1 - \Sigma p^2)d\tau_i + \int\frac{1}{2V}\Sigma i_x F_0 d\tau_i + \int\frac{1}{2}\left[\psi\Sigma pi_x - \rho\Sigma pF_0\right]d\tau_i. \quad (64)$$

In the first two terms each discontinuity is associated with its own potential. In the statical case an element of potential has the discontinuity in the numerator, and in the denominator a distance connecting two points

$$r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}.$$

With such a formula, for a denominator belonging to elements at (xyz), $(x'y'z')$, the various sections contributed to the numerator by the last term of (64) are

$$\rho'\Sigma pi_x - \rho\Sigma pi_x' + \rho\Sigma pi_x' - \rho'\Sigma pi_x;$$

and these are cancelled whether ρ and i_x exist at one place, or ρ only at one place and i'_x at another.

The form of denominator for the motional potential (to be considered more fully later) is suggested by transforming the wave-surface $\Sigma x'^2 = V^2 t^2$ to a moving standpoint $\Sigma(x + ut)^2 = V^2 t^2$, and treating this as a quadratic in t, viz.,

$$(V^2 - \Sigma u^2)t - \Sigma ux \mp \sqrt{(V^2 - \Sigma u^2)\Sigma x^2 + (\Sigma ux)^2} = 0$$

or $\quad (1 - \Sigma p^2)t - \Sigma px \mp \sqrt{(1 - \Sigma p^2)\Sigma x^2 + (\Sigma px)^2} = 0.$

If we call this $\chi = 0$, and write $R^2 = (1 - \Sigma p^2)\Sigma x^2 + (\Sigma px)^2$, it is readily verified that $R^{-1}f(\chi)$ satisfies the differential equation $\dfrac{D'^2}{Dt^2}\psi = V^2\nabla^2\psi$; and this solution replaces the

typical form $r^{-1}f(r - \mathrm{V}t)$. The quantity R is the required denominator and for points $x\,y\,z$ and $x'y'z'$,

$$\mathrm{R}^2 = (1 - \Sigma p^2)\Sigma(x - x')^2 + (\Sigma p\,(x - x'))^2.$$

R is not altered by interchanging x and x' and also not altered by reversing the sign of $(p\,q\,r)$ or $(u\,v\,w)$. Hence with this denominator the last section of (64), that containing cross-products, disappears in integration. At the same time it would seem impossible to ignore the part of the scalar potential due to $i_x...$, in determining the potential of any conductor which may be in question, unless $\Sigma p\mathrm{F}_0 = 0$ or $\Sigma p i_x = 0$. In that connexion this special case is important.

When the whole solution is expressed in terms of ψ, F_0, G_0, H_0, it stands

$$\left.\begin{aligned}
\alpha &= \frac{d\mathrm{H}_0}{dz} - \frac{d\mathrm{G}_0}{dz} + r\frac{d\psi}{dy} - q\frac{d\psi}{dz}, \quad -\mathrm{X} = (1 - \Sigma p^2)\frac{d\psi}{dx} - \frac{d}{dx}\Sigma\,p\mathrm{F}_0 \\[2mm]
-\mathrm{X} &= (1 - p^2)\frac{d\psi}{dx} - pq\frac{d\psi}{dy} - pr\frac{d\psi}{dz} - \left(p\frac{d}{dx} + q\frac{d}{dy} + r\frac{d}{dz}\right)\mathrm{F}_0 \\[2mm]
\alpha' &= \left(-pq\frac{d}{dx} + \overline{1 - q^2}\frac{d}{dy} - qr\frac{d}{dz}\right)\mathrm{H}_0 - \left(-pr\frac{d}{dx} - qr\frac{d}{dy} + \overline{1 - r^2}\frac{d}{dz}\right)\mathrm{G}_0
\end{aligned}\right\}, \quad (65)$$

in which form the æolotropic character of each solution plainly appears.

§ 19. There is an important difference in the energies attached to the two solutions, the scalar solution with ρ, ψ and the vector solution with $i_x... \; \mathrm{F}_0...$ In the solution for charge ρ essentially determines X... (in a dielectric KX,...), though the motional form is given by the intervention of other variables. The solution is therefore momental in character, and for it we write

$$\mathrm{P} = -\frac{d\mathrm{S}}{du}, \quad \text{and} \quad \mathrm{E} = \mathrm{S} - \Sigma u\frac{d\mathrm{S}}{du} = \mathrm{S} + \mathrm{T}. \quad . \quad (66)$$

In the solution for current $i_x...$ essentially determine $\alpha'...$, that is quantities with the character of velocities, and for this solution we write

$$\mathrm{P}' = \frac{d\mathrm{S}'}{du}, \quad \text{and} \quad \mathrm{E}' = \mathrm{S}' + \Sigma u\frac{d\mathrm{S}'}{du} = \mathrm{S}' + \mathrm{T}'. \quad . \quad (67)$$

The various quantities refer to integrated results. The distinction is valid apart from any translation, *i. e.* there is always a distinction between KX and X or between Mα and α, although it may be lost sight of when K and M are each written $= 1$.

If we consider the charge alone, we have $\alpha' = \beta' = \gamma' = 0$;

but $\Sigma XX' - \Sigma \alpha \alpha' = \Sigma X^2 - \Sigma \alpha^2$ always, and therefore in this case $\frac{1}{2}\Sigma XX' = \frac{1}{2}\Sigma X^2 - \frac{1}{2}\Sigma \alpha^2$, or integrating we have $S = E_e - E_m$. But $E = E_e + E_m$, and therefore

$$E_e = S + \tfrac{1}{2}T, \quad \text{and} \quad E_m = \tfrac{1}{2}T, \quad . \quad . \quad (68)$$

i. e., the integral values of E_e and E_m are separately deducible from that of S. If we consider the current solution alone, $\Sigma XX'$ is not $= 0$ at each point of the field unless $\Sigma p i_x = 0$; but its integral vanishes because the divergence of $(X\ Y\ Z)$ for this solution is

$$\left(p\frac{d}{dx} + q\frac{d}{dy} + r\frac{d}{dz}\right)\Sigma\frac{d F_0}{dx}, \quad \text{and} \quad \Sigma\frac{d F_0}{dx} = 0.$$

With $\int \Sigma XX' d\tau = 0$ goes $\int \frac{1}{2}\Sigma\alpha\alpha' d\tau = \int(\frac{1}{2}\Sigma\alpha^2 - \frac{1}{2}\Sigma X^2)d\tau$, *i. e.*, $S' = E'_m - E'_e$,

$$E'_m = S' + \tfrac{1}{2}T', \quad \text{and} \quad E'_e = \tfrac{1}{2}T'. \quad . \quad . \quad (69)$$

Before proceeding to the general solution for any ellipsoid, the simple case which corresponds to a sphere in statical potentials may be briefly noticed. With $i_x = i_y = i_z = 0$

$$-X' = \frac{d\psi}{dx}(1 - \Sigma p^2), \quad -X = (1 - p^2)\frac{d\psi}{dx} - pq\frac{d\psi}{dy} - pr\frac{d\psi}{dz}$$

$$\alpha' = 0, \quad \alpha = r\frac{d\psi}{dy} - q\frac{d\psi}{dz}; \quad \rho = \Sigma\frac{dX}{dx}.$$

For $\psi = C/R$ the surface $R = R_0$ is an equipotential surface, and may be treated as a conductor. We find $X = Cx(1 - \Sigma p^2)/R^3$, and if e is the total charge,

$$e = \int \Sigma l X dS = \frac{C(1 - \Sigma p^2)}{R_0^{\ 3}}\int \Sigma l x\, dS = \frac{3C\tau_0(1 - \Sigma p^2)}{R_0^{\ 3}} = 4\pi C,$$

since the polar and equatorial semi-axes are R_0 and $R_0/\sqrt{1 - \Sigma p^2}$ respectively. Thus

$$\psi = \frac{e}{4\pi R}, \quad \text{and then} \quad S = \frac{e}{2}\psi_s(1 - \Sigma p^2) = \frac{e^2(1 - \Sigma p^2)}{8\pi R_0}. \quad (70)$$

The potential is properly $\phi = \psi(1 - \Sigma p^2) = e(1 - \Sigma p^2)/4\pi R$, which makes $X' = -\dfrac{d\phi}{dx}$. For a uniform volume distribution $\phi_i = \dfrac{\rho}{2}(R_0^2 - \tfrac{1}{3}R^2)$, $\phi_e = \rho R_0^3/3R$; and the integration $\int \frac{1}{2}\rho\phi_i d\tau_i$ for S yields $3e^2(1 - \Sigma p^2)/20\pi R_0$.

Thus the typical elementary solution which corresponds to that of the sphere in attractions here belongs to a particular oblate spheroid.

XXXIII. *The Distribution of the Actinic Sunlight on the Northern Hemisphere at Summer Solstice.* By JOHN SEBELIEN [*].

THE distribution of the effect of the solar radiation over the different regions of the globe and at different seasons has often been subject to calculation, and attention has often been directed to the favourable position of places at high latitude during summer-time.

The laws of the intensity and quantity of solar radiation are said to have been examined for the first time by Halley in 1779, later determined by Plane [†]. Chr. Wieler [‡] published in 1879 a series of curves, showing the relative intensity of the solar radiation during its yearly periods and at various seasons for every tenth degree of latitude. His results were not corrected for atmospheric absorption, which varies for rays of different refrangibility. The calculations of Wieler show that on the 21st of June the radiation that reaches unit area of the surface of the globe during 24 hours has its maximum value at the North Pole. The relative values of the said radiation at 0°, 60°, 75°, and 90° N. lat. are as 282 : 350 : 362 : 385. The North Pole keeps its superior position not only on that single day but also during the whole period from 5th of May till the 7th of August (the longitude of the sun increasing from 45° to 135°). If the total radiation during this period is calculated, its values at 0°, 40°, 60°, and 90° N. lat. are as 710 : 886 : 846 : 896. This shows a secondary maximum at about 40°, while the absolute maximum is at 90° N. lat.

The researches of Langley on the atmospheric absorption of rays of different refrangibility having cleared this question, Spitaler [§] calculated the quantity of sunlight reaching a unit of area for each 10th degree of latitude during an average day for each of the twelve months of the year. For this calculation he assumed a coefficient of absorption 0·6, corresponding to the rays of an average refrangibility in the yellow-green part of the spectrum. The results found by Wieler, neglecting the influence of the atmosphere, were now so far modified that the maximum value of the daily quantity of sunlight at midsummer does not fall on the North Pole. Spitaler finds that for a day of June the maximum radiation is at 30° N. lat. The quantity of light

diminishes towards the equator as well as towards 70° N. lat.; from here it increases again, and reaches at 90° a secondary maximum that, however, has a smaller value than that quantity of light which on the same day reaches the regions of 60° N. lat.

But even the results of Spitaler are incomplete, since his calculations only refer to the direct insolation, without regard to the diffused daylight. Moreover, those calculations are of a purely theoretical nature, without the support of any experiment. In the fifties Bunsen and Roscoe made their famous photochemical researches [*] of the light influencing the chlorine and hydrogen explosive mixture. These experimenters found that the said activity is especially localized between the lines G and M of Fraunhofer in the solar spectrum; from here it decreases more quickly and regularly towards the red end of the spectrum than towards the ultra-violet one. Their researches showed, further, that the sun-rays, before entering the atmosphere, have a photochemical effect of 35·5 *light-metres*, that is, by complete absorption in an infinite atmosphere of chlorine-hydrogen electrolytic gas at 760 mm. pressure and 0° C., they will be able to transform during a minute a column of the said mixture of 35·5 metres height into hydrogen chloride. Having passed through the atmosphere, the sun-rays on reaching the surface of the ocean will only possess a photochemical effect of 14·4 light-metres; thus, in passing the atmosphere they will have lost about two-thirds of their photochemical effect.

Bunsen and Roscoe generally express the effects of light that they measured and calculated in *chemical photo-units*, each one of which is determined by the chemical action upon a normal explosive mixture of hydrogen and chlorine contained in an insolation vessel of such small dimensions that the variability of the extinction appearing in large vessels may be neglected when the said explosive gas is illuminated at a distance of 1 metre from a so-called normal flame, that is, a flame of burning carbonic oxide at a certain pressure issuing from a platinum burner of accurately determined dimensions. 10,000 of these photo-units are called 1 *chemical light-unit*.

The sun-rays reaching a horizontal area of the surface of the earth at an angle ϕ with the vertical will produce in one minute on each square unit of this area a photochemical effect that may be expressed in chemical light-units by the formula

$$W = 318 \cdot 3 \,.\, \cos \phi \,.\, 10^{-\frac{0 \cdot 4758 \, P}{\cos \phi}}, \quad . \quad . \quad . \quad (a)$$

* Poggendorff's *Annalen,* Bd. xcvi. &c.; Ostwald's *Klassiker Ausgaben,* Bd. xxxiv. & xxxviii.

P denoting the atmospherical pressure, and the constant 318·3 corresponding to the photochemical intensity of the sunlight at the uppermost limit of of the atmosphere, while the constant 0·4758 denotes the atmospherical extinction of the direct solar light (Poggendorff's *Annalen*, Bd. cviii. p. 257 ; Ostwald's *Klassiker Ausgaben*, No. 38, pp. 90, 91). Developing in a series with increasing powers of $\cos \phi$, and calculating the coefficients by means of the method of least squares, Bunsen and Roscoe transformed the said formula to

$$W = 31\!\cdot\!99 \cos^2 \phi + 417\!\cdot\!6 \cos^3 \phi - 248\!\cdot\!7 \cos^4 \phi. \quad . \quad (b)$$

To get the chemically active * quantity of light falling upon a horizontal element of the surface of earth for a whole day, we may substitute

$$\cos \phi = \cos \delta \cos p \cos t + \sin \delta \sin p,$$

p denoting the latitude, δ the declination of the sun, and t the hour angle of the sun. Then we may calculate the integral

$$W = \int_{-t_1}^{t_1} w \, dt,$$

where t_1 and $-t_1$ denote the hour angle of the sun at sunset and at sunrise respectively on the day in question.

For the sake of shortness put

$$\sin \delta \sin p = \alpha \quad \text{and} \quad \cos \delta \cos p = \beta,$$

when the quantity under the integral may be written

$$\left.\begin{aligned}
w = {} & 31\!\cdot\!99(\alpha^2 + 2\alpha\beta \cos t + \beta^2 \cos^2 t) \\
& + 417\!\cdot\!6(\alpha^3 + 3\alpha^2\beta \cos t + 3\alpha\beta^2 \cos^2 t + \beta^3 \cos^3 t) \\
& - 248\!\cdot\!7(\alpha^4 + 4\alpha^3\beta \cos t + 6\alpha^2\beta^2 \cos^2 t + 4\alpha\beta^3 \cos^3 t \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + \beta^4 \cos^4 t)
\end{aligned}\right\} \quad (1)$$

At the equinoxes, when $\alpha = 0$, $\beta = \cos p$, and the integration is to be performed between the limits $t = -\dfrac{\pi}{2}$ and $t = +\dfrac{\pi}{2}$, the said integral will attain a very simple form. Bunsen and Roscoe calculated it to

$$W = -11520 \cos^2 p + 127600 \cos^3 p - 67140 \cos^4 p.$$

Bunsen and Roscoe further determined the photochemical effect of the light reflected from a cloudless sky. They

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* By chemical action we understand hereafter only the action on the explosive mixture of chlorine and hydrogen.

found that the effect upon a horizontal element of the surface of the earth in one minute may be expressed [in chemical light-units by the zenith-distance of the sun ϕ according to the formula

$$w_1 = 2{\cdot}776 + 80{\cdot}849\,\cos\,\phi - 45{\cdot}996\,\cos^2\,\phi.$$

For a larger interval of time, in which the hour angle of the sun is assumed to grow from t_1 to t_2 (expressed in minutes of arc), the total photochemical effect in light-units will be

$$W_1 = \frac{12 \times 60}{\pi} \int_{-t_1}^{t_1} w_1 dt.$$

This integration may be executed with the same substitutions as above, and will then give the formula

$$W_1 = \frac{24 \times 60}{\pi} \left[2{\cdot}776 + 80{\cdot}849\alpha - 45{\cdot}996\left(\alpha^2 + \frac{\beta^2}{2}\right)\right] t_1$$
$$+ \left(80{\cdot}879\beta - \frac{3}{2} \times 45{\cdot}996\alpha\beta\right)\sin t. \quad . \quad . \quad (2)$$

Even this expression will be simplified for the equinoxes, as Bunsen and Roscoe themselves pointed out. Putting $\delta = 0$, $\alpha = 0$, $\beta = \cos p$, $t = \dfrac{\pi}{2}$, the formula (2) is transformed to

$$W_1 = 1998{\cdot}8 + 37058\,\cos\,p - 16559\,\cos^2 p.$$

By means of these formulæ Bunsen and Roscoe have calculated for a series of localities the total quantity of actinic light expressed in light-units which in the course of the day of equinox falls upon a horizontal element of surface in the form of direct solar radiation as well as in the form of reflected daylight. These numbers are reproduced in the following Table I., where I have calculated the numbers marked with an asterisk, which are not given in the original table of Bunsen and Roscoe.

This shows the extent to which the diffused daylight tends to equalize the numbers for the total quantity of light at the different latitudes. While the daily quantity of light due to direct radiation is forty times as great at the equator as it is at the pole, the quantity of diffused daylight is hardly twice as great at the equator as at the pole on the same day.

Further, the numbers in Table I. show that while at the equator the effect of the direct insolation on the said day has double the value of the daily effect of the diffused daylight, these numbers will become equal in the neighbourhood of

49° N. lat., and the further we get towards the north, the more the diffused daylight will dominate.

TABLE I.

	p.	W. Direct radiation.	W. Diffused daylight.	$W + W_1$.
	N. lat.	Light-units.	Light-units.	Light-units.
Melville Island...	74° 47′	1196	10590	11790
Reykjavik	64° 8′	5964	15020	20980
St. Petersburg ...	59° 56′	8927	16410	25340
Manchester	53° 20′	14520	18220	32740
Heidelburg	49° 24′	18240	19100	37340
Naples	40° 52′	23640	20550	47190
Cairo	30° 2′	36440	21670	58110
Bombay............	19°	43820	22234 *	66054 *
Ceylon	10°	47530	22435 *	69965 *
Equator............	0°	48940	22498 *	71438 *

The preference conferred upon the northern latitudes with regard to their actinic illumination at the equinoxes will increase with the declination of the sun, and reaches its maximum value at the summer solstice.

By means of the above-mentioned formulæ of Bunsen and Roscoe I have calculated the quantity of actinic light which on the *midsummer-day* falls upon a horizontal element of surface from sunrise to sunset for every tenth (or fifth) degree of latitude.

On the said day we shall have in formula (1)

$$\alpha = \sin 23° \, 27' \sin p \, ; \qquad \beta = \cos 23° \, 27' \cos p \, ;$$
$$\cos t = \tan 23° \, 27' \tan p,$$

and putting the constants

$$a = 31\cdot99, \qquad b = 417\cdot6, \qquad c = 248\cdot7,$$

the integration will give

$$\int_{-t}^{+t} w \, dt = 2\mathrm{A}t + 2\mathrm{B} \sin t + \mathrm{C}(\sin t \cos t + t)$$
$$+ 2\mathrm{D}\, \frac{\cos^2 t \sin t + 2 \sin t}{3} + \Sigma \frac{2 \cos^3 t \sin t + 3 \cos t \sin t + 3 t}{4},$$

2 A 2

where

$$A = \alpha^2(a + b\alpha + c\alpha^2),$$
$$B = \alpha\beta(2a + 3b\alpha + 4c\alpha^2),$$
$$C = \beta^2(a + 3b\alpha + 6c\alpha^2),$$
$$D = \beta^3(b + 4c\alpha),$$
$$E = c\beta^4.$$

Introducing in this expression various values of p we get the values for the actinic direct sunlight, put in the column W of Table II.

The values for the diffused light, which we have put in the column W_1 of Table II., we have calculated in a similar way by means of the formula (2), where the quantities α, β, and t take the same values as in the calculation of the values of W.

The sum of W and W_1 for each latitude is found in the last column of the table, and shows the total amount of the actinic light.

TABLE II.

Chemical Light-units pr. Unit Horizontal Area on Midsummer-day.

N. lat.	Direct insolation, W. Light-units.	Diffused daylight, W_1. Light-units.	Total light, $W + W_1$. Light-units.
0°	60656	22060	82716
10°	70891	23388	94479
20°	77703	24539	102242
30°	89060	25775	114835
40°	79644	27059	106701
45°	76178	27757	103935
50°	72584	28521	101105
55°	62704	28589	91293
60°	62064	30484	92548
65°	57089	32168	89257
66° 33′ *	54573	33264	87837
70°	50267	35012	85279
75°	44587	37099	81686
80°	40080	38612	78700
90°	36211	39839	76048

* The northern polar circle.

Fig. 1 gives a graphical representation of these numbers. The lower dotted curve shows the variation of the diffused daylight. We see that at the lower and middle latitudes

the curve rises rather smoothly and slowly ; at about 55° N. lat. the curve begins to get steeper with the concavity upwards, and at the highest latitudes again it turns more flat with a downward concavity. The situation of turning-point for which $\dfrac{\partial^2 W}{\partial p^2} = 0$ seems from the figure to be near $p = 70°$ N. lat.

Fig. 1.—Distribution of the violet and ultraviolet light (actinic on an explosive mixture of hydrogen and chlorine) at summer solstice at different Latitudes.

.... diffused daylight.

‑ ‑ ‑ ‑ ‑ ‑ ‑ ‑ direct sunlight.

———————— quantity of light.
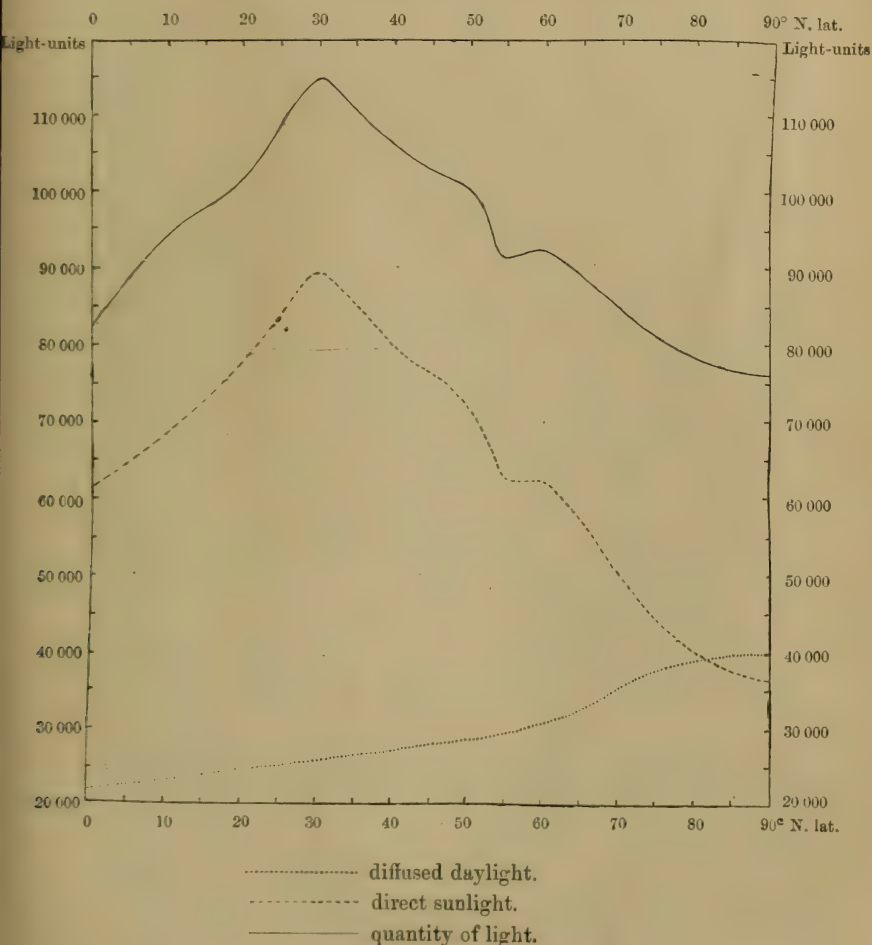
The middle curve, that indicates the distribution of the actinic light from direct radiation over various latitudes, shows quite another appearance. Starting from 0° N. lat., the curve rises quickly and reaches its maximum value at 30° N. lat. From here it decreases fast, but irregularly, towards the pole, changing its concavity several times. We see, for instance, that at 40° N. lat. there is a small upward concavity, at 50° a somewhat larger upward convexity. Between 50° and 60° we find again a very strong upward concavity, the curve completely ceasing to fall, and continuing about horizontally from 56° to 60°, that is, on a strip bordered on the south by a line traversing the northern part of Vancouver Island, the north of Newfoundland, Newcastle, the south of Fünen, and Zeeland, and on the north by a line through the southern point of Greenland, Christiania, Upsala, and St. Petersburg. From here the curve decreases again towards the higher latitudes, at last showing a marked upward concavity. At the said range from 56° to 60° the direct insolation on midsummer-day is on the whole as large as at the equator on the same day.

While the quantity of diffused chemical light at the North Pole on the said day is about double the quantity at the equator, the quantity of directly insolated chemical light has at the equator about double the value that it has at the North Pole, and at 30° N. lat. (for instance, the Canary Islands, Suez, New Orleans, Shanghai) its value is about three times as great as at the North Pole.

The value of the directly insolated chemical light (at 30° N. lat.) is on the said day more than twice the maximum value of the diffused chemical light (at 90° N. lat.), but at the North Pole the quantity of the diffused daylight (not chemical activity) will surpass the directly insolated light by about 4000 chemical light-units. At 82° N. lat. the quantities of diffused daylight and of direct solar light are equal, viz., 31,000 light-units.

The upper full-drawn curve of the figure shows the distribution of the total quantity of chemical light $(W + W_1)$. The aspect of this curve is obviously stamped by the curve for the direct insolation. However, the diffused daylight—that especially towards the north—is also greatly asserting itself. Although the total quantity of light according to this curve has still its maximum value at 30° N. lat., and the North Pole still receives less light than the equator, yet the difference between the extremes is levelled considerably. A spot at the equator receives at midsummer-day only a total quantity of 6000 chemical light-units more than the

North Pole, and at 30° N. lat. the total quantity of chemical light has not quite twice the value it has at the pole. Add to this, that at 73°·5 N. lat. (that is, more northerly than the north of Norway) the total quantity of chemical light is as great as at the equator, and further, that for a strip from 55° to 73°·5 N. lat. the daily quantity of light on the said day is as great as for an equatorial range to about 7° N. lat. When we go towards the north from the 55th degree of latitude the total quantity of chemical light will increase by some thousands of light-units until a secondary maximum at 60° N. lat. is reached. A very great decrease of about 10.000 chemical light-units, or about one-eighth of the total quantity of light at the equator, having taken place from 50° to 55° N. lat., there appears once more an increase of the quantity of light for a range reaching from the south coasts of the larger Danish isles to the Norwegian town Kristianssund. From here the quantity of light decreases further, but does not within the Scandinavian peninsula attain as low a value as at the equator. Even for the northern part of Scandinavia (70° N. lat.) the total quantity of chemical light on midsummer-day is not less than 80 per cent. of its maximum value at 30° N. lat.

We have thus shown that the northern parts of the globe, with regard to the chemical light that they receive at midsummer-time under a perfectly pure and cloudless atmosphere, are much more fortunately situated than we should have expected from the relative numbers in the table of Spitaler. According to these, we should find that the quantity of light on a day of June at 70° N. lat. would be $\frac{280}{481} \cdot 100 = 58$ per cent. of the quantity of light on the same day at 30° N. lat., where we have the maximum even according to Spitaler. In other respects, too, there are essential differences between the numbers calculated from the formulæ of Bunsen and Roscoe and those of Spitaler. Thus we do not find in the numbers for the actinic light that increase of the quantity of light that according to Spitaler should take place from 70° to 90° N. lat.

As we have already mentioned, we have only referred here to that part of the sunlight that produces the well-known effect on the explosive mixture of hydrogen and chlorine. The said effect has its maximum value in the spectral range $\frac{1}{3}GH - H$, and is produced chiefly by the violet and ultra-violet rays.

The effect of the direct sunrays of other wave-lengths is

illustrated by the formulæ given by M. Andresen[*], and similar to the formula above (*a*), viz., for the spectral fraction between the lines F and G blackening silver-chloride paper,

$$W = 4715 \times 10^{-\frac{0 \cdot 296\,P}{\cos \phi}}.$$

The sunrays passed through a solution of auramine-O, that is, the spectral fraction situated about the line D, produce an effect on a silver-bromide paper that is sensitized with rhodamine-B, expressed by the formula

$$W = 1663 \times 10^{-\frac{0 \cdot 109\,P}{\cos \phi}}.$$

and the effect of the red rays, which are situated on the other side of $\alpha \frac{2}{5} D$ (wave-length 610) is measured by a silver-bromide paper sensitized with a solution of chlorophyll, and is expressed by

$$W = 445 \times 10^{-\frac{0 \cdot 117\,P}{\cos \phi}}.$$

These formulæ may also be expanded in a series of increasing powers of $\cos \phi$, and will then give expressions similar to the formula (*b*), which may be integrated with regard to *t*. Those curves giving the distribution of the directly shining sunlight of the less refrangible spectral regions will be of the same nature as the one we have found for the more refrangible light, but they will not be equidistant, and their special points will probably not all correspond to the same latitudes.

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XXXIV. *Unrecognized Factors in the Transmission of Gases through Water.* By W. E. ADENEY, *D.Sc., Curator and Examiner in Chemistry in the Royal University, Dublin* [†].

THE question of the possibility of atmospheric gases being transmitted through water by any process other than that of diffusion, or by thermal currents, or mixing, does not appear to have been considered.

Hüfner [‡] found, in his work on the diffusion coefficients of gases in water, that, if a gas be placed above the liquid, the phenomenon of diffusion is disturbed by downward

streaming effects through the water. He explains these as being caused by the water becoming heavier as it dissolves the gas, and sinking to the bottom.

Hüfner, in order to avoid this streaming effect, employed a thin plate of the porous mineral hydrophane to hold up a column of water in an experimental tube, and so provided the means of introducing the gas at the bottom of the water and of studying the upward diffusion of the gas.

The author has met with this downward streaming in the course of some experiments recently made upon the solution of atmospheric gases by water. It did not, however, appear to him that the cause of the streaming, as suggested by Hüfner, offered a wholly satisfactory explanation of the phenomenon observed. He accordingly thought that a careful investigation of this streaming effect might be followed by results of some value.

Method of Experiment.

Glass tubes about 2 metres long and 18 millims. bore have been employed for all the experiments to be described, except for a few preliminary ones. The dimensions of these will be given with the descriptions of the experiments.

The tubes were nearly filled either with distilled water or with sea-water, and warmed and exhausted as far as desirable by means of a mercury-pump. The corks closing the lower ends of the tubes were fitted with fine glass tubes to provide the means of attachment to the mercury-pump for exhaustion, and for drawing off samples.

After the gases in solution and in the air-spaces had been abstracted, the corks closing the upper ends of the tubes were removed, and the tubes for most of the experiments were immersed to a depth of about 1300 mm. in a large water-jacket, which was kept cool by a constant flow through it of water from the main supply.

The object of the water-jacket was to prevent the formation of downward convection-currents, by keeping the lower portions of the water-columns in the tubes at a slightly lower temperature than the upper portions, which were exposed to the temperature of the laboratory.

In most of the experiments to be described, the surfaces of the water-columns were kept constantly agitated by means of currents of air drawn through their upper layers. For the few preliminary experiments to be described, the tubes were fitted with corks carrying fine glass inlet and exit tubes of 3 mm. bore, and for the later experiments with glass capillary tubes of 0·5 mm. bore. The inlet-tubes have

dipped at different depths, from 10 to 200 mm., below the surface.

Slow currents of air were employed, at the rate of 1000 c.c. per hour, with the smaller bore inlet-tubes, and at somewhat quicker rates with the tubes of the larger bore.

In a few cases, the surfaces of the water-columns were agitated by mechanically rotated glass stirrers, just dipping below the surface of the water. In these cases the experimental tube employed was of wider bore to allow of stirrers of a gridiron shape being used.

When the surfaces of the water-columns were left unbroken, the experimental tubes were loosely covered to protect their contents from dust.

The height of the columns of water were in nearly all experiments about 1800 mm. In all experiments the surfaces of the water were freely exposed to the air. Observations of barometric pressures have not been thought necessary.

At the conclusion of each experiment, layers of the water of 100 to 200 mm. thick were drawn from different depths from the surface, and stored in tubes standing over mercury until they could be analysed.

The dissolved gases in these samples were extracted by boiling with a little sulphuric acid *in vacuo*. The apparatus employed for this purpose, and for the analysis of the gases, has already been described by the author in his memoir on Dissolved Gases and Fermentative Changes [*].

Of the results of the analysis of the dissolved gases, only those for the nitrogen are recorded in this communication. These are expressed in c.c. at 0° C. and 760 mm. bar. per litre.

DESCRIPTION AND RESULTS OF EXPERIMENTS.

Experiments with Unbroken Surfaces.

No. 1.—Two tubes, each 1950 mm. long and 18 mm. bore, were filled to the height of 1800 mm. from the bottom, one with sea-water and one with distilled water, both being nearly nitrogen free, and were immersed in the water-jacket and kept undisturbed for 28 days. The temperature of the room varied between 16°·5 and 10° C., that of the jacket between 12°·9 and 8°·0 C. During the last week of the experiment the temperature of the room varied between 12° and 10° and that of the jacket between 9° and 8° C.

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[*] Trans. Royal Dublin Society, vol. v. Part 2 (1895).

Distilled water saturated at 10° C. contains 15·37 c.c. N_2 per litre. Sea-water saturated at 10° C. contains 12·47 c.c. N_2 per litre.

Depth of Layer below surface.	Sea-water.	Distilled water.
1 to 200 mm.	9·24	11·50
300 to 500 ,, ...	8·96	11·10
800 to 1000 ,, ...	8·91	8·43
1600 to 1800 ,, ...	7·89	5·84
Before aëration	1·51	1·55

No. 2.—Similar to No. 1 with the exception that the tubes were kept longer in the water-jacket, viz. for $2\frac{1}{2}$ months. Temperature of room at close of experiment 17°·8 C., of water-jacket 15°·6 C.

Distilled water, at 17°·8 C. contains 13·29 c.c. N_2 per litre. Sea-water at 17°·8 C. contains 10·80 c.c. N_2 per litre.

Depth of Layer below surface.	Sea-water.	Distilled water.
1 to 200 mm.	10·80	12·56
1600 to 1800 ,, ...	10·72	9·09

The foregoing results reveal a number of points of interest in connexion with the streaming effect above referred to.

The most noticeable is the marked difference in rate in sea- and distilled water, being much greater in the former than in the latter. Another point of interest is that the surface-layers of the sea- and distilled water, in No. 1, even after exposure to the air for 28 days, did not become saturated with nitrogen. Hence we may conclude that the streaming effect is more rapid in both sea- and distilled water than the rate of solution at the surface exposed to the air.

These experiments, so far as they go, may be regarded as supporting the suggestion of Hüfner, that the streaming is a gravitational effect due to concentrated solution currents. The following experiment, however, affords conclusive evidence that the streaming is really a gravitational effect.

No. 3.—A U-shaped tube, each limb 1600 mm. long and 50 mm. diameter, was filled with boiling sea-water and immediately closed air-tight with indiarubber corks. When the water had cooled, the tube was further exhausted by means of a mercury-pump. One limb was then uncorked, and the water in it was saturated with atmospheric gases by a current of air drawn through the

surface-layer to a depth of 50 mm. for 48 hours.   After
this, a sample was drawn from the bottom of the open
limb, and one from the lower portion of the closed limb,
about 200 mm. from the bottom. The gases were extracted
from these and the nitrogen in each determined, with the
following results :—

Open Limb.	Closed Limb.
12·12	1·49

Six months later another sample was collected from the lower
portion of the closed limb, as before, and the nitrogen in
it determined ; it was   1·73

Experiments with Broken Surfaces.

The experiment No. 1, and several others which were made
during the earlier part of this investigation, but which it is
unnecessary here to record, since the results were similar,
shows that a retardation of solution takes place at the surface
of the water exposed to the air. Experiments were accordingly
made in which the surfaces of the water exposed to the air
were kept continuously broken.

No. 4.—A tube 1100 mm. long and 50 mm. diameter was
filled with nitrogen-free sea-water, and immersed in the
water-jacket to a depth of 1025 mm.   A glass stirrer of a
gridiron shape, 40 mm. broad, was fixed to dip 25 mm.
below the surface of the water, and was mechanically
rotated for 19 hours.   At the end of this time the tem-
perature of the upper layer of the water-column above
the water-jacket was 8°·3 C., and that of the water-jacket
7°·8 C.   Samples were drawn form the top and bottom
layers of the water-column, and the nitrogen determined
in each, with the following results :—

Top layer.	Bottom layer.
12·69	12·45
At commencement.	1·51

No. 5.—A similar experiment was made with distilled water,
with results as follows :—

13·29	11·02
At commencement.	1·46

Sea-water saturated at 8°·3 C. contains 12·91 c.c. N_2
per litre.   Distilled water saturated at 8°·3 C. contains
15·92 c.c. N_2 per litre.

In order to provide in these experiments the means of detecting whether or not the mechanical agitation of the surface-layers of the water-columns caused a circulation of the water to the bottom, a large crystal of potassium bichromate was placed at the bottom of the tubes; but in no case was the dissolved bichromate observed to rise above the immediate neighbourhood of the crystal, and become mixed with the water above it.

The results of the last two experiments proved, as was anticipated, that the downward streaming effect takes place so rapidly when the surface of the water exposed to the air is continuously agitated, that long columns of water are completely saturated with atmospheric gases with comparative rapidity.

Experiments were next made in which the surfaces of the water-column were kept broken by means of slow currents of air. It was thought that by their means it would be easy to make comparative observations on the effect of agitating different thicknesses of the surface-layers of two or more columns of water by the same disturbing cause.

No. 6.—Two tubes, each 1980 mm. long and 24 mm. bore, were filled with nitrogen-free sea-water. Each tube was fitted with an indiarubber cork and fine glass inlet and exit tubes of 3 mm. bore, and they were connected in series so that the same current of air could be drawn through the surface-layers of each column of water. The inlet-tube through which the current of air was first drawn dipped 100 mm. below the surface of the first column, and the second inlet-tube dipped 10 mm. below the surface of the second column, of water. The current of air was not previously filtered. It was continued for 25 hours. At conclusion temperature of room was 19°·5 C., and of the jacket 15°·7 C.

Sea-water saturated at 19°·50 C. contains 10·50 c.c. N_2 per litre.

Depth of Layer below surface.	1.	2.
1 to 100 mm.	10·57	8·01
1500 to 1600 ,, ...	8·91	4·08
Before aëration	1·58	1·56

No. 7.—Similar to the experiment No. 6, with the exception
that the inlet-tube dipped 10 mm. below the surface in the
first column of water, and 100 mm. in the second column.
The current of air was continued for 20 hours. Tempe-
rature of room at conclusion was $18°·2$ C., and of the jacket
$14°·9$ C.

Sea-water at $18°·2$ C. contains 10·68 c.c. N_2 per litre.

Depth of Layer below surface.	1.	2.
1 to 100 mm.	10·07	9·95
1000 to 1100 ,, ...	—	6·05
1200 to 1300 ,, ...	9·61	—
1700 to 1800 ,, ...	6·26	2·15
Before aëration	1·61	1·56

The sea-water employed for the experiments thus far re-
corded was collected in Dublin Bay. It had been stored
in partially full bottles for some weeks previous to use. It
contained 19·57 parts chlorine per litre.

From the foregoing results it appears that the rate of
aëration, as a consequence of the streaming, was not materially
affected by difference of depth of the air-inlet tubes, when
two water-columns were aërated in series by the same current
of air. The inlet-tubes dipped 100 mm. and 10 mm. below
the surfaces in the columns of water 1 and 2 respectively
in experiment No. 6, and *vice versa* in experiment No. 7 ;
and yet both the columns of water which first received the
air-currents in the two experiments were found to have dis-
solved more nitrogen than either of the second columns of
water.

It appears, therefore, from these experiments that some-
thing was taken from the air by the first water-column in
each experiment in addition to the ordinary atmospheric
gases, and that very little reached either of the second columns.

The next experiments were arranged with a view to ascer-
taining whether, by using slower currents of air and fine
capillary glass inlet and exit tubes of 0·5 mm. bore, the
streaming effect could be completely confined to the first
water-column.

No. 8.—Three tubes, each containing sea-water nearly nitro-
gen-free, were fitted with indiarubber corks and air-inlet
and exit tubes, and were connected in series so that the
same current of air could be drawn through the upper
layer of each column of water to a depth of 200 mm., one
after the other.

The air-current was a slow one, at the rate of about 1 litre per hour. The aëration was continued for 48 hours and then stopped. Samples of water were then drawn from each tube as before, and stored in glass vessels standing over mercury. The nitrogen determinations are given below. The tube in which the air-current first entered was marked 1, and the one it next entered 2, and so on. The air was filtered through glass-wool.

Temperature of room 12°·8 to 10°·0 C.
Temperature of jacket 8°·6 to 8°·3 C.

Sea-water at 10° C. contains 12·47 c.c. N_2 per litre.

Depth of Layer.	1.	2.	3.
1 to 200 mm.	12·24	11·15	11·09
800 to 1000 ,, ...	8·20	2·29	2·55
1600 to 1800 ,, ...	8·13	2·18	2·54
Before aëration	1·57	2·13	2·51

No. 9.—Similar to the experiment No. 8, with the exception that the air-inlet tubes dipped only 50 mm. below the surface. The air-current was filtered through glass-wool.

Temperature of room   13°·7 to 13°·5 C.
Temperature of jacket 8°·4 to 8°·5 C.

Sea-water at 13°·5 C. contains 11·67 c.c. N_2 per litre.

Depth of Layer.	1.	2.	3.
1 to 200 mm.	10·81	7·15	6·93
300 to 500 ,, ...	10·60	1·61	1·52
800 to 1000 ,, ...	9·36	1·65	1·56
1600 to 1800 ,, ...	9·20	1·65	1·52
Before aëration	1·75	1·63	1·51

These results show that, when the current of air is sufficiently washed during its passage through the surface-layer of the first column of sea-water, it may be passed through the surface-layers of other columns of sea-water and no streaming effect occurs; although the layer of water through which the air is bubbled soon becomes saturated with atmospheric nitrogen.

It may therefore be concluded that the streaming effect is not due to concentrated solution currents, as suggested by Hüfner, but by something in the air besides its chief gaseous constituents; and that when the something is washed out, the subsequent solution of the gases is not attended by streaming.

The sea-water employed for these experiments was also

collected in Dublin Bay. It was filtered through glass-wool and stored in glass bottles for a week before use. It contained 19·45 grms. chlorine per litre.

Similar experiments were made with distilled water, as there were reasons for anticipating that the cause of the streaming would not be so rapidly washed out by distilled water as by sea-water.

Experiments with Distilled Water.

No. 10.—Similar to experiment No. 8, with the exception that distilled water was used in place of sea-water.

Temperature of room 11°·5 to 11°·8 C.
Temperature of jacket 8°·5 to 9°·0 C.

Distilled water at 11°·8 C. contains 14·83 c.c. N_2 per litre.

Depth of Layer below surface.	1.	2.	3.
1 to 200 mm.	14·60	12·27	12·20
300 to 500 ,, ...	11·62	11·10	11·18
800 to 1000 ,, ...	4·18	4·20	4·48
1600 to 1800 ,, ...	3·54	3·05	3·20
Before aëration	1·55	1·73	1·97

No. 11.—Similar to No. 10, with the exception that the air-current was not filtered.

Temperature of room 12° to 10° C.
Temperature of jacket 8°·7 to 8°·5 C.

Distilled water saturated at 10° C. contains 15·37 c.c. N_2 per litre.

Depth of Layer below surface.	1.	2.	3.
1 to 200 mm.	13·81	12·77	12·64
300 to 500 ,, ...	12·20	11·17	11·08
800 to 1000 ,, ...	5·20	5·17	5·14
1600 to 1800 ,, ...	4·15	4·06	4·00
Before aëration	1·75	1·55	1·51

These experiments confirm the conclusion which was drawn from the results of experiment No. 1, viz., that the downward aëration of distilled water by the streaming effect is not so rapid as the aëration of sea-water by the same means.

They also show that the cause of the streaming was not concentrated in the first water-column, as was the case with sea-water, but that, on the contrary, it equally affected the three water-columns. It may be inferred from this last observation that the streaming being less in distilled water than in sea-water, is due to the something causing it being more slowly taken out of the air-current by the former than by the latter.

The following experiment was also made with distilled water, wide bore, 3 mm., inlet and exit tubes being employed. The inlet-tube dipped 25 mm. below the surface of the water :—

No. 12.—A tube 1980 mm. long and 24 mm. bore was filled with nitrogen-free distilled water, and the air-current continued for 12 hours. Temperature of room 15°·5 C., of jacket 12°·9 C.

> Depth of Layer.
>
> | | |
> |---|---|
> | 0 to   200 | 10·56 |
> | 650 to   800 | 3·9 |
> | 1500 to 1600 | 2·68 |
> | Before aëration | 1·51 |

The question arises, what is the active cause of the streaming ? The author does not desire at this stage of his investigation to definitely advance any explanation of the facts observed. One, however, which is suggested by the experiments that have been described, may be pointed out. It is that the streaming is possibly caused by minute dust particles, or by other centres of condensation possibly of an electrical nature, carried by the air-current and being taken up by the water together with the gaseous constituents of the air, and that these bring about in some way a sufficient density of the dissolved or liquefied gases to render it possible for them to be drawn gravitationally downwards through the water.

The fact that the streaming is more marked in sea than in distilled water may be explained by assuming that the sea-water, from the salts which it holds in solution, is able to make more use of the centres of condensation introduced by the air which is bubbled through it. When sea-water and distilled water are separately shaken up with the air, the difference of behaviour of the two waters is, as is well known, very marked, it being easy to obtain a formation of small bubbles of air in sea-water to an extent quite impossible in distilled water.

There can be no question, from the experiments recorded, that if the air be completely washed, the surface of sea-water exposed to it will become saturated with nitrogen, but no streaming effects will take place.

It is equally evident from these experiments, that whatever may subsequently be discovered to be the true cause of the streaming, its effect in large volumes of sea- or river-water under natural conditions must be of great importance, and of such dimensions that the effect of ordinary diffusion may, in comparison, be entirely neglected.

XXXV. *Action of a Magnetic Field on the Discharge through a Gas.* *By* R. S. WILLOWS, *M.A., D.Sc.*[*]

WHEN a discharge which is passing through a gas under reduced pressure is acted on by a magnetic field, it is known, both from theory and experiment, that if the field is parallel to the discharge it causes it to pass more easily, while if the field is transverse the opposite is the case. I have shown [†], however, that below certain pressures which vary with the conditions of the experiment, a transverse field increases the current in the tube and diminishes the difference of potential at the terminals, provided it be applied near the cathode. At other points of the discharge a decrease in current is always produced by the field.

Birkeland has shown [‡] that with the lines of force parallel to the tube, supposed cylindrical, at pressures below ·012 mm., the potential-difference is made to diminish by the magnetic field, at first slowly as the intensity is increased, until a certain critical intensity is reached, when a large, abrupt diminution is obtained.

Almy [§] has studied these two effects, and has come to the conclusion that they are both due to the same cause. He concludes that the action of the magnet is simply to concentrate the discharge so that it passes through the gas by a sort of brush or arc rather than in the usual manner, and that this brings about an increase in the conductivity.

Further study of these effects was my object in starting the experiments described in the following paper.

When the negative glow is the part of the discharge acted on by a transverse field, an increase in potential at the terminals always takes place; but it has not been determined previously whether the results depend on whether the magnet acts at the surface of the cathode or at any other point in the dark space.

In the earlier paper I was fortunate enough to be able to use a large accumulator battery to produce the discharge, and so both voltage at the terminals and current through the tube could be measured. In the present case a coil, driven by a mechanical interrupter, was used, and the voltage only was measured by means of a multicellular voltmeter. Where the range of the voltmeter was not great enough, the tube was shunted by a liquid high resistance, and a fraction of the total voltage taken or the whole measured in steps. An electromagnet with pole-pieces formed so as to give a very local

---

field was used, and the strength of the field at different distances from the line joining the poles, and for different currents, was found by means of a flat coil and ballistic galvanometer.

A long cylindrical tube of 25 mms. diameter, provided with disk electrodes, was pumped down until the Crookes dark space was 3 cms. long, and the transverse field applied successively at the cathode and at distances of 1, 2, 3, and 4 cms. away from it in the direction of the anode. The observations showed that when equal decreases in potential were produced, the field at the cathode surface was practically the same for all positions of the magnet. We may therefore conclude that it is at, or near, the surface of the cathode that the action arises. The field shortened the dark space from 3 cms. to about 2 mms.

If Almy's explanation is the correct one, the critical pressure, above which the field causes an increase and below which a decrease in the terminal voltage, might be expected to be more or less irregular and ill-defined. In any case, it was thought worth while to study this pressure more carefully with different transverse fields, and an observation of interest resulted. Starting from a pressure of 3–4 mms., the volts at the terminals were measured with the magnet off and on at the cathode. As the pressure is continually lowered, the magnet being off, it is, of course, known that the voltage decreases, reaches a minimum, and then rapidly rises. The value of the critical pressure corresponding to minimum volts depends on the diameter of the tube, the nature of the gas, and perhaps also on the current. The last point I could not investigate with a coil. It was found that *above this critical pressure the field caused an increase in the voltage at the terminals, while below it caused a decrease.* At pressures much above 5 mms. very little effect was noticed. This result was confirmed by experiments on tubes with lengths varying from 50 cms. to 10 cms., diameters from 25 mms. to 3 mms., and with disk or wire electrodes. By using hydrogen, carbon dioxide, and air in these tubes, the critical pressure could be shifted from ·3 mm. to about 1 mm. and the observation still held, although with hydrogen the results were not so regular as with the other gases.

This critical pressure is much more sharply marked in narrow tubes. The series of observations given below will serve as an example of the results obtained. They were taken with a tube 3 mms. in diameter, 10 cms. long, filled with air, and a field of about 600 lines per sq. cm. A fraction only of the voltage is given.

VOLTS.

Magnet off.	Magnet on.	Magnet off.	Magnet on.
290	305	218	200
262	272	248	218
252	260	268	235
208	198	270	242
190	197	285	265
184	191	295	275
186	187 (?)	305	285
195	190		

The pressure was gradually reduced from the beginning to the end of the series, the minimum voltage is 184–186 ; before this is arrived at the magnet causes an increase, after it is passed a decrease in the volts, but it never causes the voltage to fall below 186. The field therefore does not reduce the absolute minimum voltage causing the discharge. This is worthy of note, as Carr [*] has shown that the minimum sparking potential is a constant for the gas, and there is doubtless a close connexion between the minimum sparking potential and the minimum voltage required to maintain the discharge.

With tubes much shorter than 10 cms., the pressure had to be lower than the minimum voltage pressure before the magnet caused a decrease. This is no doubt due to the fact that the field could never be sufficiently localized to act at the cathode only, and where it acts at other points of the discharge it brings about a rise in voltage.

It would appear, therefore, from my experiments, that the transverse magnetic effect is closely connected with other features of the discharge, and hence is probably not due to a concentration of the luminous portion into a brush or arc.

I tried next the effect of a longitudinal field on several of the tubes, but, down to a pressure ·01 mm., a certain decrease in the terminal voltage was never obtained. The iron core of the magnet employed to produce the field was of greater diameter than the largest cathode used ; the field had a maximum value of 900, it was therefore more intense than that used by Almy or Birkeland. I am unable to account for the discrepancy between my results and those of these physicists.

One tube was about 2 cms. long, so that the longitudinal field could act along the whole length of the discharge. Still no effect could be found, but if the same field was applied transversely at the cathode it produced a fall of 30 per cent., and this although it was not specially concentrated. The

[*] Phil. Trans. cci. p. 403 (1903).

transverse and longitudinal effects are therefore of a different nature.

The dependence of the fall in voltage on the strength of the field when the pressure in the tube is kept constant, can be seen from the following series of readings. The gas-pressure was ·052 mm.

Field.	Per cent. decrease in volts caused by field.
159	1·1
238	3·2
318	4·3
398	6·5
477	7·1
589	8·7
685	10·6
844	13·9
1050	13·8

These numbers are shown graphically in fig. 1. The effect reaches a limiting value in this case ; in others it reached a maximum, and then slightly decreased as the field continually increased ; this usually took place in the wider tubes.

Fig. 1.
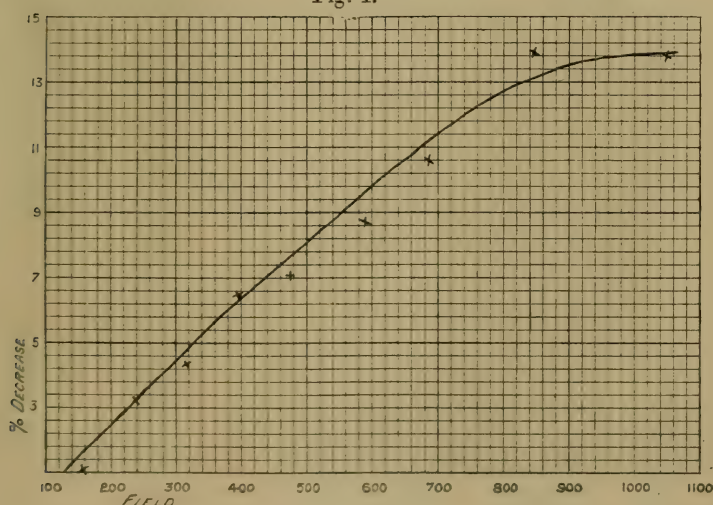
The fall in volts does not increase indefinitely as the pressure is decreased. In all the cases tried, it appeared either to reach a limiting value or much more generally to fall off, and in some cases, at the lowest pressures reached,

·002 mm., a rise in volts was brought about by the field. The results, however, at the lowest pressures were very irregular, and it may be that Almy's supposition as to the concentration of the discharge may have some weight here.

When the transverse field is applied, the Crookes dark space is greatly shortened on that side of the tube to which the discharge is deflected, very little alteration in its length is seen on the other side. The glow at the surface of the cathode (called by German writers the first negative layer) is also affected. The figures adjoining (fig. 2) show this.

Fig. 2.

WITHOUT MAGNET   WITH MAGNET
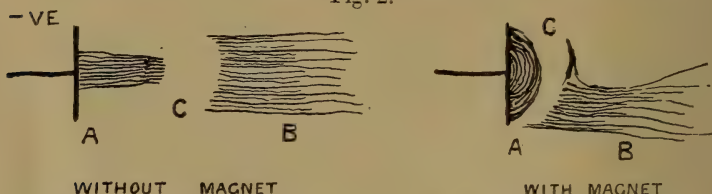
When the field is off, the first negative layer, A, covers the centre only of the cathode and is cylindrical in section; with the field on, its length is considerably less and it covers much more of the cathode area.

Since this layer is supposed to be due to ionization caused by canal-rays striking the cathode, an experiment was performed to see whether the path of these rays was altered by the field. A tube, as shown in fig. 3, with an aluminium wire projecting across the surface of the cathode was used.

The surface of the cathode behind this wire is devoid of all glow as shown by various investigators. The magnet caused no shift of this shadow, so that the fields used were not sufficient to deflect the canal-rays.

Townsend * has given a complete explanation of the fact that there exists a pressure for which the sparking potential between two fixed electrodes is a minimum. The fact that there exists a pressure for which the E.M.F. required to *maintain* the discharge is a minimum yet awaits a detailed explanation. The following observation on this point is of interest.

Fig. 3.
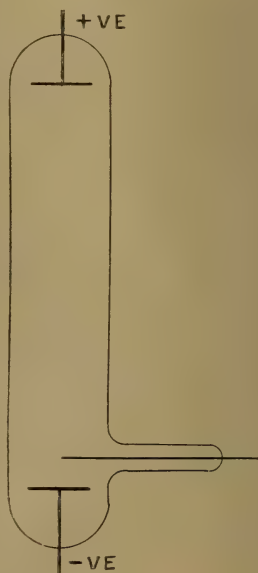
* Phil. Mag. Nov. 1903, p. 598.

If a curve be plotted showing the relation between terminal voltage and pressure of the gas in the discharge-tube, then, as the latter is diminished, starting from a few mms., the relation between the two is a linear one. As soon, however, as striæ appear at the cathode end of the positive column the voltage decreases less quickly than before ; when the whole length of the positive column is striated, the voltage reaches its minimum value. If the pressure is further decreased, the voltage required to maintain the discharge rapidly increases. This observation was confirmed on tubes up to 50 cms. in length, of diameters ranging from 3–25 mms., and on air, carbon dioxide, and hydrogen. The last gas was irregular.

It appears, therefore, that the positive column striates because, by that means, the discharge is enabled to pass most readily. After it is completely striated, the magnetic field causes the current to pass more easily.

The field always causes more striæ to appear. Measurements were made on the longer tubes to see if it caused an alteration in the distance between the striæ when they were so far away from the cathode as to be unaffected by the field directly. The results obtained with a coil as source of current were too irregular to admit of conclusions being drawn.

That the positive column is differently affected in different gases may be inferred from the following experiment. A long tube filled with air giving striæ at a suitable pressure was used. If a current was then sent through the electromagnet and gradually increased, from each of the original striations a smaller one emerged on the side remote from the anode ; with a further increase of field, this smaller one eventually coalesced with the original stria next nearest the cathode, and this could be repeated several times by continuously increasing the field. Below is a set of observations.

Current in magnet in amperes.	Appearance of striæ.
0	Fairly clear.
1·4	Very distinct double, one small one appearing exactly midway between two large ones.
1·65	No doubling.
2·4	Clear.
2·9	Double.
5·8	Very clear.
6·6	Very double.
7·3	Not so double.

Crookes * has shown that the double striæ frequently met with in hydrogen are due to the current being carried partly by the hydrogen and partly by mercury vapour. A similar explanation would probably apply to the above results.

The appearance with hydrogen in the tube was very striking at certain pressures. When the magnet was off, the bright parts of the striæ were very narrow and pink in colour, the spaces between being hazy.   On putting on the magnet the voltage was reduced, the striæ became very distinct and steady, double as in air and the interspaces much less hazy. After a few seconds each small stria retreated into the adjoining large one next nearest to the anode. After another interval of a few seconds the voltmeter became very unsteady, the positive column changed from pink to grey, the striæ assumed the form they had before the magnet was put on, and the voltage rose to its original value.   If the coil was stopped and turned on again after the lapse of 15 seconds or more, the whole of the changes were reproduced ; if turned on at a shorter interval, any particular stage was at once produced according to the time that had elapsed since the stoppage.

The magnet evidently causes the current to be carried entirely by one gas ; this change occupies several seconds, and the tube must rest for 15 secs. in order to allow the ions of one gas to disappear so that the other may carry its share of current.

The lower the pressure the smaller the field required to produce double striæ.

I next sought to determine whether the decrease in potential brought about by the transverse field takes place at the cathode, or whether some of this might not be found in the positive column, owing to the increased number of striæ found there when the field is on.   A tube as shown in fig. 4 was used.

The side electrodes B, C, were thin pointed wires fastened in with sealing-wax joints.   B was always in the positive column when A was the anode; at the lower pressures C was in the Faraday dark space or the negative glow.

The difference of potential between A and the other electrodes was measured with the voltmeter both when the magnet was on at D, and when it was off.   A large fall in potential between A–B and A–C was always brought about by the field below the critical pressure, the percentage fall over these portions of the tube being much greater than over the whole.   By far the greatest potential fall (magnet off) takes place at the cathode, so that the actual fall in volts

* Proc. Roy. Soc. lxix. p. 399 (1902).

between A–C brought about by the magnet was not sufficient to account for the whole decrease for the tube. When the magnet caused the potential between A–D to increase, it frequently effected a fall between A–B or A–C if the gas-pressure was not above about 2 mms.

The results given by this tube are, however, not altogether trustworthy, for the sealing-in of the electrodes B, C caused the magnet to influence the tube as a whole at a much lower pressure than previously; that is to say, the positive column was striated for some time before the magnet caused the
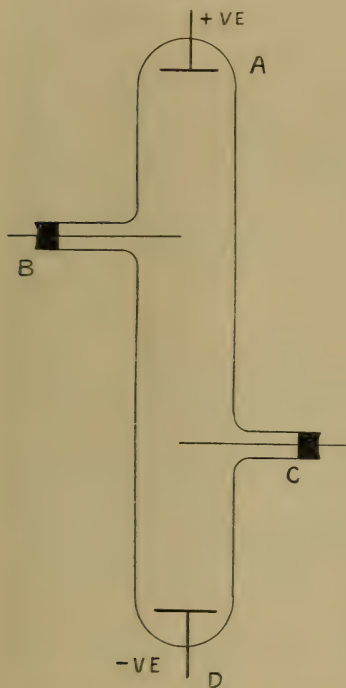
Fig. 4.   Fig. 5.
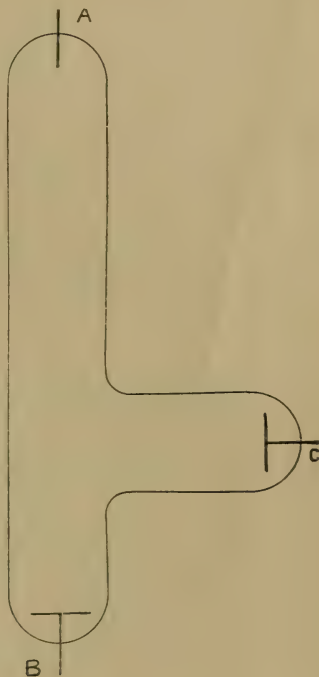
voltage to drop. This may be due to the distortion of the discharge by the wires, but it is much more likely on account of B and C acting as secondary anodes and cathodes according to which is connected with the voltmeter. It is certain that a dark space was developed round them in some instances.

A tube was next tried in which the development of the positive column could be controlled while everything else remained the same. This is shown in fig. 5.

A was a pointed aluminium wire and was always made the

anode. B, C were disks, and either could be made the cathode at will ; the limbs containing them were made as nearly as possible equal.   At low pressures, when B was cathode, the positive column was very short; when the glass round B gave a green fluorescence scarcely any positive column was present. Under similar conditions with C cathode the column extended to the side limb.   With the field on at either cathode, the positive column extended nearly the whole length of the tube. In spite of this great difference in development, no difference could be found in the behaviour of B or C with the magnet, at a pressure when the latter brought about a large decrease in the voltage.

We may therefore conclude that the fall in potential has its chief seat near the cathode.

As to the cause of this fall, it may be suggested that the negative ions are caused to travel over greater paths in the dark space, because the magnetic field causes them to move obliquely through it.   In this space, therefore, they produce more ions by collision, and the discharge being previously poor in ions at this point, as shown by numerous experiments, this causes the fall in voltage between the terminals.

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XXXVI. *Action of Radium on the Electric Spark.   By* R. S. Willows, *M.A., D.Sc., and* J. Peck, *B.Sc.*[*]

IT is well known that radioactive substances, when placed near a spark, in general cause the discharge to pass more readily by the formation of ions in the electric field. The action on a long spark is, however, different, and appears, as far as our knowledge goes at present, somewhat irregular. In some cases the discharge passes more readily, in others with greater difficulty [†].

The present paper describes experiments which were undertaken with the object of studying the origin of this influence.

The spark was produced between two brass spheres of unequal diameters, one 27 mm. the other 48 mm., by means of a Wimshurst machine carrying three pairs of plates of 2 feet diameter.   The radium bromide, 5 mgm., was contained in a small capsule closed with mica.   The strength of the

sample was not known, but in most of the experiments it was not of the highest activity.

The action was altogether different according to the direction of the discharge. Using a spark-gap longer than 2 cms., and making the larger knob positive, the radiations had practically no influence. With the smaller sphere positive the radium in most cases extinguished the spark, the visible portion of the discharge being confined to a glow at the surface of both electrodes. With the machine used a torrent of sparks could be kept up with a gap of 3 cms. ; in these circumstances the spark was least sensitive to the radium.

As the knobs were gradually pulled apart, a good deal of brush discharge became mixed with the spark, and the effect was more easily observed. With still longer discharges there appeared only a positive brush, beyond this a dark space, and on the surface of the cathode a faint glow. In this condition the discharge is very susceptible to influence, it may be extinguished instantaneously by the radium held at a distance of 1 metre.

The following series of observations and the accompanying curve (fig. 1) plotted from the numbers, show how the sensibility increases with the spark-length. The two spheres were set apart so that their nearest points were distant from each other the lengths given in the first column, the machine was put in action and the radium gradually approached along a line at right-angles to the discharge. The numbers in column 2 give the distances at which the discharge was just extinguished ; in each case the numbers are the mean of several observations.

Distance apart of knobs. Smaller sphere positive.	Distance of radium from spark-gap when the discharge is just put out.
cms.	cms.
4	6*
5	12
6	15
7	17·5
8	20
9	24
10	28
11	29·7
13	39
15	66

* The discharge in this case is mixed brush and spark. Most of the brush is stopped at 10 cms., the spark also at 6 cms.

The curve is practically a straight line up to a spark-length of 12 cms., after which the sensibility rapidly increases. It is at this point also that the dark space first appears.

Fig. 1.
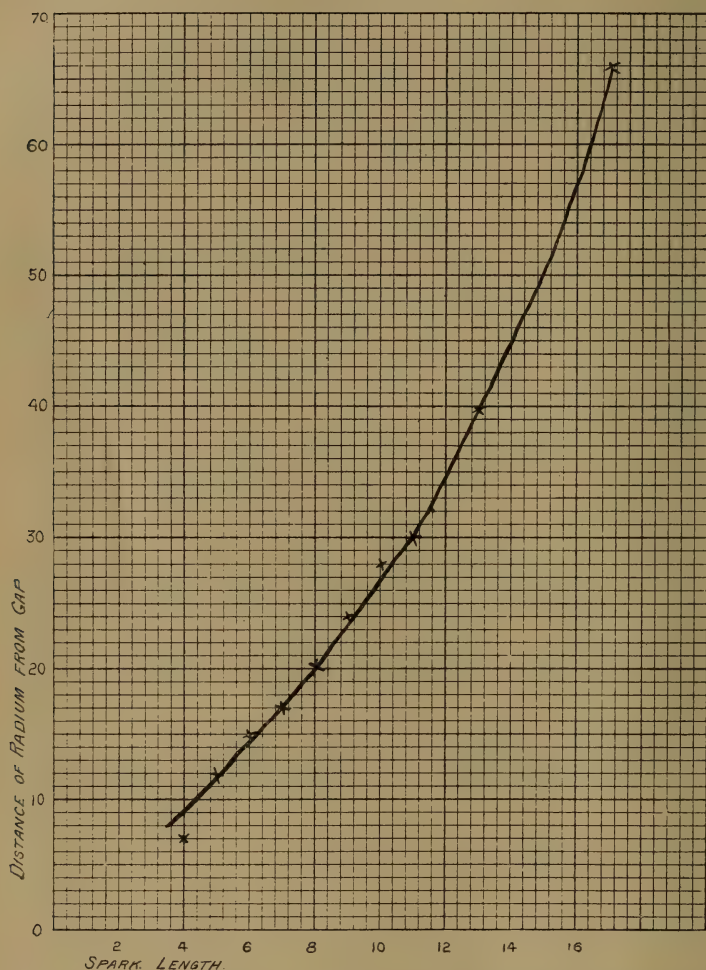
The appearance of the discharge with spark-lengths 4, 7, and 13 cms. is shown in the sketches A, B, &c. respectively of fig. 2.

When the anode consisted of a piece of No. 22 wire, with which the discharge passed nearly entirely in brush form, the radium had no visible effect. Both this and the fact that

the radiation has no influence when the discharge passes from the larger to the smaller sphere, are consistent with the theory that a minimum discharge potential is necessary for the

Fig. 2.
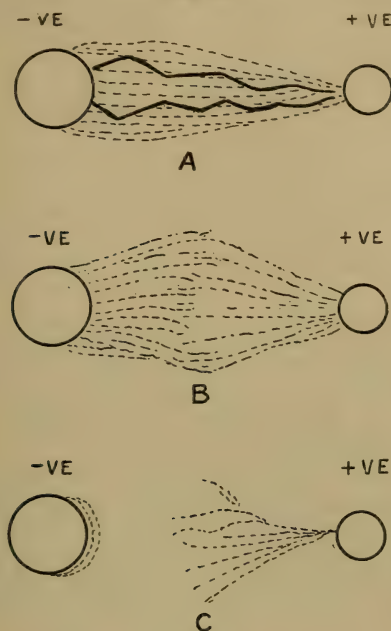
effect to be produced. In each of these cases, the mean potential gradient is less than it is when the spark is in its sensitive condition.

Although the visible discharge is extinguished, it does not follow that the current actually passing is diminished; the conductivity might be so largely increased that a non-luminous discharge takes place. It was found possible to measure this current, the numbers, although variable, being sufficiently accurate for the purpose of the experiment. To do this, the positive side of the spark-gap was connected directly to the machine, the other side to one terminal of a dead-beat D'Arsonval galvanometer, the other terminal of which was earthed. The negative pole of the machine was also earthed. The current thus passed from the anode across to the cathode, and through the galvanometer to earth.

It was found that the extinction of the spark corresponded with a decrease in the current. Further, by encasing the radioactive salt in a thick lead case, so that a narrow beam

only of rays issued, it was possible to investigate the effect at different points of the discharge. The lead case was closed by a thick lid of lead, working on a hinge, which could be opened or closed by the observer at a distance.

The discharge had the form shown in C, fig. 2. The two sets of numbers given below will show the results and the order of accuracy obtained.

	Near positive knob.	Middle of discharge.	Near negative knob.
Per cent. decrease of current.	52	45·5	33
	50·5	41	30

Distance of radium, 20 cms.

The numbers are the mean of a large number of separate observations. It is seen therefore that the positive end is the most sensitive under these conditions. With shorter discharges, the currents were too irregular to admit of conclusions being drawn.

A reference to C, fig. 2, shows that the discharge near the anode is concentrated into a single branch very narrow in section. It hence seems probable that the most sensitive part is where the electric field is greatest. We found it impossible to measure this field at different points of the gap.

Fig. 3.
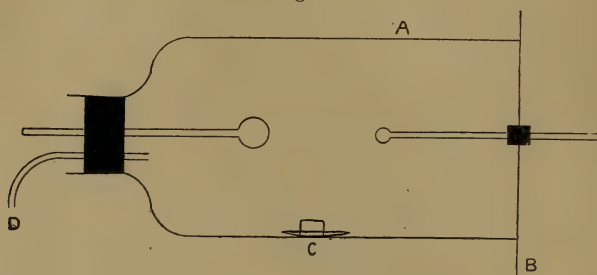
The apparatus shown in fig. 3 was used to investigate the effect of altering the pressure of the gas in which the discharge took place. The bell-jar A was 25 cms. long and 12 cms. in diameter. One end was closed with a brass plate, B, through which passed a rubber bung; D was connected with a water-pump and gauge. The radium capsule, C, was fastened to a piece of soft iron which allowed of it being shifted to any

part of the jar by means of an external magnet. The spark-gap was about 6 cms. long. The results are as follows :—

Pressure in cms. of mercury.	Effect of radium.
77	Visible discharge (spark and brush) stopped.
51	Discharge (all brush) stopped.
40	Discharge (all brush) concentrated but not extinguished.
28	No effect.

The action of the radium is therefore less marked as the pressure is reduced. This may be due to the reduction of the voltage required to maintain the current.

A very much weaker sample of the salt enclosed in a glass capsule showed quite an appreciable effect when the discharge was in a fairly sensitive condition. Thus the action cannot be due to the α rays, since these would be absorbed by the glass. It must be due to either the β or γ rays. Since the latter are supposed to be identical with Röntgen rays, the influence of these on the spark was tried, but they brought about no stoppage of the discharge. The ionizing powers of the Röntgen rays and of the rays from the radium on the air in the spark-gap were of altogether different orders. Thus the radium at 50 cms. readily extinguished the spark, a Röntgen-ray tube 25 cms. away had no effect ; but if the gap was replaced by the knob of a charged electroscope, the latter was completely discharged in less than 1 sec. when the tube was working, while with the radium complete discharge took 5 min. to accomplish.

The result obtained cannot therefore be ascribed to the γ rays, nor does it depend directly on the number of ions produced by the ionizing agent, but rather is a special action of the β rays.

The negative result obtained with Röntgen rays also precludes the following explanation :—The ions in the discharge are produced in two ways, directly by the field and by collision of ions already present with neutral atoms if the field is sufficiently intense. If the ionizing agent, rays of any kind, produces sufficient ions, the voltage between the electrodes may be so reduced that the second method of formation is impossible, and the final result may be a decrease in the current which passes. The effect would then be more marked as the external agent produces more ions, but, as is seen, this is not the case.

It remains only to show directly that the effects observed are due to the β rays. The radium was enclosed in a lead

capsule of the shape shown in fig. 4, and was placed near
enough to the spark to extinguish it.
The capsule was further placed between
the poles of a strong electromagnet,
so orientated that the β rays would be
driven into the lead by its action. Imme-
diately the magnetic field was created,
the discharge passed again with un-
diminished intensity, while when the
current in the magnet was stopped the
visible discharge also stopped, showing
clearly that it was the easily deflectable

Fig. 4.

β part of the rays that was concerned in the extinction.
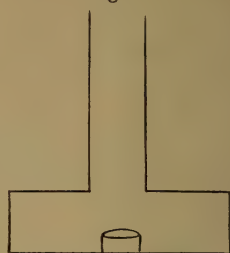
As these rays are of the same nature as Lenard rays, the
action on the spark of the latter was also tried. With
the aluminium window of the tube distant from the spark
about 10 cms., an appreciable effect could be seen ; when the
tube was in good working order and the spark in its most
sensitive condition, extinction of the latter could be readily
brought about. If the cathode rays inside the tube were
deflected from the window by means of a magnet, so that no
Lenard rays existed outside, no action on the spark could be
seen. In fact the discharge from the Wimshurst could
be started and stopped quite readily by simply approaching
a magnet to the cathode of the Lenard tube.

The ionizing action of the rays as tested by an electroscope
10 cms. away from the window, was more than that of the
radium 50 cms. away, but less than that of the Röntgen rays
when the tube emitting the latter was 25 cms. away.

This experiment also shows that the failure of the Röntgen
rays to put out the spark cannot be ascribed to their
intermittence, as both Lenard and Röntgen tubes were
worked by the same coil. It would also tend to prove that
the effects are dependent on the high velocity of the β rays,
rather than directly on the number of ions they produce in
the gap ; for the ionization produced by the Lenard rays is
here greater than that produced by the radium, while it
is well known that their velocity is much less than that of
the β rays that the radium emits.

In conclusion we must express our gratitude to Mr. J. J.
Vezey for the loan of some radium bromide, with which the
preliminary experiments were performed, and also to the
Governors and Principal of the Institute for providing us
with the radium bromide necessary to continue those ex-
periments. We are also indebted to Mr. F. G. Bratt for
considerable help in the experimental part of the investigation.

Sir John Cass Technical Institute, Jewry St., E.C.
November 10, 1904.

XXXVII. *Notices respecting New Books.*

An Introduction to the Theory of Optics. By ARTHUR SCHUSTER,
  Ph.D., Sc.D., F.R.S., *Professor of Physics at the University of
  Manchester.* London : Edward Arnold. 1904. Pp. xv + 340.

THIS important work is divided into two parts. In Part I. the
  author deals with the more elementary aspects of the subject,
while Part II. is intended as an introduction to the higher
branches of it.

The first two chapters are of an introductory nature, and deal
with simple periodic disturbances and their composition, and with
the kinematics and kinetics of wave motion. In Chapter III. the
author deals with some general principles, optical length and optical
distance, Fermat's principle, and the principle of reversibility.
Chapter IV. is devoted to interference effects, Chapter V. to dif-
fraction, and Chapter VI. to diffraction-gratings. The theory of
optical instruments, with special reference to resolving power and
brightness, is considered in Chapter VII. In Chapter VIII. we
have an account of the propagation of light in crystalline media,
while Chapter IX., which is the concluding chapter of Part I., deals
with the interference of polarized light.

In Part II. the author gives a sufficiently complete account of
the matters dealt with to place the reader who has mastered this
portion of the book in a position to study with profit the writings
of the leading authorities on the subject. Chapter X. begins with
a general mathematical account of strains and stresses, and then
deals with the equations of motion in a disturbed medium, and the
electromagnetic theory of light. Dispersion and absorption are
next considered in Chapter XI. Chapter XII. contains an account
of rotatory effects, including the rotation of the plane of polar-
ization, rotatory dispersion, and the Zeeman effect. In Chapter
XIII. the author considers the propagation of energy through
a medium, and in the concluding Chapter XIV. the nature of
light.

The work is a contribution to optical literature for which all
earnest advanced students of the subject will feel deeply grateful
to the author: it combines clearness of expression, comparative
simplicity and striking originality of treatment, with a keenness of
critical insight but seldom to be found in works of a similar scope.

The Electric Furnace. By HENRI MOISSAN, *Membre de l'Institut,
  Professor of Chemistry at the Sorbonne.* Translated by A. T.
  DE MOUILPIED, B.Sc., Ph.D., Assistant Lecturer in Chemistry in
  the University of Liverpool. London : Edward Arnold. 1904.
  Pp. xii + 308.

CHEMISTS and electro-chemists whose knowledge of French is too
imperfect to enable them to read Professor Moissan's great work
on the electric furnace in the original will welcome the appearance
of an English translation of it. The name of Moissan is so well

known in connection with his epoch-making researches on the behaviour of various substances at extremely high temperatures, and the formation, at such temperatures, of previously unknown compounds, possessing many remarkable properties, that an account of his investigations will be full of interest to many readers who do not intend making a special study of the subject, but who would, nevertheless, like to acquire some knowledge of the methods employed and the results achieved by the great French investigator.

In Chapter I. the author gives an interesting account of the various types of electric furnace devised by him, and of the special class of work to which each type is best adapted. Chapter II. contains a systematic and exhaustive account of the three varieties of carbon. Chapter III. deals with the preparation, by means of the electric furnace, of a number of elements—chromium, manganese, molybdenum, tungsten, uranium, vanadium, zirconium, titanium, silicon, and aluminium. Chapter IV. is devoted to an account of the new class of compounds formed at these extremely high temperatures—carbides, silicides, borides, phosphides, and arsenides. Of particular interest in this chapter are the suggestions put forward by the author regarding the origin of certain kinds of petroleum and the cause of volcanic eruptions as being due to the action of water on easily decomposable metallic carbides. Chapter V. contains an account of the most recent researches on carbides, silicides, and borides.

Valuable as is the service rendered by the translator, we regret to say that the work of translation leaves, in our opinion, a good deal to be desired. The determination, on the part of the translator, to follow the original with absolute slavishness has in a number of instances resulted in extreme clumsiness and obscurity of expression, and on page 4 we actually read of " a *force* of 2000 H.P.," a phrase which is surely absolutely inexcusable in any writer with the slightest pretensions to scientific accuracy.

Index of Spectra. Appendix O. *By* W. MARSHALL WATTS, D.Sc., F.I.C. Manchester: Abel Heywood & Son. 1904. Pp. 40.

THIS appendix to the author's *Index of Spectra* contains the arc-spectrum of Molybdenum, and the spark-spectra of Calcium, Scandium, Indium, Beryllium, Lithium, Thallium, Potassium, Caesium, Antimony, Arsenic, and Radium.

Grundzüge der Theorie der Optischen Instrumente nach Abbe. Von Dr. SIEGFRIED CZAPSKI. Zweite Auflage. Unter Mitwirkung des Verfassers und mit Beiträgen von M. von ROHR, herausgegeben von Dr. O. EPPENSTEIN. Leipzig: Johann Ambrosius Barth. 1904. Pp. xvi + 480.

THE first edition of this well-known standard work, which was a reprint of the articles contributed by Dr. Czapski to Winkelmann's *Handbuch der Physik*, appeared about eleven years ago. When a new edition of the latter work was called for, Dr. Czapski found

himself unable personally to undertake the task of revision, and
this was entrusted to Dr. Eppenstein, with the co-operation of
M. von Rohr. The new edition fully maintains the reputation of
its predecessor, and is sure to find an honoured place on the
shelves of every physical library.

One of the most important features of the work is the extreme
generality of the theory of image formation as developed by Abbe.
It is interesting to point out the fact, mentioned in the present
edition, that the germs of this theory are contained in a neglected
paper of Clerk Maxwell's, " On the General Laws of Optical
Instruments," contributed to the Philosophical Magazine in 1856.

It will be convenient to present in brief outline the contents of
the present volume. The introductory section deals with the laws
of geometrical optics. This is followed by Abbe's geometrical
theory of image formation. Then comes a section dealing with the
formation of optical images by means of thin axial and oblique
pencils. Spherical and chromatic aberrations are next considered.
The formation of images by means of prisms and prism systems,
and the conditions of achromatism, are then dealt with. The
next few sections are devoted to the important subjects of apertures
and the effects depending on them, the optical system of the eye
and vision. This latter subject is dealt with by M. von Rohr, who
also contributes the two succeeding sections on the photographic
objective and spectacles. The sections on the magnifying-glass
and the compound microscope are due to Czapski, and are followed
by two sections on projection systems and methods of illumination,
contributed by Eppenstein. The telescope receives very careful
treatment at the hands of Czapski, to whom is also due the
excellent concluding section on the methods of determining
experimentally the constants of optical instruments.

The book forms a veritable mine of information on many
important points either not dealt with at all or else very perfunc-
torily in the ordinary type of text-book, and it contains numerous
references to the literature of the subject. The publishers are to
be congratulated on the clearness of the type and excellence of
the illustrations.

La Statique Chimique. Basée sur les deux Principes Fondamentaux
de la Thermodynamique. Par le Lieutenant-Colonel E. ARIÈS.
Paris: A. Hermann. 1904. Pp. viii + 251.

THIS work is intended as an introduction to the methods of modern
theoretical chemistry, in which thermodynamical principles play so
important a part. The author deals only with reversible operations,
and confines his attention to those cases in which temperature and
pressure are the two independent variables. The wisdom of the
restrictions which the author thus voluntarily imposes on himself
is fully apparent in the extreme lucidity of treatment which
characterises the book. And by the thorough discussion of a
number of comparatively simple concrete cases, the reader is enabled
to gain an insight into the princples of the science which a more

abstruse and general discussion would fail to instil into him. We can strongly recommend the book to all students of chemical theory.

The first five chapters deal with general theoretical considerations, in which the notion of chemical potential plays an important part. The remainder of the book is devoted to applications and illustrations of the theory. In Chapters VI. to XI. the author studies changes of physical state and analogous phenomena, various cases of dissociation, solution, systems having only a single phase of variable composition, and mixtures which arrange themselves in layers. The last six chapters (XII. to XVII.) are devoted to the theory of gases, dilute solutions, and osmosis.

Percentage Tables for Elementary Analysis. By LEO F. GUTTMANN, *Ph.D., A.C.G.I., A.I.C.* London: Whittaker & Co. 1904. Pp. vii + 43.

THESE tables will be found extremely useful in calculating the percentage of carbon and hydrogen in a substance which on combustion yields carbon dioxide and water. We would suggest the omission of the " so " before " multiply " and " divide " on p. vii, this form of expression being a German idiom but not an English one.

Mémoire sur la Reproduction Artificielle du Rubis par Fusion. Par A. VERNEUIL. Paris: Gauthier-Villars. 1904. Pp. 30.

THIS pamphlet, which is reprinted from the *Annales de Chimie et de Physique,* contains an interesting account of the author's experiments on the artificial production of rubies. The problem, it seems, is an extremely difficult one, and although very small rubies were obtained whose physical and chemical characteristics were indistinguishable from those of the natural product, yet all attempts to produce larger sizes failed.

XXXVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 288.]

December 21st, 1904.—J. E. Marr, Sc.D., F.R.S., President, in the Chair.

THE following communications were read:—

1. 'On Certain Genera and Species of Lytoceratidæ.' By S. S. Buckman, F.G.S.

2. 'The Leicester Earthquakes of August 4th, 1893, and June 21st, 1904.' By Charles Davison, Sc.D., F.G.S.

The earthquake of 1893 was a twin, with its principal epicentre between Markfield and Woodhouse Eaves, and the other near Tugby, about 17 miles to E. 34° S. Its disturbed area contains about 2200 square miles. In the north-western focus, the originating fault hades to the north-east, and is probably one of the group of

faults on the north-east side of the Charnwood anticlinal axis, which appear to hade in the direction given. In the south-eastern focus, the fault hades in the opposite direction, the change of hade taking place a short distance to the south-east of Leicester.

On June 21st, 1904, two shocks were felt : the first, a very slight one, at about 3.30 A.M.; the second at 5.28 A.M. The epi-centre of the earlier shock was in the neighbourhood of Markfield and Groby, or near the south-eastern margin of the north-western focus of 1893. The second shock disturbed an area of about 1200 square miles, and was a double shock, not a twin, for the foci were overlapping. It originated in a fault running nearly north-west and south-east, hading to the south-west, and passing a short distance from Tugby, and therefore in all probability coincident with the fault in action in 1893. The distance between the epi-centres of the earthquakes of 1904 was about 12 miles. Thus, the foci of 1904 appear to have occupied the nearer margins of the foci of 1893.

3. 'The Derby Earthquakes of July 3rd, 1904.' By Charles Davison, Sc.D., F.G.S.

Although weaker than the earthquake of March 24th, 1903, this shock, owing to its occurrence at 3.21 on a Sunday afternoon, was felt over a much wider area (about 25,000 square miles). As in 1903, the earthquake was a twin, the epicentres being almost exactly coincident with those of that year, one being situated near Ashbourne, and the other, about 6 or 7 miles from it, near Wirks-worth and Matlock Bath. The impulse at the south-western or Ashbourne focus was slightly stronger than the other, and took place a second or two later. The principal slip was preceded by a slighter one in the north-eastern focus at 2.28 P.M., and was followed, as in 1903, though after a shorter interval, by a slip in the inter-focal region of the fault, at 11.8 P.M.

4. 'Twin-Earthquakes.' By Charles Davison, Sc.D., F.G.S.

In a twin-earthquake, the shock consists of two maxima of intensity, or of two distinct parts separated by a brief interval of rest and quiet. In Great Britain, one in every twenty earth-quakes is a twin, and our strongest shocks (the Colchester earth-quake of 1884, the Hereford earthquake of 1896, etc.) belong to the same class.

The two parts of a twin-earthquake differ in their order of intensity, both in different earthquakes and in different parts of the disturbed area of the same earthquake. The interval between the two parts varies on an average from 2 to $3\frac{1}{2}$ seconds; and, although the twin-shock is felt over a very wide area (sometimes over nearly the whole of the disturbed area), there may exist within it a band (the synkinetic band) in which the two parts coalesce and form a single shock.

These phenomena show that twin-earthquakes cannot be caused by reflection or refraction of the earth-waves, nor by the separation of the waves of direct and transverse vibrations, nor by the repetition

of the impulse within the same or an overlapping focus. They must therefore be due to impulses in two detached, or practically detached, foci; and it is shown that all the known phenomena of twin-earthquakes can be thus accounted for.

In British twin-earthquakes, the distance between the epicentres varies from 4 to 23 miles, the average for seven recent earthquakes being between 10 and 11 miles. As a rule, the foci are elongated approximately in the direction of the line joining them, showing that they are portions of the same fault. The foci appear to be situated at different depths, and, in two cases, the fault probably changes hade in the region between them.

The existence of the synkinetic band and the brevity of the interval between the two parts of the shock, show that, when the impulses are not absolutely simultaneous, the second takes place as a rule before the vibrations from the first focus have time to reach the other. Thus, twin-earthquakes are due to a single generative effort; and they are caused, it is urged, by the growth of a crust-fold cut transversely by a fault. If the movement is such that the anticline of the fold is made more anticlinal and the syncline more synclinal, the middle limb will remain practically undisturbed; so that, by a single effort, there will be two impulses in detached foci situated at different depths. The average distance between successive anticlines and synclines in France along several lines lies between 9 and 12 miles, which agrees closely with the average distance between the epicentres of twin-earthquakes.

January 4th, 1905.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

The following communications were read :—

1. 'The Marine Beds in the Coal-Measures of North Stafford-shire.' By John T. Stobbs, Esq., F.G.S.

2. 'The Geology of Cyprus.' By Charles Vincent Bellamy, M.Inst.C.E., F.G.S. With Contributions by Alfred John Jukes-Browne, B.A., F.G.S.

The following are the chief formations described :—

PLEISTOCENE.	Sandy limestones. Sands and conglomerates.	Up to 50 feet.
PLIOCENE.	Shelly limestones, with calcareous sands and sandstones.	100 to 150 feet.
Break.	Intrusion of igneous rocks.	
IDALIAN (OLIGOCENE).	White chalky limestones and marly chalks, with layers and nodules of flint.	About 1000 feet.
	Grey and yellowish marls, with beds of gypsum.	400 feet.
KYTHRÆAN (UPPER EOCENE).	Grey felspathic sandstones and sandy shales.	? 1500 feet.
Break.		
TRYPANIAN (? CRETACEOUS).	Grey limestones and dolomites; white and pink marbles, both massive and laminated.	Possibly 6000 feet,

The Kyrenia Mountains form the northern rampart of the Central Plains, and they rise abruptly to heights of over 3000 feet from the plain to the south and the sea to the north. They are composed of rocks tilted into a vertical position, much altered by compression and intrusion, and are devoid of fossils. They are referred by Prof. Gaudry to the Cretaceous Period, and are compared by him with the Hippurite-Limestones of Attica. They are pierced and marmorized by igneous rocks of quartz-felsite type. In some places they are schistose. Similar rocks are found in the extreme south-west of the island.

The Kythræan rocks attain a great development along the southern side of the Kyrenia Mountains. They are based on breccias and conglomerates made up of fragments of the Trypanian Limestones, and are thus probably unconformable. No fossils, except a few small tests of *Globigerina* and a few fragments possibly of shells, have been found in this Series, which consists entirely of volcanic débris embedded in a calcitic matrix, like the 'Macignos' of Italy.

The Idalian Series appears to rest conformably on the last. The gypsum-beds are largely developed in the south ; the white chalky marls and limestones extend over nearly one half of the island, and are always conspicuous from their intense whiteness. Fora-minifera are abundant, and other fossils have been found which indicate that the beds are mainly of Oligocene age.

Igneous rocks are most conspicuous in the centre of the island, forming a mass of mountainous ground. They are intrusive into the formations already mentioned. The rocks include augite-syenite, rhyolite, liparite, olivine-dolerite, basalt, augitite, and several varieties of serpentine. Considerable metamorphism and much jasperization occur in association with them.

While most of the older sediments were formed in deep water, the fauna of the Neogene rocks is that of shallow-water deposits. Miocene rocks have only been recognized in the south-east of the island. The Pliocene strata lie in horizontal or slightly-inclined beds, resting unconformably upon all older rocks. The Pleistocene rocks comprise :—(1) Littoral limestones and pebble-beds, (2) soft shelly sands, (3) inland surface-conglomerates, (4) cave-earths ; and (5) alluvial deposits. The cave-earths have yielded *Hippopotamus minutus* and *Elephas Cypriotes* to Miss D. M. Bate. An account of the chief economic mineral products of the island is given, including the marbles, clays, limestones, gypsum, building- and paving-stones, road-metals, metalliferous minerals, pozzuolana, terre-verte ; and a note on the springs, salt-lakes, and soils. Descriptions of some of the rocks, a note on the Miocene rocks, and a sketch of the physical history of the island are contributed by Mr. Jukes-Browne.

January 18th.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

The following communication was read :—

' On the Geology of Arenig Fawr and Moel Llyfnant.' By
William George Fearnsides, M.A., F.G.S.

This paper contains a detailed description of the succession of
beds in Sedgwick's typical area of development of his Arenig
Series. The following beds are described :—

Orthis-Limestone.
Thin ashy shales.
Upper ashes of Arenig and Llyfnant.
Daerfawr Shales, with a few graptolites, apparently of the *Didymograptus-*
  Murchisoni Zone.
Upper platy ashes, passing down into :— .
Agglomerate.
Lower platy ashes.
Didymograptus-bifidus Beds : blue shales with occasional ashes, passing
  down into : —
Didymograptus-hirundo Beds : dark calcareous shales, with some ashes.
Calymene-Ashes, with some calcareous material and occasional shaly beds.
Didymograptus-extensus Flags : quartzose flags with shaly partings.
Basal grit.

    Unconformity,

Shumardia-Shales, passing down into :—
Asaphellus-Flags.
Bellerophon-Beds : hard shales, with *Parabolinella triarthrus* in the upper
  part.
Dictyonema-Band, } not well exposed.
Niobe-Beds,   }
Peltura-Beds : *Peltura scarabæoides* abundant in the middle part. These
  are separated by thin beds with *Orthis lenticularis* from :—
Parabolina-Beds, with *P. spinulosa.*
Lingula-Beds, with *Lingulella Davisii.*

The unconformity marks the base of the Arenig Series. The
Orthis-Limestone yields Lower Caradoc-Sandstone forms, and dips
beneath the great Black-Shale Series, which occupies the adjoining
belt of country.

The author discusses the relationship of these divisions to
corresponding beds of other areas. He gives a description of
the intrusive igneous rocks, and some account of the structure
of the district and the nature of its glaciation.

February 1st..—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

The following communication was read :—

' On the Sporangium-like Organs of *Glossopteris Browniana*,
Brongn.' By E. A. Newell Arber, M.A., F.L.S., F.G.S.

High-Frequency Alternator.   Two-fifths full size.
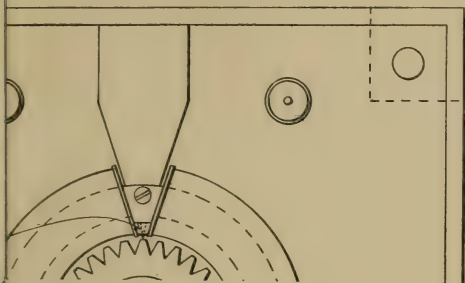
Fig. 1.—High-Frequency Alternator. Two-fifths full size.

Sight Feed Lubricator
Screws in here

The Reduction Gearing
for the Speed Indicator
is Lubricated by means
of a Pad in the end of
this Tube.

Spindle to drive
Speed Indicator

PART SECTIONAL ELEVATION

Fig. 2.—High-Frequency Alternator. Two-fifths full size.

Armature Coil
on Pole Tip

PLAN
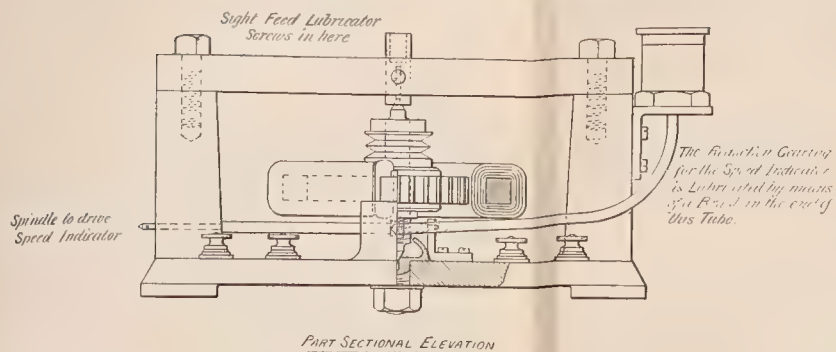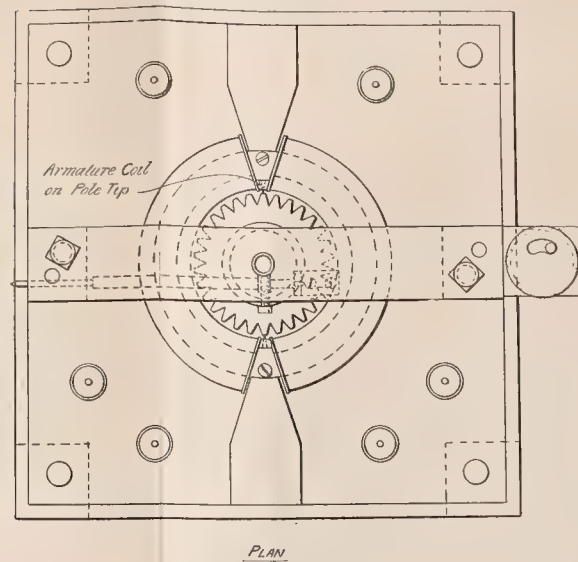
Fig. 3.—First Method of driving High-Frequency Alternator. 1 inch = 1 foot.
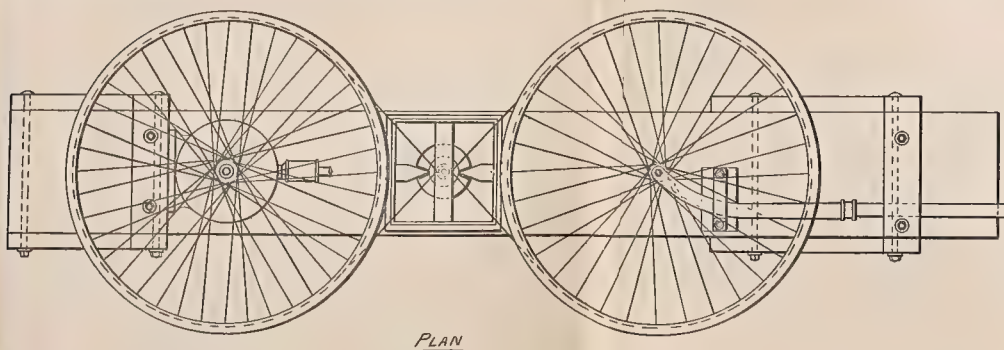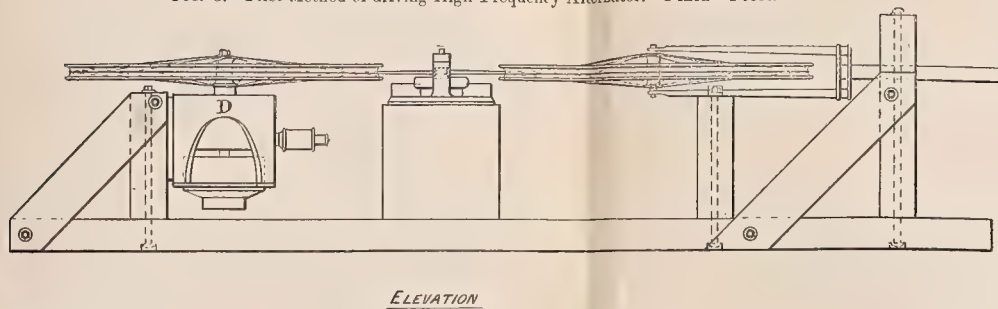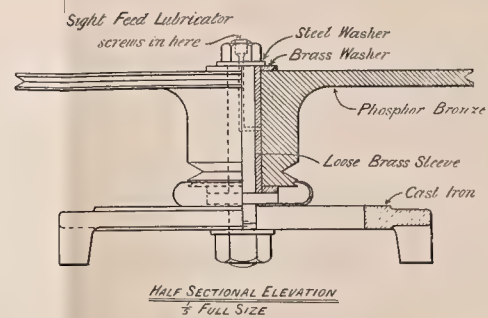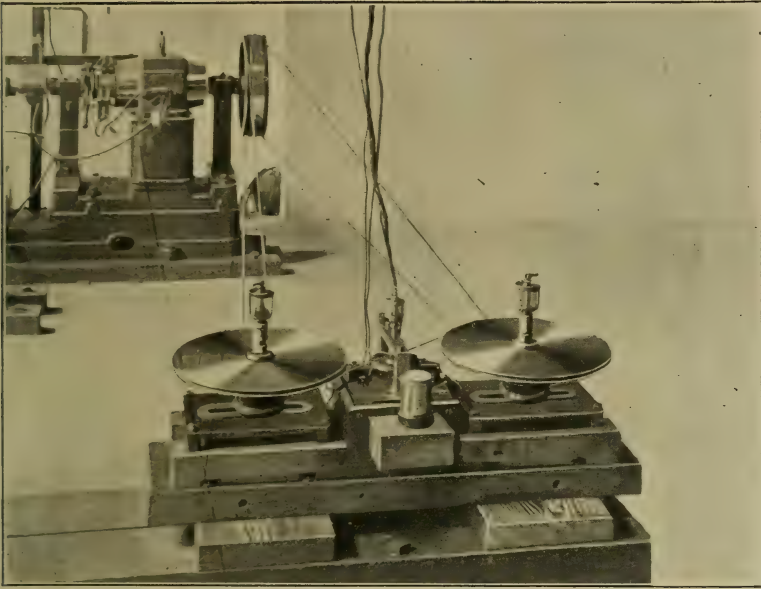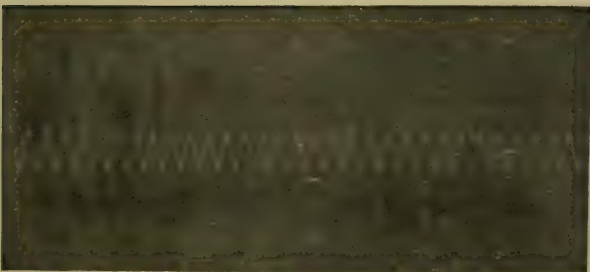
D

ELEVATION

PLAN

Fig. 5.

Sight Feed Lubricator
screws in here

Steel Washer
Brass Washer

Phosphor Bronze

Loose Brass Sleeve

Cast Iron

HALF SECTIONAL ELEVATION
½ FULL SIZE

Fɪɢ. 4.

Fɪɢ. 6.

XXXIX. *Radiation Pressure.*
By J. H. POYNTING, *Sc.D.*, *F.R.S.*[*]

A HUNDRED years ago, when the corpuscular theory held almost universal sway, it would have been much more easy to account for and explain the pressure of light than it is today, when we are all certain that light is a form of wave-motion. Indeed, on the corpuscular theory it was so natural to expect a pressure that numerous attempts were made [†] in the eighteenth century to detect it. But the early experimenters had a greatly exaggerated idea of the force they looked for. Even on their own theory it would only have double the value which we now know it to possess, and their methods of experiment were utterly inadequate to show so small a quantity. But had these eighteenth-century philosophers been able to command the more refined methods of today, and been able to carry out the great experiments of Lebedew and of Nichols and Hull, and had they further known of the emission of corpuscles revealed to us by the cathode stream and by radioactive bodies, there can be little doubt that Young and Fresnel would have had much greater difficulty in dethroning the corpuscular theory and setting up the wave theory in its place.

The existence of pressure due to waves, though held by

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[*] Communicated by the Physical Society; being the Presidential Address, delivered at the Annual General Meeting, February 10, 1905.

[†] Some account of these methods is given by Nichols and Hull in "The Pressure due to Radiation," Proc. Am. Ac. xxxviii. No. 20, p. 559. See also Priestley, "On Vision," p. 385.

Euler and used by him 160 years ago to explain the formation of comets' tails by repulsion, seems to have dropped out of sight, till Maxwell, in 1872, predicted its existence as a consequence of his Electromagnetic Theory of Light. It is remarkable that it should have been brought to the front through the investigation of such a special type, such an abstruse case, of wave-motion, and that it was not seen that it must follow as a consequence of any wave-motion, whatever the type of wave we suppose to constitute Light. I believe that the first suggestion that it is a general property of waves is due to Mr. S. Tolver Preston, who in 1876 * pointed out the analogy of the energy-carrying power of a beam of light with the mechanical carriage by belting, and calculated the pressure on the surface of the Sun by the issuing radiation, obtaining a value equal to the energy-density in the issuing stream, without assumption as to the nature of the waves. But though the analogy is valuable, I confess that Mr. Preston's reasoning does not appear to me conclusive, and I think it still remains an analogy. There is, I suspect, some general theorem yet to be discovered, which shall relate directly the energy and the momentum issuing from a radiating source. It seems possible that in all cases of energy transfer, momentum in the direction of transfer is also passed on, and therefore there is a back pressure on the source. Such pressure certainly exists in material transfer, as in the corpuscular theory. It exists too, as we now know, in all wave transfer. From the investigation below (p. 397) it appears to exist when energy is transferred along a revolving twisted shaft. In heat-conduction in gases, the kinetic theory requires a carriage of momentum from hotter to colder parts; so that there is some ground for supposing the pressure to exist in all cases.

Though we have not yet a general and direct dynamical theorem accounting for radiation pressure, Professor Larmor† has given us a simple and most excellent indirect mode of proving the existence of the pressure, which applies to all waves in which the average energy-density for a given amplitude is inversely as the square of the wave-length. Let us suppose that a train of waves is incident normally on a perfectly reflecting surface. Then, whether the reflecting surface is at rest, or is moving to or from the source, the perfect reflexion requires that the disturbance at its surface shall be annulled by the superposition of the direct and reflected trains. The two trains must therefore have

* Engineering, 1876, vol. xxi. p. 83.
† *Encyc. Brit.* xxxii. Radiation, p. 121.

equal amplitudes. Suppose now that the reflector is moving forwards towards the source. By Doppler's principle, the waves of the reflected train are shortened, and so contain more energy than those of the incident train. This extra energy can only be accounted for by supposing that there is a pressure against the reflector, that work has to be done in pushing it forward. When the velocity of the reflector is small, the pressure is easily found to be equal to $E\left(1 + \dfrac{2u}{U}\right)$, where $\dfrac{E}{2}$ is the energy-density just outside the reflector in the incident train, U is the wave-velocity, and u the velocity of the reflector. If $u = 0$, the pressure is E; but it is altered by the fraction $\dfrac{2u}{U}$ when the reflector is moving, and the alteration changes sign with u. A similar train of reasoning gives us a pressure on the source, increased when the source is moving forward, decreased when it is receding.

It is essential, I think, to Larmor's proof that we should be able to move the reflecting surface forward without disturbing the medium except by reflecting the waves. In the case of light-waves it is easy to imagine such a reflector. We have to think of it as being, as it were, a semipermeable membrane, freely permeable to æther, but straining back and preventing the passage of the waves. In the case of sound-waves, or of transverse waves in an elastic solid, it is not so easy to picture a possible reflector. But for sound-waves I venture to suggest a reflector which shall freeze the air just in front of it, and so remove it, the frozen surface advancing with constant velocity u. Or perhaps we may imagine an absorbing surface which shall remove the air quietly by solution or chemical combination. In the case of an elastic solid, we may perhaps think of the solid as melted by the advancing reflector, the products of melting being passed through pores in the surface and coming out to solidify at the back.

Though Larmor's proof is quite convincing, it is, I think, more satisfying if we can realize the way in which the pressure is produced in the different types of wave-motion.

In the case of electromagnetic waves, Maxwell's original mode of treatment is the simplest, though it is not, I believe, entirely satisfactory. According to his theory, tubes of electric and of magnetic force alike, produce a tension lengthways and an equal pressure sideways, equal respectively to the electric and magnetic energy-densities in the tubes. We regard a train of waves as a system of electric and magnetic tubes transverse to the direction of propagation, each kind

pressing out sideways—that is, in the direction of propagation. They press against the source from which they issue, against each other as they travel, and against any surface upon which they fall. Or we may take Professor J. J. Thomson's point of view *. " Let us suppose that the reflecting surface is metallic ; then, when the light falls on the surface, the variation of the magnetic force induces currents in the metal, and these currents produce opposite effects to the incident light, so that the inductive force is screened off from the interior of the metal plate : thus the currents in the plate, and therefore the intensity of the light, rapidly diminish as we recede from the surface of the plate. The currents in the plate are accompanied by magnetic force at right angles to them ; the corresponding mechanical force is at right angles both to the current and the magnetic force, and therefore parallel to the direction of propagation of the light." In fact, we have in the surface of the reflector a thin current-sheet in a transverse magnetic field, and the ordinary electrodynamic force on the conductor accounts for the pressure.

In sound-waves there is at a reflecting surface a node—a point of no motion, but of varying pressure. If the variation of pressure from the undisturbed value were exactly proportional to the displacement of a parallel layer near the surface, and if the displacement were exactly harmonic, then the average pressure would be equal to the normal undisturbed value. But consider a layer of air quite close to the surface. If it moves up a distance y towards the surface, the pressure is increased. If it moves an equal distance y away from the surface, the pressure is decreased, but by a slightly smaller quantity. To illustrate this, take an extreme case, and for simplicity suppose that Boyle's law holds. If the layer advances half way towards the reflecting surface, the pressure is doubled. If it moves an equal distance outwards from its original position, the pressure falls, but only by one-third of its original value ; and if we could suppose the layer to be moving harmonically, it is obvious that the mean of the increased and diminished pressures would be largely in excess of the normal value. Though we are not entitled to assume the existence of harmonic vibrations when we take into account the second order of small quantities, yet this illustration gives the right idea. The excess of pressure in the compression half is greater than its defect during the extension half, and the net result is an average excess of pressure—a quantity itself of the second order—on the reflecting surface.

* Maxwell's ' Electricity and Magnetism,' 3rd edition, vol. ii. p. 441, footnote.

This excess in the compression half of a wave-train is connected with the extra speed which exists in that half, and makes the crests of intense sound-waves gain on the troughs.

Lord Rayleigh[*], using Boyle's Law, has shown that the average excess on a surface reflecting sound-waves should be equal to the average density of the energy just outside ; and I think the same result can be obtained by his method if we use the adiabatic law. But the subject is full of pitfalls, and I am by no means sure that the result is to be obtained so easily as it appears to be. It is perhaps worth while to note one of these pitfalls, of which I have been a victim. It is quite easy to obtain the pressure against a reflecting surface by supposing that the motion just outside it is harmonic. But the result comes out to $(\gamma + 1)$ energy-density, where γ is the ratio of the specific heats. Lord Rayleigh kindly pulled me out of the pit into which I fell, pointing out that when we take into account second-order quantities the ordinary sound equation does not hold. In fact we cannot take the disturbance as harmonic, and the simple mode of treatment is illusory.

The pressure in transverse waves in an elastic solid is, I think, to be accounted for by the fact that when a square,

Fig. 1.
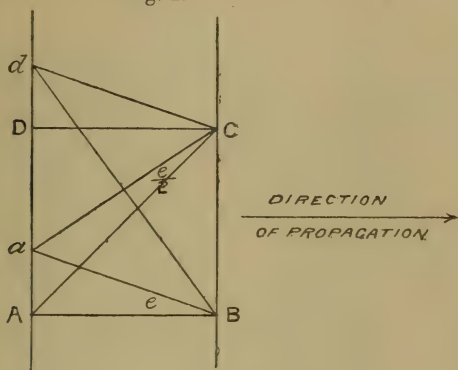
ABCD, is sheared into the position aBCd (fig. 1) through an angle e, the axes of the shear, aC and Bd, no longer make $45°$ with the planes of shear AD, BC. Since $ACa = \dfrac{e}{2}$, the pressure-line aC is inclined at $45° - \dfrac{e}{2}$ to the direction of propagation, and the tension-line at $45° + \dfrac{e}{2}$ to that line.

[*] Phil. Mag. iii. 1902, p. 338, " On the Pressure of Vibrations."

The result is a small pressure perpendicular to the planes of shear, that is, in the direction of propagation; and this small pressure is just equal to the energy-density of the waves.

For let PQR (fig. 2) be a small triangular wedge of the solid, PQ being a plane of shear perpendicular to the direction of propagation. Let this wedge have unit thickness perpendicular to the plane of the figure. Let PR be along a pressure-line and QR along a tension-line, and let pressure and tension each be P. Resolve the forces on PR and QR perpendicular to PQ. Then we have a force from right to left,

Fig. 2.

$$\text{P . QR} \cos \text{PQR} - \text{P . PR} \cos \text{QPR}$$

$$= \text{P . PQ} \left(\cos^2 \left(45^\circ - \frac{e}{2} \right) - \cos^2 \left(45^\circ + \frac{e}{2} \right) \right)$$

$$= \text{P . PQ} \cdot e.$$

Thus, to prevent motion in the direction of propagation there must be a pressure on PQ equal to $\text{P}e = ne^2$, where n is the rigidity modulus. But the strain-energy per unit volume is $\frac{ne^2}{2}$, and the kinetic energy is equal to it. The total energy-density is therefore ne^2, and the pressure is equal to this.

The pressure of elastic solid waves appears to be beyond experimental verification at present. But that of sound-waves has been demonstrated most successfully by Altberg[*], working in Lebedew's laboratory at Moscow.

A small wood cylinder, 21 mm. diameter, was suspended at one end of a torsion arm, with its axis horizontal and transverse to the arm. One end of the cylinder occupied a circular hole in the middle of a board, there being just sufficient clearance to allow it to move; and the plane end was flush with the outer surface of the board. When very intense sound-waves 10 cm. in length, from a source 50 cm. distant, impinged on the board, the cylinder was pushed back, the pressure sometimes rising to as much as 0·24 dyne/sq. cm. The intensity of the sound was measured independently by the vibrations of a telephone-plate, in a manner devised by M. Wien; and through a large range it was found that the pressure on the cylinder was proportional to the intensity indicated by the telephone manometer.

Just lately Professor Wood[†] has devised a strikingly

[*] *Ann. der Physik*, xi. 1903, p. 405.
[†] *Phys. Zeitschrift*, 1 Jan. 1905, p. 22.

simple experiment to illustrate sound-pressure. The sound-waves from strong induction-sparks are focussed by a concave mirror on a set of vanes like those of a radiometer, and when the focus is on the vanes as they face the waves the mill spins round.

Theory and experiment, then, justify the conclusion that when a source is pouring out waves, it is pouring out with them forward momentum as well as energy, the momentum being manifested in the reaction, the back pressure against the source, and in the forward pressure when the waves reach an opposing surface. The wave train may be regarded as a stream of momentum travelling through space. This view is most clearly brought home, perhaps, by considering a parallel train of waves which issues normally from a source for one second, travels for any length of time through space, and then falls normally on an absorbing surface for one second. During this last second, momentum is given up to the absorbing surface. During the first second, the same amount was given out by the source. If it is conserved in the meanwhile, we must regard it as travelling with the train.

Since the pressure is the momentum given out or received per second, and the pressure is equal to the energy-density in the train, the momentum-density is equal to the energy-density $\div$ wave-velocity.

This idea of momentum in a wave train enables us to see at once what is the nature of the action of a beam of light on a surface where it is reflected, absorbed, or refracted, without any further appeal to the theory of the wave-motion of which we suppose the light to consist [*].

It is convenient to consider the energy per linear centimetre in the beam, and the total pressure force, equal to this linear energy-density, so as to avoid any necessity for taking into account the cross section of the beam.

Thus, in total reflexion, let a beam AB (fig. 3) be reflected along BC, and let AB = BC represent the momentum in each in length V equal to the velocity of light.

Produce AB to D, making BD = AB.

Then DC represents the change in the momentum per second due to the reflexion—the force on the beam, if such language is permissible ; and CD is the reaction, the total light-force on the surface.

If there is total absorption, let AB (fig. 4) represent the

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[*] A discussion, on the electromagnetic theory, of the forces exerted by light is given by Goldhammer, *Ann. der Phys.* 1901, 4. p. 483.

momentum of the incident beam. Resolve AB into AE parallel and EB normal to the surface. Then, since the momentum AB disappears as light-momentum, there must

Fig. 3.
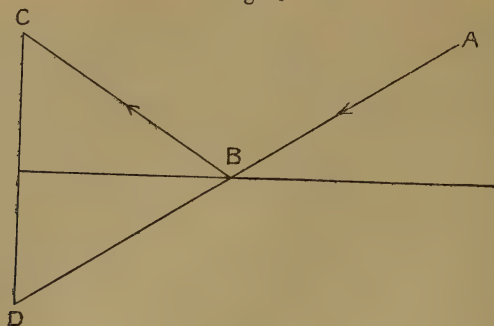
Fig. 4.
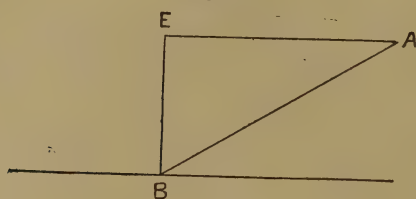
Fig. 5.
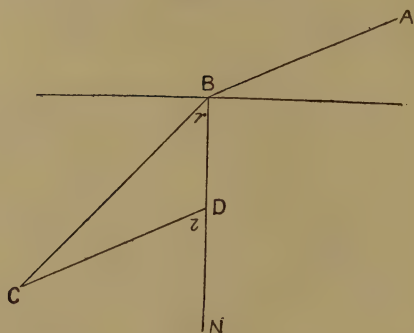
be a normal force EB on the surface and a tangential force AE parallel to the surface. I have lately* described an experiment which shows the existence of the tangential force AE.

* Phil. Mag. Jan. 1905, p. 169.

If there is total refraction, let AB (fig. 5) be refracted along BC. If E is the energy in unit length of AB, and if E′ is the energy in unit length of BC, the equality of energy in the two beams is expressed by

$$VE = V'E'.$$

But if M is the stream of momentum passing per second along AB, and if M′ is that along BC,

$$M = E \quad \text{and} \quad M' = E'.$$

Whence $\qquad VM = V'M'$

and $\qquad M' = \dfrac{V}{V'} M = \mu M.$

Let AB = M, and BC along the refracted beam = M′ = μM = μAB.

Draw CD parallel to BA, meeting the normal BN in D. Then

$$CD = CB \sin r / \sin i = \frac{CB}{\mu} = AB = M.$$

Hence by the refraction, momentum DC has been changed to momentum BC, or momentum BD has been imparted to the light. There is therefore a reaction DB on the surface. The force DB may be regarded as a pull-out or a pressure from within, and it is along the normal *.

If the refraction is from a denser to a rarer medium, CB will now represent the incident stream and BA or CD the refracted stream. BD is the stream added to CB to change it to CD, and DB is the force on the surface, again a force outwards along the normal.

In any real refraction with ordinary light, there will be reflexion as well as refraction. The reflexion always produces a normal pressure, and the refraction a normal pull. But with unpolarized light, a calculation shows that the refraction pull, for glass at any rate, is always greater than the reflexion push, even at grazing incidence.

The following table has been calculated from Fresnel's formula for unpolarized light by Dr. Barlow :—

* It has been pointed out by J. J. Thomson, 'Electricity and Matter,' p. 67, "that even when the incidence of the light is oblique, the momentum communicated to the substance is normal to the refracting surface." The change of momentum of a beam of light is, it may be noted, the same on the wave and on the corpuscular theory.

P = total pull on surface.
M = momentum per second in incident beam.
R = reflexion coefficient for angle i.
$\mu = 1\cdot5$.

i.	R.	P/M.
0	·0400	·4000
20	·0402	·4240
40	·0458	·4925
50	·0572	·5310
60	·0893	·5720
65	·1205	·5771 Maximum.
70	·1710	·5683
75	·2531	·5329
80	·3878	·4521
89	·9044	·0738
$90 - d\theta$	......	$2\mu^2 d\theta$
90	1·0000	·0000

If a ray of light passes obliquely through a parallel plate, there is a normal pull outwards at incidence and a normal pull outwards at emergence ; and if the refraction were total, this would result in a couple. But since some of the light returns into the first medium, it is easy to see that the net result is a normal repulsion and a couple.

An experiment which I have lately made in conjunction with Dr. Barlow will serve as an illustration of the idea of a beam of light regarded as a stream of momentum. A rectangular block of glass, 3 cm. × 1 cm. × 1 cm., was suspended

Fig. 6.—Plan.
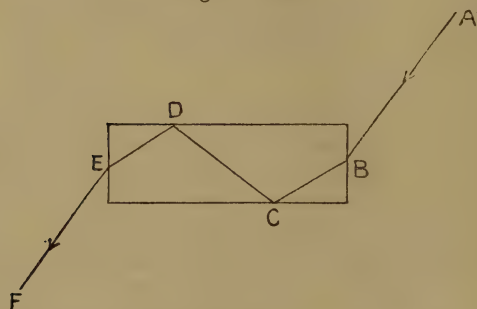
by a quartz fibre so that the long axis of the block was horizontal. It hung in a case with glass windows, which was exhausted to about 15 mm. of mercury. A horizontal beam of light from either a Nernst lamp or an arc was directed on to one end of the block so that it entered centrally as AB

in fig. 6, and at an angle of incidence about 55°. After two internal reflexions it emerged centrally as EF from the other end. Thus a stream of momentum AB was shifted parallel to itself into the line EF, or a counter-clockwise couple acted on the beam. The reaction was a clockwise couple on the block. Using mirror, telescope, and a millimetre-scale about 184 cm. distant, a very small deflexion could just be detected with the strongest light and in the right direction. But the quartz fibre was rather coarse, indeed needlessly strong ; and as the time of vibration was only 39 seconds, the deflexion was very minute. To render the effect more evident we used intermittent passage of the beam, sending it in during the half-period of vibration while B was moving from A, and shutting it off while B was moving towards A. The swings then always increased. When the beam was sent in during the approaching half and shut off during the receding half, the swings always decreased, and always rather more rapidly than they increased during the first half. For in the first case the natural damping acted against the light couple, and in the second with it. In one experiment the average increase was ·55 scale-division and the average decrease ·61 per period, and was fairly regular in each case. The mean was ·58. The steady deflexion is half this, or 0·29 division, giving a couple 11×10^{-6} cm.-dyne. We made a measurement of the energy in the beam by means of the rate of rise of a blackened silver disk ; but it was necessarily very inexact, as we had no means of securing constancy in the arc used in this experiment. This energy measurement gave as the value of the couple 6×10^{-6}, and the agreement is sufficient to show that the order of the result is right.

An analysis of this experiment shows that the couple was really due to the pressures at the two internal reflexions ; for, as we have seen, the forces at incidence at A and emergence at E are normal and produce no twist.

Another experiment which we have made is, I think, more interesting, in that it brings into prominence the pull outwards or push from within occurring on refraction. Two glass prisms, each with refracting angle 34°, another angle being a right angle, and with refracting edge 1·6 cm. long, were arranged as in fig. 7 (which shows the plan) at the ends of a thin brass torsion-arm suspended at its middle point from a quartz fibre in the same case as that used in the last experiment. The two inner faces were 3 cm. apart, and their width was 1·85 cm. A mirror gave the reflexion of a millimetre-scale 171·4 cm. distant. The moment of inertia of the system was 48 gm.-cm.², and the time of vibration was 317 seconds.

The air-pressure was reduced as before. When a beam of light from a Nernst lamp was sent through the system, as shown in the figure, it was shifted parallel to itself through a distance about 1·64 cm. The torsion-arm moved round clockwise by

Fig. 7.—Plan.
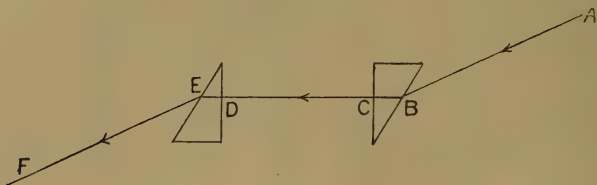
an easily measurable amount. In one experiment the deflexion was 3·3 scale-divisions, indicating a couple $1·84 \times 10^{-5}$ cm.-dyne. The same beam directed on to the blackened silver disk gave the linear energy-density as $9·8 \times 10^{-6}$, which should have given a couple $1·6 \times 10^{-5}$. Though the agreement is perhaps accidentally close, yet, as we could use a Nernst lamp, the measurements were much more trustworthy than in the last experiment.

The interesting point here is that the effect could only be produced by a force outwards at B and E. Whatever forces exist at C and D would be normal to the surfaces and would give no twist.

A very short experience in attempting to measure these light-forces is sufficient to make one realise their extreme minuteness—a minuteness which appears to put them beyond consideration in terrestrial affairs, though I have tried to show* that they may just come into comparison with radiometer-action on very small dust particles.

In the Solar system, however, where they have freer play and vast times to work in, their effects may mount up into importance. Yet not on the larger bodies ; for on the earth, assumed to be absorbing, the whole force of the light of the sun is only about a 50 million-millionth of his gravitation-pull. But since the ratio of radiation-pressure to gravitation-pull increases in the same proportion as the radius diminishes if the density is constant, the pressure will balance the pull on a spherical absorbing particle of the density of the earth if its radius is a 50 billionth that of the earth—a little over a hundred-thousandth of a centimetre, say, if its diameter is a hundred-thousandth of an inch.

* 'Nature,' Dec. 29, 1904, p. 200.

We may illustrate the possible effects of radiation pressure without proceeding to such fineness as this. Let us imagine a particle of the density of the earth, and a thousandth of an inch in diameter, going round the sun at the earth's distance. There are two effects due to the sun's radiation. In the first place, the radiation-push is $\frac{1}{100}$ of the gravitation-pull; and the result is the same as if the sun's mass were only 99/100 of the value which it has for larger bodies like the earth. Hence the year for such a particle would be longer by $\frac{1}{200}$, or about 367 instead of $365\frac{1}{4}$ days. In the second place, the radiation absorbed from the sun and given out again on all sides is crushed up in front as the particle moves forward and is opened out behind. There is thus a slightly greater pressure due to its own radiation on the advancing hemisphere than on the receding one, and this appears as a small resisting force in the direction of motion. Through this the particle tends to move in a decreasing orbit spiralling in towards the sun, and at first at the rate of about 800 miles per annum.

Further, if there be any variation in the sun's rate of emitting energy, there will be a corresponding variation in the increase of the year and the decrease of the solar distance, and the particle, if we could only observe it, would form a perfect actinometer.

Though, unfortunately, we cannot observe the motion of independent small particles circling round the sun at the distance of the earth, there is good reason to suppose that some comets at least are mere clouds of dust. If we are right in this supposition, they should show some of these effects. Encke's comet at once suggests itself as of this class; for, as everyone knows, it shortens its journey of $3\frac{1}{2}$ years round the sun on every successive return, and on the average by about $2\frac{1}{2}$ hours each revolution. Mr. H. C. Plummer [*] has lately been investigating this comet's motion; and he finds that if it were composed of dust particles, each of the earth's density and about $\frac{1}{50}$ mm. or rather less than a thousandth of an inch in diameter, the resisting force due to radiation pressure would account for its accelerating return. But the sun's effective mass would be reduced by about 1/80; and on certain suppositions he finds that the assumed mean distance as calculated from Kepler's law, without reference to radiation, is greater than the true mean distance by something of the order of 1 in 400, and he thinks such a large error is hardly possible. So

[*] Monthly Notices R.A.S., Jan. 1905, "On the Possible Effects of Radiation on the Motion of Comets, with special reference to Encke's Comet."

that radiation pressure has not yet succeeded in fully explaining the eccentricities of this comet. But comets are vague creatures. As Mr. Plummer suggests, we hardly know that we are looking at the same matter in the comet at its successive returns; and I still have some hope that the want of success is due to the uncertainty of the data.

There is one more effect of this radiation pressure which is worthy of note : its sorting action on dust particles. If the particles in a dust cloud circling round the sun are of different sizes or densities, the radiation accelerations on them will differ. The larger particles will be less affected than the smaller, will travel faster round a given orbit, and will draw more slowly in towards the sun. Thus a comet of particles of mixed sizes will gradually be degraded from a compact cloud into a diffused trail lengthening and broadening, the finer dust on the inner and the coarser on the outer edge.

Let us imagine, as an illustration of this sorting action, that a planet, while still radiating much energy on its own account, while still in fact a small sun, has somehow captured and attached to itself as satellite a cometary cloud of dust. Then, if the cloud consists of particles of different sizes, while all will tend to draw in to the primary, the larger particles will draw in more slowly. But if the larger particles are of different sizes among themselves, they will have different periods of revolution, and will gradually form a ring all round the planet on the outside. Meanwhile the finer particles will drift in, and again difference in size will correspond to difference in period and they too will spread all round, forming an inner fringe to the ring. If there are several grades of dust with gaps in the scale of size, the different grades will form different rings in course of time. Is it possible that here we have the origin of the rings of Saturn ?

The Radiation Theory is only just starting on its journey. Its feet are not yet clogged by any certain data, and all directions are yet open to it. Any suggestion for its future course appears to be permissible, and it is only by trial that we shall find what ways are barred. At least we may be sure that it deals with real effects and that it must be taken into account.

XL. *The Discharge of Electricity from Hot Platinum in Phosphorus Vapour.* By O. W. RICHARDSON, *M.A., D.Sc., Clerk Maxwell Student and Fellow of Trinity College, Cambridge* *.

IN the Philosophical Magazine for July 1903 it was shown that the positive leak from a fresh, clean platinum wire, maintained at a constant temperature *in vacuo,* fell off asymptotically with the time until it reached values of the order of one-thousandth of that it originally possessed. More recently † the author has been investigating the conditions under which a wire would recover its discharging power, with a view to explaining, if possible, the abnormal properties of new wires. Among other results, it has been found that a wire does not show any appreciable recovery if left *in vacuo* for a fortnight, or if exposed to the vapours from tap-grease (a mixture of vaseline and solid paraffin) or mercury. Further exposure to air at atmospheric pressure for periods of the order of one day only produces very small and rather uncertain effects.

In one case, however, an apparatus which had been showing no recovery when connected with one pump and phosphorus-pentoxide bulb, was found to behave quite differently when it was cut down and subsequently sealed on to another pump. As the possible sources of contamination appeared to be only tap-grease or phosphorus-pentoxide, and the former had been previously found to be harmless, experiments were tried to see whether the recovery was caused by some peculiarity in the latter.

To test this point, the pentoxide-bulb was heated by surrounding it with an asbestos-lined brass cylinder, which could be heated by a Bunsen burner. At the commencement of the experiments, the pressure inside the apparatus was very low ; and even after the bulb had been heated for some time it did not rise to more than about a hundredth of a millimetre. Thus, any increase in the current could not be due to ionization by collisions. The tube containing the wire was separated by nearly a metre of glass tubing which included a tap from the pentoxide-bulb, so that there was no danger of any of the contents of the latter being conveyed mechanically to the former. The platinum wire was maintained at a constant temperature, as measured by its resistance ; so that any alteration in the leak from it could

* Communicated by the Author.
† See Phil. Mag. [6] vol. viii. p. 400.

only be due to changes in the surrounding gas produced by heating the pentoxide-bulb.

Throughout this paper, in describing the leaks the unit of current is taken as 3.3×10^{-11} ampere. In measuring the leak the platinum wire was charged to a potential of ± 80 volts, and the neighbouring electrode connected to one quadrant of an electrometer, the other being earthed. The effect of heating the pentoxide-bulb is indicated by the following table :—

TABLE I.

Time.	Temperature of P_2O_5 bulb.	Current.	
		Wire at + 80 volts.	Wire at −80 volts.
3.38	18° C.	·35	·036
3.50	94° C.	20·4	
3.56	94° C.	......	·025
3.58	94° C.	55·2	
4.06	124° C.	26·8	
4.10	125° C.	......	·025
4.16	145° C.	......	·030
4.20	144° C.	87·5	
4.26	164° C.	133·0	

The table shows that the positive leak from the hot platinum is greatly increased by something given off when the P_2O_5 is heated, whereas the negative leak is unaffected. As a matter of fact, the values of the negative leak given above are just about equal to the insulation leak obtained with the wire cold, so that there is no evidence of the existence of any negative leak from a platinum wire in phosphorus vapour at this temperature. Throughout the experiments described in this paper, the temperature of the hot wire was always about 600°–700° C. The table also shows that the value of the positive leak increased with the temperature of the P_2O_5 bulb: any irregularities in this respect are probably to be attributed to the fact that the temperature of the bulb was not absolutely steady, but was sometimes going up and sometimes going down as the observations were taken, so that the effect in the testing arrangement might lag behind a good deal.

At the end of an experiment of this kind, it was found that the wire generally broke at some point, and on more minute examination it presented a beaded appearance. This was due to the formation of globules of a fusible substance which collected at regular intervals on the wire. Evidently a vigorous chemical action had occurred between the wire and the surrounding vapour ; the alteration in the properties of the wire caused by this prevented any very accurate measurements being taken.

It has been pointed out that this effect was first detected owing to a wire which had been left cold inside the apparatus gradually recovering its lost discharging power. As was to be expected, this phenomenon was shown in a higher degree if the pentoxide-bulb was heated. In one case, after the bulb had been heated and the wire allowed to remain cold for several days, the leaks at different times were found to be those given in the following table :—

TABLE II.

Time wire heated (minutes)......	0	3	6	9	15	25	30	36	42	46	52	53·5	55·5	58	61
Positive leak	142	86	57	36	28	17	11·5	6	7	5·2	1·2	2·9	2·1	1·9	1·8

The effect appears to fall off with time in the same sort of way that the positive leak from a new wire does. The striking alteration which gradually takes place in the properties of the wire, leads one to attribute the falling off in the leak rather to an absorption of the vapour by the wire, than to an emission of something previously absorbed. The wire was not subjected to quite the same conditions during the whole of the time represented by the last table. After taking the readings corresponding to 30 and 46 minutes, the temperature of the wire was raised, for the space of about two minutes, from about 700° C. to about 1200° C. It was subsequently reduced to the previous temperature, and the readings taken as usual. It will be noticed that in each case the leak starts by having a lower value than what we should have expected if the wire had not been heated ; the value subsequently increases for the space of two or three minutes, and finally decreases with time as before. These results can be reconciled with the view that the effect is due to the absorption of the vapour by the metal, if one supposes that the rate of absorption is greater at high than at low temperatures.

On examining the pentoxide-bulb which had been heated,

it was found to contain one or two yellowish patches in the part which had been converted into phosphoric acid by the absorption of water. This led to the suspicion that the effect was not due to phosphorus pentoxide, but to the lower oxides or to phosphorus itself. A new bulb was therefore set up containing the best phosphoric anhydride on the market. This was found to give an effect which, though easily capable of detection, was insignificant compared with that previously obtained ; clearly showing that the effect is not due to P_2O_5 itself. A blank bulb containing nothing at all was also tried, and found to give no effect even when heated to 400° C. The pure P_2O_5 was heated to about 270° C. and, subliming over the apparatus, caused the gas-pressure to rise to about $\frac{1}{40}$ mm., but only produced a slight increase in the positive leak. It was thought that possibly light as well as heat would be able to drive off the stimulating vapour ; and the effect of an arc-light on the original bulb was tried, but no effect was obtained.

An experiment was next tried in which a small quantity of phosphorus itself was introduced into the apparatus. A piece of yellow phosphorus was placed in a small bulb which was sealed on to the apparatus beyond the pentoxide-bulb (see figure). It was found that the wire, when set up under these conditions, exhibited a big initial leak. This result cannot, however, be regarded as decisive ; since it is often found that the wire possesses an abnormal emissibility for positive electricity after any great change has been made in the apparatus. The author hoped to obtain an unobjectionable result by heating the wire until as much as possible of this initial leak had been destroyed, and then heating the lump of phosphorus. It was thought that any increase in the leak thus produced could only be due to the effect of the phosphorus. Unfortunately, the action of the vapour from the cold phosphorus on the hot wire was so great that the wire came to grief long before any readings could be taken.

The above experiment made it evident that to be decisive this test must satisfy two conditions :—(1) It must be capable of being made very rapidly ; and (2) the only cause tending to give an increase of ionization at the moment of making the test must be the presence of phosphorus vapour. To satisfy these conditions, the apparatus was arranged so that the platinum wire could be heated in a vacuum quite independent of that containing the phosphorus vapour. In this way, the initial leak could be got rid of and the wire brought to the condition for making the test without the wire being rotted by the action of the phosphorus vapour. Then, on turning a

tap, the phosphorus vapour could be allowed to come in
contact with the hot wire, and the test made immediately.
In carrying out this experiment, it is important that the gas-
pressure on the two sides of the tap should be very low, say
about a hundredth of a millimetre, otherwise disturbances
might arise due to ionization by collision or—what it is still
more necessary to guard against—changes in the temperature
of the wire.

The general arrangement of the apparatus will be made
clearer by reference to the accompanying figure. The tap A
separates the right-hand portion containing the phosphorus,
P, from the left-hand portion containing the hot wire, W.
B and B′ are phosphorus-pentoxide drying-bulbs; the main
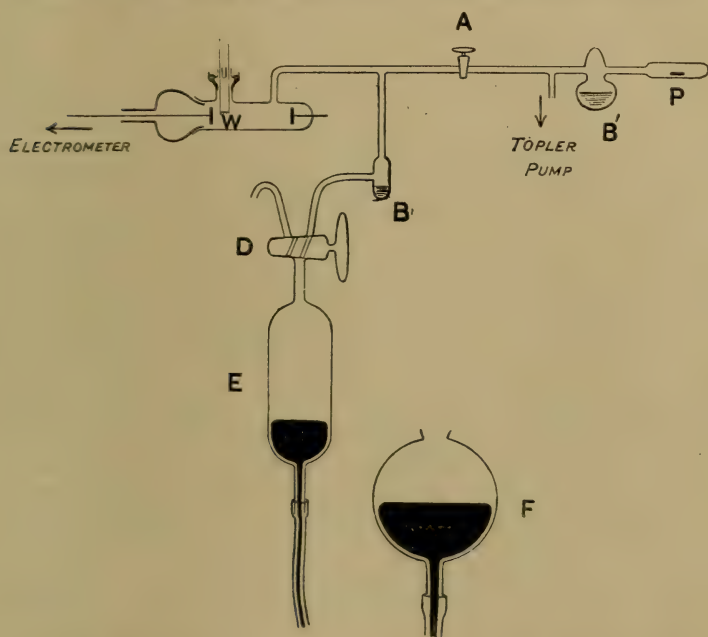
pump is not shown. The left-hand side of the apparatus was
exhausted by means of the tap and bulb arrangement DEF.
The action of this is sufficiently obvious from the figure; it
has been found very convenient where a pump is required
for a purely temporary purpose.

The following numbers refer to an experiment under these
conditions. A new platinum wire, cleaned by dipping into
nitric acid and heating in the Bunsen flame, on heating in a
good vacuum at a temperature of 680° C., gave an initial leak

of 312 of the units previously employed. After maintaining the wire at this temperature for 52 minutes, the leak had fallen to 7·5 units. On lowering the temperature to 610° C., the leak fell to 1·3 units. With the wire at this temperature the tap was turned and the phosphorus vapour admitted. The leak was now determined as quickly as possible, and found to be equal to 2175 units. Thus the phosphorus vapour increased the leak from the hot wire to seven times the value obtained from the wire when new, and to nearly two thousand times the value it gave immediately before the phosphorus was admitted. The phosphorus was now rapidly heated to 200° C., when the leak from the wire was found to have risen to 16,900 units, or more than fifty times the initial leak from a new wire. Before any more readings could be taken the wire broke.

This effect of phosphorus on a hot platinum wire is of especial interest as being one of the few instances where the leak from that metal is affected in a definite and metrical way by a foreign chemical substance. The above experiments are not complete enough to determine the nature of the action giving rise to the positive leak with absolute certainty, but there are some points which are worthy of further consideration. For instance, the results given in Table II. show that the effect accumulates if a cold wire is left exposed to the vapour. This might be due either to an absorption of the phosphorus by the cold platinum, or to a gradual formation of an increasing quantity of vapour inside the apparatus. In the latter case, the falling-off with time at constant temperature would probably be due to the gradual absorption of the vapour by the wire. But if the effect were proportional to the amount of a vapour which gradually accumulated, we should expect pumping to reduce it in the same ratio as it reduced the gas-pressure. Pumping was found to reduce the effect slightly, but not to this extent. For instance, in Table II. the pump was worked twice between the readings for 6 and 9 minutes. This corresponded to reducing the pressure to less than one half ; but the corresponding fall in the leak was only equal (proportionally) to that in the preceding interval of time, when no pumping had been done. If the pumping had not been done in the 6–9 minutes interval, the ratio of the final to the initial leak would have been slightly greater than in the preceding interval, owing to the rate of decay falling off with time. Another wire which had been heated, in presence of a pentoxide-bulb giving off phosphorus vapour, until the leak ceased to fall away perceptibly with time, gave the following values of the leak after successive strokes of the pump,

viz. :—·50, ·45, ·41, ·40, ·345, ·35,· 35 ; showing that, beyond
a certain extent, the leak is not further reduced by pumping
out the vapour. These experiments indicate that most of the
effect is not due to the phosphorus vapour surrounding the
wire, but rather to that which has been absorbed by the
superficial portions of the platinum. This view appears,
without violent straining, to offer a satisfactory explanation
of all the effects which have been observed ; but it does not
seem advisable to lay much stress on these somewhat refined
theoretical points, as it is scarcely possible to test them
experimentally as yet.

It might be thought that this effect was only a case of the
ordinary leak produced when phosphorus oxidizes in air, due
to traces of air left inside the apparatus. This view seems
to be rendered quite untenable by the fact that the hot
platinum discharges only positive, and not negative electricity.
In addition to this, it takes no account of the very marked
chemical action between the phosphorus and the hot metal.

In conclusion the author wishes to thank Prof. Thomson
for his valuable advice and criticism during the course of
these experiments, which were carried out in the Cavendish
Laboratory.

XLI. *Determination of Young's Modulus (Adiabatic) for
Glass. By* Chichester A. Bell, *M.B. With an Appendix
by* C. Chree, *F.R.S.**

ALTHOUGH the elastic constants of glass have been
the subject of many investigations, it is not easy to
make use of the published results in any experimental
inquiry into which these enter. The characteristics of the
specimens examined are usually so vaguely given, and the
composition of the material is so variable, that it is never safe
to assume the identity of any sample purchased in the shops
with one of ascertained properties.

This difficulty was acutely felt by the writer in the course
of a series of experiments in which a tolerably exact knowledge
of the value of Young's Modulus (adiabatic) for different
kinds of glass was all-important. The results of these experi-
ments showed variations which it was impossible to account
for by any justifiable assumptions, and accordingly it became
necessary to determine the required constant by experiments
on specimens of the glass actually employed, the method
selected for this purpose being the simplest of all—the
acoustical.

So far as can be judged from a cursory examination of the literature of the subject, this method has been but little used in the case of glass, doubtless because of the well-known difficulty of obtaining perfectly uniform rods of this material of any considerable length. The original plan of these experiments was simply to determine the tones given by longitudinal vibration of a number of rods (or tubes) of each kind of glass; to cut each of them into ten approximately equal segments; and by weighing and measuring the segments to construct for each rod a diagram representing the variations in its cross-section from end to end. It was expected that, of the more nearly uniform of the original rods, a few might thus be selected, from the tones of which Young's Modulus might be calculated.

Inspection of the diagrams showed, however, that of the many rods examined not one showed such an approach to uniformity as to be suitable for the object in view. But, taking as a standard for comparison the mean value of the product, number of vibrations per second × twice length of rod, for each bundle, it was possible in a general way to connect the variation from this mean exhibited by each rod with its most marked irregularities; and this observation suggested that it might be possible, by the aid of the diagrams, to calculate for each rod an " addition to length " which would enable the velocity of sound through it to be calculated from its rate of vibration as if the cross-section were constant—a correction analogous to that given by Lord Rayleigh * for fluid columns of slightly variable section. Trial showed that this was certainly possible; and on mentioning the matter to Dr. Chree, to whom I am indebted for his cordial interest, he kindly undertook to look into it, and has given in the Appendix to this Note the mathematical proof that the correction, somewhat empirically applied, was strictly in accordance with theory. The necessary correction is, in fact, given by Lord Rayleigh's formula *with reversed sign*.

In the experiments quoted below, tuning-forks of high frequency were not available; the product $n \times 2l$ for the glass rods was therefore determined as follows. Bessemer steel rods of remarkable uniformity as regards diameter and texture are now easily procurable. Three such rods from the same drawing, $\frac{3}{8}$ of an inch in diameter, when cut to exactly the same length, about 11 feet, and clamped centrally by means of corks in a vertical position †, gave, on being struck

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* 'Theory of Sound,' 1st edition, vol. ii. p. 61.

† The note given by a rod of these dimensions is distinctly raised in pitch when it is fixed horizontally.

endwise, notes so nearly identical in pitch that beats were scarcely perceptible. This trial was carried out in the manner described farther on, and every precaution taken to secure uniformity of temperature. In the course of subsequent experiments many opportunities presented themselves of comparing the notes given by pieces of these bars of approximately equal length, and the differences not accounted for were always so small as to be unimportant, the only serious difference noted amounting to about six vibrations per second in the case of two pieces of exactly equal length, about 40 cms.

To determine the velocity of sound in the steel rods, the length of one of them was cautiously altered by cutting and filing until its note nearly coincided with that of a König fork giving 1024 double vibrations per second ; and, subsequently, taking advantage of a spell of cold weather, when room temperature had been long constant at 0°, rod and fork were compared together, beats being counted for 20 seconds. According to König the rate of his forks, standardized at 20° C., increases by ·000118 per cent. for each degree below standard temperature. With these data, the product $n \times 2l$ for the steel bar was found to be

$$516040 \text{ cms. per second at } 0°.$$

Similarly favourable opportunities did not occur for testing the other bars at 0°; but repeated comparisons of the bars with each other, and with 768 and 896 König forks at other temperatures, left no doubt of the substantial accuracy of this determination.

Some very careful determinations of the velocity of sound in rods of various metals and of St. Gobain glass, at temperatures ranging from 0° to 100° C., have been made by Mayer[*], who found for a rod of Bessemer steel at 0° $V = 515090$ cms. per sec. In view of possible variations in the composition of the metal, the agreement must be considered satisfactory.

According to Mayer also the velocity of sound in such a rod diminishes by ·2 per cent. when the temperature rises from 0° to 20°, or by nearly 50 cms. per sec. for each degree.

With these data the rates of vibration of the steel rods here employed were calculated, measurements of length being throughout made by a steel metre-scale, the very trifling errors of which had been determined at Kew.

In the experiments with glass, a steel rod was first cut giving a note higher in pitch than any one of the bundle of

[*] "Researches in Acoustics," by Alfred M. Mayer, Phil. Mag. [5] vol. xli. p. 168.

rods or tubes under examination ; and the lengths of these were gradually reduced by cutting and grinding, until each gave a note differing from that of the steel by about 4 or 5 vibrations per second. In cutting an ordinary glass knife was used, but the trimming of the ends, and small alterations of length, were effected by lightly pressing the ends against a sheet of moderately coarse emery-cloth clamped on the face of a circular wooden disk which was rapidly rotated in a lathe. During the comparison of tones, the steel rod was simply balanced on the corner of a table, and the glass rod held at its centre in the hand, both rods being struck endwise by a light wooden mallet.

After this preliminary adjustment, the bundle of rods was laid aside until its temperature might be assumed to be that of the room. Each rod was then passed through a hole in a short cylindrical cork placed at such a point—not necessarily the centre—that the vibration was of maximum persistence, and mounted on a board side by side with the similarly supported steel rod.

A cone or funnel was fixed on the board so that its wider end embraced one pair of the rod ends, while its narrower end was connected by rubber tubing with a short length of glass tube pushed well into the ear-cavity. Beats due to the rod vibrations were thus heard with the greatest distinctness. To throw the rods into vibration, two small "pneumatic hammers" were arranged to deliver light blows on their distant ends. Each hammer was formed of a cylinder of wood about 5 cms. in length, cut from an ordinary pen-handle, lying within a slightly wider tube of glass in which it could just slide freely. A perforated cork or a section of rubber tubing, forced into the glass tube, kept the cylinder lying normally just within one of its ends ; the other drawn-out end was connected by thin rubber tubing with a bag such as is used for pneumatic bells, which could be placed in any convenient position. By *tapping* on this bag with the finger the little wooden cylinder was sharply projected from, and as sharply drawn back into, the glass, after impact on the rod end ; whilst by *pressing* gently on the bag the hammer was caused to rest against the rod, at once checking its motion. The rods could thus be thrown into vibration either alternately or simultaneously, the force of the blows delivered being under complete control. This little device, which I have found superior to any other mechanical or electrical arrangement, has proved of eminent service in a number of acoustical experiments. Thus in working with tuning-forks it makes it unnecessary for the operator to approach them—

sometimes a matter of importance ; whilst in experiments on rods, hammer and rod may be enclosed in a tube, which is then immersed in a water or steam bath.

The beats produced by the simultaneous vibration of the glass and steel rods were, as a rule, simply estimated by comparison with those of a watch. The most successful of more exact methods tried consisted in timing the oscillations of a heavy ball attached to a string to coincide with the period of the beats, by altering the length of the string, the rate of the pendulum being subsequently ascertained by means of a watch. But a moderate amount of practice enables one to make a pretty exact estimate of the frequency of the beats when this is in the neighbourhood of four or five per second. It is not advisable to attempt to bring steel and glass rod more closely into unison : as a rule the vibrations of thin glass rods, especially if these are rather irregular, die out so rapidly that the apparent absence of beats is no criterion of exact synchronism.

In every case, the sense of the difference between steel and glass rod was positively ascertained in the usual way by means of a spring-brass rider which could be fixed on the steel bar. This was more than usually important, since the fundamental tone of an irregular rod is certain to be accompanied by anharmonic overtones.

In the way described, the product $n \times 2l$ is easily and rapidly found for a number of thin rods. The correction, Δl, to be applied to the observed length l of a free-free irregular rod vibrating longitudinally in its gravest mode, is (see Appendix)

$$\Delta l = \int_0^l \frac{\delta S}{S_0} . \cos \frac{2\pi z}{l} dz,$$

where δS is the difference, positive or negative, between the cross-section at the point z and its mean value, S_0, for the whole rod.

Now an exact determination of S throughout even a single rod of considerable length would be a matter of some difficulty ; and as rods and tubes are never quite free from knots and streaks, very often not homogeneous throughout, and only exceptionally properly annealed, would in the end lead to no high degree of accuracy. In these experiments I have, as a rule, been satisfied to divide each rod into *ten* nearly equal lengths ; after suitable trimming of the ends of each piece on the emery disk, to determine its length to the nearest tenth millimetre, and its weight to within one milligramme ; and to take $\frac{\text{weight}}{\text{length}}$ as a measure of S, supposed

to be *uniform* throughout the segment, S_0 being similarly determined for the whole rod.   The values of the integral

$$\int \cos \frac{2\pi z}{l} \cdot dz$$

between the limits $z=0$, $z=\frac{1}{10}l$; $z=\frac{1}{10}l$, $z=\frac{2}{10}l$, and so on, are, to the degree of approximation considered necessary, and in order, $\frac{l}{2\pi}$ multiplied by the fractions

$$\overset{+}{\cdot 59} \quad \overset{+}{\cdot 36} \quad 0 \quad \overset{-}{\cdot 36} \quad \overset{-}{\cdot 59} \quad \overset{-}{\cdot 59} \quad \overset{-}{\cdot 36} \quad 0 \quad \overset{+}{\cdot 36} \quad \overset{+}{\cdot 59}.$$

The total correction thus becomes

$$\Delta l = \frac{l}{2\pi}\left(\frac{\cdot 59(S_1-S_0)+\cdot 36(S_2-S_0)-\cdot 36(S_4-S_0)\ldots+\cdot 59(S_{10}-S_0)}{S_0}\right)$$

the third and eighth sections being omitted.   Inspection of the diagrams will show at once if the omission is justified.

The lists below include three rods in which the subdivision was subsequently made into *twenty* parts.   In two cases the final result was practically unchanged ; in the third the value of a was raised only 3 metres per second.   On the whole, further subdivision did not appear to be necessary.

Finally, we get for the " velocity of sound "

$$a = 2n(l+\Delta l),$$

and for Young's Modulus (adiabatic)

$$E' = a^2\rho,$$

where ρ is the density referred to water at 4° C.

In the tables are given the values of a and E' reduced to 0°. According to Mayer (*loc. cit.*) Young's Modulus for St. Gobain glass ($\rho = 2\cdot 545$) diminishes by 1·16 per cent. for a rise of temperature from 0° to 100° C.   In a special experiment with the Gallenkamp soft German glass, a diminution of 1·12 per cent. from 5° to 100° was observed, closely agreeing with Mayer's determination.   With Powell's flint-glass, on the other hand, the curious result was obtained that Young's Modulus is *sensibly constant* between the same temperature limits.   This observation is quite in harmony with the results obtained by Winkelmann and Schott * for glass of the same general character.   In fact, if two rods of this glass be " tuned " to the same pitch, one of them may be made very hot by passage through a gas-flame before any distinct alteration of its note can be observed.   Owing to the brilliance

* " Ueber die Elasticität, etc., verschiedener Gläser," Wiedemann's *Annalen*, Bd. li. p. 697 (1894).

and persistence of the tones obtained from stout and fairly uniform rods of this glass, the low value of a and the absence of a temperature-coefficient, it is admirably adapted for standards of high pitch. The end of the rod on which blows are delivered may be fused, to diminish the risk of chipping, and the tuning effected by grinding the other end.

All results are given in absolute units. The density at $0°$ has been determined from selected samples. As it was found to be slightly variable, especially in the case of flint-glass, it is given to only four figures.

It is perhaps well to point out that the occasional close agreement between the corrected and uncorrected values of $n \times 2l$ does not imply approximate uniformity in the corresponding rod. This is far indeed from being the case.

Soft German Glass.

1. From Messrs. Gallenkamp & Co.

Mean $n \times 2l$ for eight rods at $14° = 527280$ cms. per sec. Of these five of the best were selected for examination. Lengths about 150 cms. ρ_0 (mean) $= 2\cdot508$.

$n \times 2l$ observed.	a (corrected) cms. per sec.
527620	525930
523920	526030
517240	525940
525850	526610
528100	526820

$$\text{Mean} = 526260 \text{ at } 6° \text{ C.}$$
$$= 526400 \text{ at } 0°$$

Young's Modulus (adiabatic) or E'
$$= 6\cdot95 \times 10^{11} \text{ dynes per sq. cm. at } 0°.$$

Eight *selected* thin tubes of this glass gave a mean value of $n \times 2l$ at $6° = 525500$ cms. per sec. These were not cut up.

2. From Messrs. C. E. Müller & Co.

Mean product for 12 carefully chosen rods at $14° = 524800$. Of these six examined. Lengths about 150 cms. $\rho_0 = 2\cdot507$.

$n \times 2l$ observed.	a (corrected).
529000	525490
523020	524780
521270	525400
526020	524760
526020	524720
524580	525100

$$\text{Mean} = 525050 \text{ at } 9°$$
$$= 525250 \text{ at } 0°.$$
$$E' = 6\cdot91 \times 10^{11} \text{ at } 0°.$$

3. From Messrs. Baird & Tatlock.

Mean product for nine rods at $9° = 525990$.   Lengths about 143 cms. $\rho_0 = 2·488$.

$n \times 2l$ observed.	a (corrected).
524310	527360
530640	527180
527900	526710
527480	526900
517660	526370

Mean a at $6° = 527040$ cm. per sec.
at $0° = 527170$.
$E' = 6·91 \times 10^{11}$ at $0°$.

The last result here has been rejected. For this very irregular rod Δl was $+2·36$ cms.

Bohemian Glass.

From Messrs. Baird & Tatlock. This glass was obtainable only in thin tubes, very irregular, much streaked and full of knots.

Mean product for six tubes $= 513740$ at $6°$.   Lengths about 138 cms. $\rho_0 = 2·394$.

$n \times 2l$ observed.	a (corrected).
524580	518780
511690	515430
511740	516950
511040	517260
512050	516220
511360	515800

Mean a at $6° = 516740$ cm. per sec.
at $0° = 516880$.
$E' = 6·39 \times 10^{11}$ at $0°$,
or, rejecting the first result, $6·38 \times 10^{11}$.

The agreement here is not nearly so good as usual, as indeed was anticipated.

Gauge-tube Glass.

A hard, ferruginous, light-green glass, almost infusible, manufactured by Messrs. Powell & Sons of Whitefriars, specially for gauge-tubes. Four carefully-chosen tubes, about 112 cms. in length, were examined. The external diameter

of these tubes, about 1·3 cm., was rather greater, in proportion to length, than was desirable, but narrower or much longer tubes were not obtainable. $\rho_0 = 2\cdot525$.

$n \times 2l$ observed.	a (corrected).
542380	539820
539690	539350
540570	539850
541420	539560

Mean a at $9° = 539650$ cm. per sec.
at $0° = 539850$.
$E' = 7\cdot36 \times 10^{11}$ at $0°$.

"Soft" Jena Glass.

Stout tubes, somewhat narrower than the foregoing, of length about 144 cms. $\rho_0 = 2\cdot499$.

$n \times 2l$ observed.	a (corrected).
533120	535370
535650	535650
531510	535200
527800	534860

Mean a at $7° = 535270$ cm. per sec.
$= 535400$ at $0°$.
$E' = 7\cdot16 \times 10^{11}$ at $0°$.

Flint Glass.

From the Whitefriars Glass-works. Mean product for ten thin rods at $3° = 409910$. Lengths about 91 cms. ρ_0 (mean) $= 3\cdot164$.

$n \times 2l$ observed.	a (corrected).
405360	408740
410400	409550
412680	409820
407300	409780
410510	409610

Mean $a = 409500$ cm. per sec.
Not changing with temperature.
$E' = 5\cdot30 \times 10^{11}$ at $0°$.

For this glass, the value of E' depends greatly on the amount of lead contained. For example, in two other

specimens from the same works, but of density 3·136 at 10°/4°, an indirect determination of E′ gave 5·39 × 10^{11}.

The foregoing list includes *all* the rods and tubes that have been examined, with only one exception, where some gross blunder had evidently been committed. Inspection shows that discrepancies between the uncorrected products $n \times 2l$ are reduced by 90 per cent. in the corrected values of a, and it is highly probable that the calculated values of E′ are correct to three figures.

When a glass can be obtained in thin rods or tubes—a diameter of 5 or 6 mms. seems most desirable—the experiments are easily and rapidly carried out, since no great accuracy is required in the weighing or measurement of the segments. Thick rods or tubes naturally give more trouble, especially when of hard glass, and here some care in selecting suitable specimens is desirable. Tubes which do not give persistent notes when struck should be at once rejected. Perfect uniformity is not to be expected; failing this, the best results are obtained with rods or tubes which taper continuously from one end to the other.

Appendix *by* C. Chree, *F.R.S.*

Taking the axis of the tube as axis of z, the origin being at one end, we have for the longitudinal displacement in the fundamental free-free vibration

$$w = A \cos kt \cos \pi z/l, \quad \ldots \quad (1)$$

where A is a constant, l the length, and $k/2\pi$ the frequency.

In reality w varies slightly over the cross-section, but for the purpose of the present correction we may neglect this variation and also the energy answering to the lateral motion.

If the section S were uniform we should have—neglecting the correction for finite area of section—

$$k/\pi = a/l, \qquad \text{where } a = \sqrt{E/\rho}.$$

Here ρ denotes the density, E Young's modulus, and a the "velocity of sound" in the nomenclature employed by Lord Rayleigh.

S, it should be noticed, means the sectional area of the *material*; in a circular tube whose outer and inner surfaces are circles of radii r and $r′$, $S = \pi(r^2 - r'^2)$.

Let

$$S = S_0 + \delta S, \quad \ldots \ldots \quad (2)$$

where S_0 is constant, and $\delta S/S_0$ is supposed small everywhere along the tube.

Let T denote the kinetic, V the potential energy, then, as in Lord Rayleigh's 'Theory of Sound,'

$$T = \int_0^l \tfrac{1}{2}\rho \dot{w}^2 S \, dz,$$

$$V = \int_0^l \tfrac{1}{2} E \left(\frac{dw}{dz}\right)^2 S \, dz.$$

Thence from (1) and (2)

$$T = \tfrac{1}{2}\rho A^2 k^2 S_0 \sin^2 kt \int_0^l \cos^2 (\pi z/l)(1 + \delta S/S_0) dz,$$

$$V = \tfrac{1}{2} A^2 E (\pi/l)^2 \cos^2 kt \int_0^l \sin^2 (\pi z/l)(1 + \delta S/S_0) dz.$$

But $T + V$ is constant throughout the motion, and so the coefficients of $\sin^2 kt$ and $\cos^2 kt$ are equal. Equating them, we find

$$k^2 = (E/\rho)(\pi/l)^2 \frac{\int_0^l \sin^2 (\pi z/l)(1 + \delta S/S_0) dz}{\int_0^l \cos^2 (\pi z/l)(1 + \delta S/S_0) dz},$$

or

$$k^2 = (E/\rho)(\pi/l)^2 \frac{1 + \dfrac{2}{l}\int_0^l \sin^2 (\pi z/l)(\delta S/S_0) dz}{1 + \dfrac{2}{l}\int_0^l \cos^2 (\pi z/l)(\delta S/S_0) dz}. \quad (3)$$

As $\delta S/S_0$ is by hypothesis everywhere small, we may neglect squares of either integral as compared to unity, and so deduce

$$k^2 = \frac{E}{\rho}\left(\frac{\pi}{l}\right)^2 \left\{ 1 + \frac{2}{l}\int_0^l \left(\sin^2 \frac{\pi z}{l} - \cos^2 \frac{\pi z}{l}\right)\frac{\delta S}{S_0} dz \right\}$$

$$= \frac{E}{\rho}\left(\frac{\pi}{l}\right)^2 \left\{ 1 - \frac{2}{l}\int_0^l \frac{\delta S}{S_0} \cos \frac{2\pi z}{l} \, dz \right\}. \quad . \quad . \quad . \quad (4)$$

Now suppose that the rod has the same period as if it possessed a uniform section but were of slightly different length $l + \delta l$. Then

$$k^2 = \frac{E}{\rho} \frac{\pi^2}{(l + \delta l)^2} = \frac{E}{\rho}\frac{\pi^2}{l^2}\left\{ 1 - \frac{2\delta l}{l} \right\} . \quad . \quad . \quad (5)$$

Comparing the identities (4) and (5) we deduce at once

Mr. Bell's formula

$$\delta l = \int_0^l \frac{\delta S}{S_0} \cos \frac{2\pi z}{l} \, dz. \quad . \quad . \quad . \quad . \quad (6)$$

This gives the correction δl necessary to obtain the length of the equivalent rod of uniform section.

The proof does not assume S_0 to be the *mean* area of the cross section. But the assumptions that $\delta S/S_0$ is everywhere very small, and that the squares of the integrals in (3) are negligible, will be in general most satisfactorily fulfilled when S_0 is the mean section.

When S is more than usually variable, a slight increase in accuracy would probably be obtained by determining k directly from (3). In this event it would be convenient that S_0 should be the mean section, as we should then have

$$\int_0^l \left(\cos^2 \frac{\pi z}{l} + \sin^2 \frac{\pi z}{l} \right) \frac{\delta S}{S_0} \, dz = 0,$$

and so

$$\int_0^l \cos^2 \frac{\pi z}{l} \cdot \frac{\delta S}{S_0} \, dz = -\int_0^l \sin^2 \frac{\pi z}{l} \cdot \frac{\delta S}{S_0} dz = \frac{1}{2} \int_0^l \cos \frac{2\pi z}{l} \cdot \frac{\delta S}{S_0} dz.$$

We should thus have still really only one integral to evaluate.

If the tube were fixed-fixed instead of free-free, the origin being still at an end, we should obtain for the correction to the length

$$\delta l = -\int_0^l (\delta S/S_0) \cos (2\pi z/l) dz. \quad . \quad . \quad (7)$$

This is numerically equal but opposite in sign to the correction obtained for the free-free rod; thus by combining results from the two species of vibrations we could eliminate the effects of variation of section, assuming them of course to be small.

In a fixed-free bar, $z = 0$ being the fixed end, we find

$$\delta l = -\int_0^l (\delta S/S_0) \cos (\pi z/l) dz. \quad . \quad . \quad (8)$$

In all these cases an increase of S near a fixed end or node tends to make δl negative or raises the pitch, while an increase of S near a free end tends to make δl positive. In other words, an increase of material stiffens the bar elastically when near a node, but acts mainly as a load when occurring near a section of maximum amplitude of motion.

XLII. *Radiation and Electromagnetic Theory.*
II. *Æolotropic Potential. By* R. HARGREAVES [*].

Solution for Ellipsoids of the Æolotropic form of the Equations of Laplace and Poisson.

§ 20. In the following analysis for ellipsoids a general æolotropic notation is used. The ellipsoid is first treated as a conductor, and then as having a uniform distribution through its volume; and the equations are taken to be

$$-X' = \frac{d\phi}{dx}, \quad -kX = p\frac{d\phi}{dx} + r'\frac{d\phi}{dy} + q'\frac{d\phi}{dz}, \quad \Sigma\frac{dX}{dx} = \rho$$

so that

$$\Sigma p\frac{d^2\phi}{dx^2} + 2\Sigma p'\frac{d^2\phi}{dy\,dz} + k\rho = 0, \quad \text{say } \nabla_e^2\phi + k\rho = 0. \tag{71}$$

The passage to the motional case is made by writing

$$\text{for} \quad \begin{array}{cccccc} p, & q, & r, & p', & q', & r' \\ 1-p^2, & 1-q^2, & 1-r^2, & -qr, & -rp, & -pq \end{array} . \tag{72}$$

In the motional case k is $1-\Sigma p^2$, and in the general case we may put

$$k = \begin{vmatrix} p & r' & q' \\ r' & q & p' \\ q' & p' & r \end{vmatrix} = \Delta_p.$$

The successive rows of this determinant are the coefficients appearing in X Y Z.

Associated with this is the problem presented by the

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* Communicated by the Author.

equations]

$$\alpha = \frac{dH}{dy} - \frac{dG}{dz},$$

$$\alpha' = \left(r'\frac{d}{dx} + q\frac{d}{dy} + p'\frac{d}{dz} \right)H - \left(q'\frac{d}{dx} + p'\frac{d}{dy} + r\frac{d}{dz} \right)G, \dots$$

with

$$\frac{i_x}{V} = \frac{d\gamma'}{dy} - \frac{d\beta'}{dz}, \dots \quad \text{and therefore} \quad \nabla_e^2 F + i_x/V = 0 ;$$

(73)

which will be solved for i_x a linear function of xyz within the ellipsoid.

Take the problem of the conductor first. If ϕ depends on a single parameter λ,

$$\frac{d\phi}{d\lambda}\left(\Sigma p\frac{d^2\lambda}{dx^2} + 2\Sigma p'\frac{d^2\lambda}{dy\,dz} \right) + \frac{d^2\phi}{d\lambda^2}\left(\Sigma p\left|\frac{d\lambda}{dx}\right|^2 + 2\Sigma p'\frac{d\lambda}{dy}\cdot\frac{d\lambda}{dz} \right) = 0,$$

say

$$\frac{d\phi}{d\lambda}\nabla_e^2\lambda + \frac{d^2\phi}{d\lambda^2}\cdot(D_e\lambda)^2 = 0.$$

It is proposed to determine the coefficients in the quadric

$$u \equiv \Sigma ax^2 + 2\Sigma a'yz = 1$$

as functions of λ in such a way as to make the surface equipotential. If the precise functions were known, λ would be defined as a function of $x\,y\,z$ by this equation.

Use $\dot{\alpha}$ for $\dfrac{d\alpha}{d\lambda}$, $\dot{u}$ for $\Sigma\dot{\alpha}x^2 + 2\Sigma\dot{\alpha}'yz$, then

$$\dot{u}\frac{d\lambda}{dx} + \frac{\partial u}{\partial x} = 0 ; \quad \dots \quad \dots \quad (74)$$

and since $\dfrac{d\dot{u}}{dx} = \ddot{u}\dfrac{d\lambda}{dx} + \dfrac{\partial\dot{u}}{\partial x}$, the differentiation of (74) gives

$$\dot{u}\frac{d^2\lambda}{dx^2} + \ddot{u}\left(\frac{d\lambda}{dx}\right)^2 + 2\frac{d\lambda}{dx}\frac{\partial\dot{u}}{\partial x} + 2\alpha = 0.$$

$$\dot{u}\frac{d^2\lambda}{dx\,dy} + \ddot{u}\frac{d\lambda}{dx}\frac{d\lambda}{dy} + \frac{d\lambda}{dx}\frac{\partial\dot{u}}{\partial y} + \frac{d\lambda}{dy}\frac{\partial\dot{u}}{\partial x} + 2\gamma' = 0.$$

The sum of these equations with multipliers $p, q, r, 2p', 2q', 2r'$ gives

$$\dot{u}\nabla_e^2\lambda + \ddot{u}(D_e\lambda)^2 + 2\Sigma p\frac{d\lambda}{dx}\frac{\partial\dot{u}}{\partial x} + 2\Sigma r'\left(\frac{d\lambda}{dx}\frac{\partial\dot{u}}{\partial y} + \frac{d\lambda}{dy}\frac{\partial\dot{u}}{\partial x} \right) + 2\Sigma(p\alpha + 2p'\alpha') = 0;$$

or if we multiply by $\dot{u}$ and quote (74),

$$\dot{u}^2\nabla_e^2(\lambda) + \dot{u}\,\ddot{u}(D_e\lambda)^2 + 2\dot{u}\Sigma(p\alpha + 2p'\alpha')$$

$$= 2\Sigma p\frac{\partial u}{\partial x}\frac{\partial\dot{u}}{\partial x} + 2\Sigma r'\left(\frac{\partial u}{\partial x}\frac{\partial\dot{u}}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial\dot{u}}{\partial x} \right). \quad \dots \quad (75)$$

By (74) $\quad \dot{u}^2 (D_e\lambda)^2 = \Sigma\, p\left(\dfrac{\partial u}{\partial x}\right)^2 + 2\Sigma\, p'\dfrac{\partial u}{\partial y}\dfrac{\partial u}{\partial z},$

which is quadratic in $x\,y\,z$; the isotropic solution suggests that it may be made proportional to $\dot{u}$, in which case

$$\dot{u}^2(D_e\lambda)^2 = \Sigma\, p\left(\frac{\partial u}{\partial x}\right)^2 + 2\Sigma\, p'\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} = 4A\dot{u}. \quad . \quad . \quad (76)$$

The second and third members have the coefficients of each power and product (as regards $x\,y\,z$) alike, if six conditions are fulfilled, belonging to the types

$A\dot{\alpha} = p\alpha^2 + q\gamma'^2 + r\beta'^2 + 2p'\beta'\gamma' + 2q'\alpha\beta' + 2r'\alpha\gamma',$

$A\dot{\alpha}' = p\beta'\gamma' + q\alpha'\beta + r\alpha'\gamma + p'(\beta\gamma + \alpha'^2) + q'(\alpha'\beta' + \gamma\gamma') + r'(\gamma'\alpha' + \beta\beta').$

Differentiate the second and third members of (76) with regard to λ only, as it appears in the coefficients; if A is taken to be constant, the result is

$$2\Sigma p\frac{\partial u}{\partial x}\frac{\partial \dot{u}}{\partial x} + 2\Sigma p'\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial y}\right) = 4A\ddot{u} = \dot{u}\,\ddot{u}(D_e\lambda)^2.$$

These terms then are cancelled in (75), which becomes

$$\dot{u}\nabla_e^2\lambda + 2\Sigma\,(p\alpha + 2p'\alpha') = 0. \quad . \quad . \quad . \quad (77)$$

The effect of making A constant is that we secure the parameter λ commonly used in the isotropic case, instead of a function of that parameter; the constant A determines the scale of the parameter, and the value $A = -1$ gives the precise usage in the isotropic case. The relation (76) is then

$$-\dot{u}(D_e\lambda)^2 = 4 = \frac{\partial u}{\partial x}\left(p\frac{d\lambda}{dx} + r'\frac{d\lambda}{dy} + q'\frac{d\lambda}{dz}\right)$$

$$+ \frac{\partial u}{\partial y}\left(r'\frac{d\lambda}{dx} + q\frac{d\lambda}{dy} + p'\frac{d\lambda}{dz}\right) + \frac{\partial u}{\partial z}\left(q'\frac{d\lambda}{dx} + p'\frac{d\lambda}{dy} + r\frac{d\lambda}{dz}\right). \quad . \quad (78)$$

Then (77) and the first part of (78) make the equation in ϕ

$$2\frac{d^2\phi}{d\lambda^2}\Big/\frac{d\phi}{d\lambda} + \Sigma(p\alpha + 2p'\alpha') = 0. \quad . \quad . \quad . \quad (79)$$

Thus ϕ is to be found in terms of λ through the medium of $\alpha \ldots \alpha' \ldots$, which are connected with λ by six equations, the types of which are here rewritten with $A = -1$,

$\dot{\alpha} + \alpha(p\alpha + r'\gamma' + q'\beta') + \gamma'(r'\alpha + q\gamma' + p'\beta') + \beta'(q'\alpha + p'\gamma' + r\beta') = 0$

$\left\{ \begin{array}{l} \dot{\alpha}' + \gamma'(p\beta' + r'\alpha' + q'\gamma) + \beta(r'\beta' + q\alpha' + p'\gamma) + \alpha'(q'\beta' + p'\alpha' + r\gamma) = 0 \\ \text{or } \dot{\alpha}' + \beta'(p\gamma' + r'\beta + q'\alpha') + \alpha'(r'\gamma' + q\beta + p'\alpha') + \gamma(q'\gamma' + p'\beta + r\alpha') = 0 \end{array} \right\} (80)$

$2\text{ F }2$

§ 21. The solution of these equations can be shown in a constructive manner with much less labour than the original straightforward process by which it was obtained.

Denote by Δ_a the determinant of the ellipsoid u_a with reference to which a solution is sought, use $A \ldots A' \ldots$ for the minors of Δ_a, and $P \ldots P' \ldots$ for the minors of Δ_p. Then consider the equations

$$
\left.
\begin{aligned}
\alpha(A + p\lambda\Delta_a) + \gamma'(C' + r'\lambda\Delta_a) + \beta'(B' + q'\lambda\Delta_a) &= \Delta_a \\
\alpha(C' + r'\lambda\Delta_a) + \gamma'(B + q\lambda\Delta_a) + \beta'(A' + p'\lambda\Delta_a) &= 0 \\
\alpha(B' + q'\lambda\Delta_a) + \gamma'(A' + p'\lambda\Delta_a) + \beta'(C + r\lambda\Delta_a) &= 0
\end{aligned}
\right\}, \quad . \quad (81)
$$

and two other triads, the first of which has the variables $\gamma'\beta\alpha'$ on the left, and Δ_a on the right transferred to the second equation; the other has $\beta'\alpha'\gamma$, and Δ_a appears in the third equation. With $\Delta(\lambda)$ for the determinant of (81), and $A(\lambda) \ldots$ for its minors, the solution is expressed by

$$
\alpha\Delta(\lambda) = \Delta_a A(\lambda), \quad \alpha'\Delta(\lambda) = \Delta_a A'(\lambda). \quad . \quad . \quad . \quad (82)
$$

Differentiate (81) with regard to λ, and write it

$$
\dot{\alpha}(A + p\lambda\Delta_a) + \dot{\gamma}'(C' + r'\lambda\Delta_a) + \dot{\beta}'(B' + q'\lambda\Delta_a) + \Delta_a(p\alpha + r'\gamma' + q'\beta') = 0
$$
$$
\dot{\alpha}(C' + r'\lambda\Delta_a) + \dot{\gamma}'(B + q\lambda\Delta_a) + \dot{\beta}'(A' + p'\lambda\Delta_a) + \Delta_a(r'\alpha + q\gamma' + p'\beta') = 0
$$
$$
\dot{\alpha}(B' + q'\lambda\Delta_a) + \dot{\gamma}'(A' + p'\lambda\Delta_a) + \dot{\beta}'(C + r\lambda\Delta_a) + \Delta_a(q'\alpha + p'\gamma' + r\beta') = 0.
$$

Now solve these as a system to determine $\dot{\alpha}\,\dot{\gamma}'\,\dot{\beta}'$: for $\dot{\alpha}$ we get

$$
\dot{\alpha}\Delta(\lambda) + \Delta_a\big[A(\lambda)(p\alpha + r'\gamma' + q'\beta') + C'(\lambda)(r'\alpha + q\gamma' + p'\beta')
$$
$$
+ B'(\lambda)(q'\alpha + p'\gamma' + r\beta')\big] = 0.
$$

When $A(\lambda), \ldots$ are cleared by the use of (82) the resulting equation is the first of (80). In the same way the solution for $\dot{\gamma}'$ gives one form of the second of (80), and the solution which belongs to the second triad gives the other form. The triads are consistent but not wholly independent.

When $\Delta(\lambda)$, $A(\lambda)$, $A'(\lambda)$ are fully evaluated we find

$$
\Delta(\lambda) = \Delta_a^2[1 + \lambda\Sigma(pa + 2p'a') + \lambda^2\Sigma(PA + 2P'A') + \lambda^3\Delta_p\Delta_a],
$$
$$
A(\lambda) = \Delta_a[a + \lambda(qC + rB - 2p'A') + \lambda^2 P\Delta_a],
$$
$$
A'(\lambda) = \Delta_a[a' + \lambda(q'C' + r'B' - p'A - pA') + \lambda^2 P'\Delta_a];
$$

and the solution, written at length, is

$$
\left.
\begin{aligned}
J\alpha &= a + \lambda(qC + rB - 2p'A') + \lambda^2 P\Delta_a \\
J\alpha' &= a' + \lambda(q'C' + r'B' - pA' - p'A) + \lambda^2 P'\Delta_a \\
J &= 1 + \lambda\Sigma(pa + 2p'a') + \lambda^2\Sigma(PA + 2P'A') + \lambda^3\Delta_p\Delta_a
\end{aligned}
\right\}. \quad (83)
$$

It contains the correct number of constants, and these are the values of $\alpha \ldots$ for $\lambda = 0$.

§ 22. The mode of construction of the equipotential surfaces corresponding to the ellipsoid $u_a = 1$ as a conductor, is readily shown. Reciprocate $u_a = 1$ with regard to the sphere $x^2 + y^2 + z^2 = k^2$, and the result is $\Sigma(\mathrm{A}x^2 + 2\mathrm{A}'yz) = k^4 \Delta_a$. Then reciprocate

$$\Sigma\{(\mathrm{A} + p\lambda\Delta_a)x^2 + 2(\mathrm{A}' + p'\lambda\Delta_a)yz\} = k^4\Delta_a$$

with regard to the same sphere, and the reciprocal is

$$\Sigma(\mathrm{A}(\lambda)x^2 + 2\mathrm{A}'(\lambda)yz) = \Delta(\lambda)/\Delta_a,$$

which, since $\mathrm{A}(\lambda)\Delta_a = \alpha\Delta(\lambda)$, is the surface $u_a = 1$.

In the isotropic problem the ultimate form of the equipotential surfaces of a conducting ellipsoid is a sphere, and the interval between the two forms is bridged by surfaces confocal with the ellipsoid. In the æolotropic problem the ultimate form is an ellipsoid reciprocal to the ellipsoid constant $= u_p$, having the coefficients of the æolotropic differential equation. The interval between the primary form $u_a = 1$ and the ultimate form $u_p = \lambda\Delta_p$ is bridged over by the system of quadrics used here. They are not confocal: it is proposed to call them *æolian* quadrics.

We now give some properties of the coefficients $\alpha \ldots$, which it is convenient to place together, though not all are immediately required.

First, then, we have the determinant properties

$$\Delta_\alpha \Delta(\lambda) = \Delta_a^3, \quad \mathrm{A}_\alpha\Delta(\lambda) = \Delta_a^2(\mathrm{A} + p\lambda\Delta_a). \quad . \quad (84)$$

If we introduce J as defined above,

$$\left.\begin{array}{l} \mathrm{J}\Delta_\alpha = \Delta_a, \quad \mathrm{J}\Delta_a^2 = \Delta(\lambda), \\ \mathrm{J}\mathrm{A}_\alpha = \mathrm{A} + p\lambda\Delta_a, \; i.\,e. \; \mathrm{J}(\beta\gamma - \alpha'^2) = \mathrm{A} + p\lambda\Delta_a, \\ \qquad \text{and} \quad \mathrm{J}(\beta'\gamma' - \alpha\alpha') = \mathrm{A}' + p'\lambda\Delta_a, \end{array}\right\} \quad . \quad . \quad (85)$$

the first showing the mode of dependence of the volume of $u_a = 1$ on λ.

We next prove the important relations

$$\mathrm{J}\Sigma(p\alpha + 2p'\alpha') = \frac{d\mathrm{J}}{d\lambda}, \quad \text{and} \quad \mathrm{J}\Sigma(\mathrm{A}\alpha + 2\mathrm{A}'\alpha') = \Delta_a\left(3\mathrm{J} - \lambda\frac{d\mathrm{J}}{d\lambda}\right) \quad (86)$$

The mode of differentiating a determinant gives

$$\frac{d\Delta(\lambda)}{d\lambda} = \Delta_a\Sigma\{p\mathrm{A}(\lambda) + 2p'\mathrm{A}'(\lambda)\} = \Delta(\lambda)\Sigma(p\alpha + 2p'\alpha')$$

by (82); and this is the equivalent of the first of (86). Again,

$$3\Delta(\lambda) = \Sigma\{A(\lambda)(A + p\lambda\Delta_a) + 2A'(\lambda)(A' + p'\lambda\Delta_a)\},$$

or

$$3\Delta_a = \Sigma\{\alpha(A + p\lambda\Delta_a) + 2\alpha'(A' + p\lambda\Delta_a)\};$$

which is the relation got by adding the second of (86) to the first multiplied by $\lambda\Delta_a$.

Again, if we add the members of the triad (81) with multipliers $\alpha\gamma'\beta'$, we get by taking account of (80) the first of the relations

$$\left.\begin{aligned}
\Delta_a(\alpha + \lambda\dot\alpha) &= A\alpha^2 + B\gamma'^2 + C\beta'^2 + 2A'\beta'\gamma' + 2B'\alpha\beta' + 2C'\alpha\gamma' \\
\Delta_a(\alpha' + \lambda\dot\alpha') &= A\beta'\gamma' + B\alpha'\beta + C\alpha'\gamma + A'(\beta\gamma + \alpha'^2) \\
&\quad + B'(\alpha'\beta' + \gamma\gamma') + C'(\gamma'\alpha' + \beta\beta').
\end{aligned}\right\} \quad (87)$$

The second is got in the case of γ' from the same triad by the use of multipliers $\gamma'\beta\alpha'$.

§ 23. Combining now the first relation of (86) with (79) we have

$$2\frac{d^2\phi}{d\lambda^2}\Big/\frac{d\phi}{d\lambda} + \frac{dJ}{d\lambda}\Big/J = 0, \text{ and therefore } \sqrt{J}\frac{d\phi}{d\lambda} = \text{ constant.} \quad (88)$$

To complete the solution for a conductor the constant is to be found in terms of the total charge which is given by $e = \int\Sigma lX dS$, where $(l\,m\,n)$ are direction-cosines of a normal. Now

$$-kX = p\frac{d\phi}{dx} + r'\frac{d\phi}{dy} + q'\frac{d\phi}{dz} = \frac{d\phi}{d\lambda}\Big(p\frac{d\lambda}{dx} + r'\frac{d\lambda}{dy} + q'\frac{d\lambda}{dz}\Big),$$

and if ϖ is a perpendicular from the centre on the tangent plane, $l = \dfrac{\varpi}{2}\dfrac{\partial u}{\partial x}, \dots$; thus

$$ke = k\Sigma lX dS = -\frac{1}{2}\int\frac{d\phi}{d\lambda}\varpi dS\Sigma\frac{\partial u}{\partial x}\Big(p\frac{d\lambda}{dx} + r'\frac{d\lambda}{dy} + q'\frac{d\lambda}{dz}\Big),$$

and quoting (78)

$$= -2\int\frac{d\phi}{d\lambda}\varpi dS = -6\tau\frac{d\phi}{d\lambda}.$$

The constancy of $\tau\dfrac{d\phi}{d\lambda}$, τ being the volume of $u_a = 1$, is a consequence of (85) and (88). The constant is fixed by taking $\lambda = 0$, when $J = 1$; thus

$$ke = -6\tau_0\Big(\frac{d\phi}{d\lambda}\Big)_{\lambda=0}, \text{ and } \sqrt{J}\frac{d\phi}{d\lambda} = \Big(\frac{d\phi}{d\lambda}\Big)_{\lambda=0} = -\frac{ke}{6\tau_0}.$$

Therefore

$$\phi = \frac{ke}{6\tau_0}\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}} = \frac{k\rho}{6}\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}}, \quad . \quad . \quad . \quad (89)$$

the second, with $e = \rho\tau_0$, written for comparison with a volume-distribution. The field-integral $\int \frac{1}{2}\Sigma XX' d\tau_e$ extending to the whole space outside the conductor depends on the surface value of ϕ, and is

$$S = \frac{e}{2}\phi_s = \frac{ke^2}{12\tau_0}\int_0^\infty \frac{d\lambda}{\sqrt{J}} = \frac{k\rho^2\tau_0}{12}\int_0^\infty \frac{d\lambda}{\sqrt{J}}; \quad . \quad . \quad (90)$$

§ 24. For the volume-distribution write

$$\psi = \phi - \Sigma Lx^2 - 2\Sigma L'yz,$$

in which for external values ϕ has the functional value just used, and L .. are functions of λ which make

$$\dot\phi = \Sigma \dot L x^2 + 2\Sigma \dot L' yz$$

agree with the form found above for an æolian of the ellipsoid, *i.e.* $\dot L = \alpha\dot\phi, \quad \dot L' = \alpha'\dot\phi, \ldots$ Then

$$\frac{d\psi}{dx} = -2(Lx + N'y + M'z),$$

$$\frac{d^2\psi}{dx^2} = -2L - 2\frac{d\lambda}{dx}(\dot Lx + \dot N'y + \dot M'z) = -2L - \dot\phi\frac{d\lambda}{dx}\frac{\partial u}{\partial x},$$

$$\frac{d^2\psi}{dx\,dy} = -2N' - \dot\phi\frac{d\lambda}{dy}\frac{\partial u}{\partial x} = -2N' - \dot\phi\frac{d\lambda}{dx}\frac{\partial u}{\partial y}.$$

Use the multipliers $p \ldots r'$ and add, then

$$\nabla_e^2\psi = -2\Sigma(pL + 2p'L') - \dot\phi\left\{\frac{d\lambda}{dx}\left(p\frac{\partial u}{\partial x} + r'\frac{\partial u}{\partial y} + q'\frac{\partial u}{\partial z}\right)\right.$$
$$+ \frac{d\lambda}{dy}\left(r'\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} + p'\frac{\partial u}{\partial z}\right) + \frac{d\lambda}{dz}\left(q'\frac{\partial u}{\partial x} + p'\frac{\partial u}{\partial y} + r\frac{\partial u}{\partial z}\right)\right\}$$
$$= -2\Sigma(pL + 2p'L') - 4\dot\phi \quad \text{by (78)}.$$

Thus for the external solution $\nabla_e^2\psi = 0$,

if $\qquad \Sigma(pL + 2p'L') + 2\dot\phi = 0. \quad . \quad . \quad . \quad . \quad (91)$

If we differentiate (91) with regard to λ, and use $\dot L = \alpha\dot\phi, \ldots$ we get $\Sigma(p\alpha + 2p'\alpha') + 2\ddot\phi/\dot\phi = 0$, which is (79); thus (91) will be satisfied if a proper arrangement as to the constant of integration is made. Write then $\psi = C\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}}(1 - u_a)$ for the external potential, and for the internal potential put

$\lambda = 0$ in the lower limit, so that $\phi_0 \, L_0 \ldots$ are constants; then the form of $\dfrac{d\psi}{dx}$ above shows that first differential coefficients are continuous at the boundary $\lambda = 0$. The constant is given by $k\rho = 2\Sigma \, (p L_0 + 2p' L_0') = -4\dot\phi_0$ if (91) is true;

and
$$\dot\phi_0 = -\frac{C}{\sqrt{J_0}} = -C, \quad i.e. \ C = \frac{k\rho}{4}; \text{ and}$$

$$\psi = \frac{k\rho}{4} \cdot \int_\lambda^\infty \frac{d\lambda}{\sqrt{J}} \, (1 - u_x) \ . \ . \ . \ . \ . \ (92)$$

is the value of the external potential. To verify the observance of the condition (91) we have

$$\Sigma p \mathrm{L} + 2p' \mathrm{L}' = \frac{k\rho}{4} \int_\lambda^\infty \frac{\Sigma(p\alpha + 2p'\alpha')}{\sqrt{J}} = \frac{k\rho}{4} \int_\lambda^\infty \frac{dJ}{d\lambda} \cdot \frac{d\lambda}{J^{\frac{3}{2}}} \text{ by (86)}, = \frac{k\rho}{2J^{\frac{1}{2}}}.$$

Since $\dot\phi$ in (92) is $-\dfrac{k\rho}{4J^{\frac{1}{2}}}$, the condition is satisfied.

The field integral is given by

$$\mathrm{S} = \int \frac{1}{2} \rho \psi d\tau_i = \frac{k\rho^2\tau_0}{8} \int_0^\infty \frac{d\lambda}{\sqrt{J}} \, (1 - \Sigma\alpha\overline{x}^2 - 2\Sigma\alpha'\overline{yz})$$

$$= \frac{k\rho^2\tau_0}{8} \int_0^\infty \frac{d\lambda}{\sqrt{J}} \left\{ 1 - \frac{1}{5\Delta_a}\Sigma(\mathrm{A}\alpha + 2\mathrm{A}'\alpha') \right\},$$

where

$$\overline{x}^2 = \frac{1}{\tau_0}\int x^2 \, d\tau_i = \frac{\mathrm{A}}{5\Delta_a}, \quad \text{and} \quad \overline{yz} = \frac{1}{\tau_0}\int yz \, d\tau_i = \frac{\mathrm{A}'}{5\Delta_a}.$$

But by (86)

$$\frac{1}{\Delta_a} \int_0^\infty \frac{d\lambda}{\sqrt{J}} \Sigma(\mathrm{A}\alpha + 2\mathrm{A}'\alpha') = \int_0^\infty \frac{d\lambda}{\sqrt{J}} \left(3 - \frac{\lambda}{J}\frac{dJ}{d\lambda}\right)$$

$$= \int_0^\infty \frac{d\lambda}{\sqrt{J}} + 2\int_0^\infty \frac{d}{d\lambda}\left(\frac{\lambda}{\sqrt{J}}\right) d\lambda = \int_0^\infty \frac{d\lambda}{\sqrt{J}}, \quad (93)$$

since $\lambda / \sqrt{J}$ vanishes at each limit. Hence

$$\mathrm{S} = \int \frac{1}{2} \rho \psi d\tau_i = \frac{k\rho^2\tau_0}{10} \int_0^\infty \frac{d\lambda}{\sqrt{J}} = \frac{ke^2}{10\tau_0} \int_0^\infty \frac{d\lambda}{\sqrt{J}} \ . \ . \ (94)$$

For the geometry of potentials it is often convenient to write the constant of the internal potential separately, as

$$\psi_i = \frac{k\rho}{4} \, (\phi_0 - \Sigma \mathrm{L}_0 x^2 - 2\mathrm{L}_0'yz),$$

and then

$$\Sigma(\mathrm{A}\mathrm{L}_0 + 2\mathrm{A}'\mathrm{L}_0') = \Delta_a\phi_0, \quad \text{and} \quad \Sigma(p\mathrm{L}_0 + 2p'\mathrm{L}_0') = 2$$
$$\left. \right\} \ . \ (95)$$

replace (93) and (91). Thus ϕ_0 is an area, and $L_0 \ldots$ are ratios: and $\phi_0 = \int_0^\infty d\lambda / \sqrt{J}, \quad L_0 = \int_0^\infty a\,d\lambda / \sqrt{J}.$

§ 25. The connexion of the divisors 10 and 12 for volume and surface distributions deserves a word of comment. The difference depends in the main on the fact that the field-integral ($i.\,e.\ \frac{1}{2}\int \Sigma XX' d\tau$) is for a conductor limited to the space outside : for the volume distribution there is no such limitation. Thus for the sphere in the isotropic case, charge gives the same potential and the same field-integral outside the sphere, whether it is uniformly distributed over the surface or through the volume, and the extra fifth part is the internal integral for the latter case. With $\phi_i = \rho\left(\dfrac{a^2}{2} - \dfrac{r^2}{6}\right)$, and $\phi_e = \dfrac{\rho a^3}{3r}$, we have

$$\frac{1}{2}\int_{r=0}^a \left(\frac{d\phi_i}{dr}\right)^2 d\tau_i = \frac{2\pi\rho^2 a^5}{45}, \quad \text{and} \quad \frac{1}{2}\int_{r=a}^\infty \left(\frac{d\phi_e}{dr}\right)^2 d\tau_e = \frac{2\pi\rho^2 a^5}{9}.$$

The ellipsoid is not itself a potential surface with an even volume distribution, and the ratio of the inner to the outer field-integral turns on the ratio of a quadratic function of $L_0 \ldots$ to ϕ_0 (*vide infra*).

For the maintenance of the ratio 6 : 5 of the totals, simple reasons can be given. For a cone of angle $d\omega$ the potential at the vertex is $\rho d\omega \int r^2 dr / r = \rho r^2 d\omega / 2$. Applying this to the potential of an ellipsoid at its centre, and noting $r^3 d\omega = p\,dS$, we have $\int \dfrac{\rho r^2 d\omega}{2} = \int \dfrac{\rho p\,dS}{2r}$. This last gives the potential at the centre, of a surface-charge with element $\rho p\,dS/2$, and therefore total charge $3\rho\tau_0/2$ or $3e/2$. Such a distribution makes the surface equipotential, and its potential is that of its centre. Thus the potential of a volume distribution at the centre is $3/2$ times that of an equal charge distributed on the surface so as to make it equipotential. Combine this with the fact that the mean of the potential for volume distribution taken through the ellipsoid is $4/5$ of its value at the centre, as follows from (95), and we have the explanation of the ratio $6/5$. In æolotropic work we have $(u_P)^{\frac{1}{2}}$ or R in the denominator (*vide infra*), and its ratio to r is the same for different distances in one direction, so that $\int \rho\, d\omega \int r^2 dr / \mathrm{R} = \rho \int p\,dS/2\mathrm{R}$.

In æolotropic potentials the surface $u_P = \epsilon \Delta_p$ has the special characteristics which belong to a sphere in isotropic potentials. In the solution (83) write $\mathrm{P}/\epsilon\Delta_p$ for $a, \ldots$; then

$$\mathrm{A} = p/\epsilon^2\Delta_p, \quad \Delta_a = 1/\epsilon^3\Delta_p, \quad \mathrm{J} = (1 + \lambda/\epsilon)^3, \quad \text{and} \quad \alpha = \mathrm{P}/\Delta_p(\epsilon + \lambda).$$

Thus the æolian surfaces are similar quadrics, viz.

$$u_\mathrm{P} = (\epsilon + \lambda)\Delta_p,$$

and the external potential is

$$\psi_e = \frac{k\rho\epsilon^{\frac{3}{2}}}{4}\int_\lambda^\infty \frac{d\lambda}{(\epsilon+\lambda)^{\frac{3}{2}}}\left[1 - \Sigma(\mathrm{P}x^2 + 2\mathrm{P}'yz)/\Delta_p(\epsilon+\lambda)\right].$$

But

$$\int_\lambda^\infty \frac{d\lambda}{(\epsilon+\lambda)^{\frac{5}{2}}} = \frac{1}{3(\epsilon+\lambda)}\int_\lambda^\infty \frac{d\lambda}{(\epsilon+\lambda)^{\frac{3}{2}}},$$

which with $\Sigma(\mathrm{P}x^2 + 2\mathrm{P}'yz) = (\epsilon+\lambda)\Delta_p$ gives

$$\psi_e = \frac{k\rho\epsilon^{\frac{3}{2}}}{6}\int_\lambda^\infty \frac{d\lambda}{(\epsilon+\lambda)^{\frac{3}{2}}} = \frac{k\rho\epsilon^{\frac{3}{2}}}{3(\epsilon+\lambda)^{\frac{1}{2}}} = \frac{k\rho\epsilon^{\frac{3}{2}}\Delta_p^{\frac{1}{2}}}{3\{\Sigma(\mathrm{P}x^2+2\mathrm{P}'yz)\}^{\frac{1}{2}}}$$

$$= \frac{k\rho\tau_0}{4\pi\{\Sigma(\mathrm{P}x^2+2\mathrm{P}'yz)\}^{\frac{1}{2}}}. \quad \cdot \quad \cdot \quad \cdot \quad (96)$$

Thus the external potential is the same as for a conductor with equal charge, and the action is that of a point charge at the centre under the law by which $k/u_\mathrm{P}^{\frac{1}{2}}$ replaces $1/r$. The use of the infinite upper limit requires ϵ to be positive.

§ 26. We now prove a theorem akin to Maclaurin's theorem in isotropic potentials. Let u_{α_1}, u_{α_2} be two æolians derived from u_a by the parameters λ_1, and $\lambda_1 + \lambda_2$ respectively. We shall show that α_2 may be derived from α_1 by the parameter λ_2, α_1 and λ_2 taking the places of a and λ in the scheme (81); and shall then establish a relation between the discriminants used in getting at α_2 according as we set out from a or from α_1. For $\alpha_2\ldots$ as derived from the base a by the parameter $\lambda_1 + \lambda_2$, the first equation of (81) is

$$\alpha_2(\mathrm{A} + p\lambda_1\Delta_a + p\lambda_2\Delta_a) + \gamma_2'(\overline{\mathrm{C}' + r'\lambda_1\Delta_a} + r'\lambda_2\Delta_a)$$
$$+ \beta_2'(\overline{\beta' + q'\lambda_1\Delta_a} + q'\lambda_2\Delta_a) = \Delta_a.$$

Multiply each member by Δ_a^2, and use both relations (84) with λ_1 for the λ there; then removing the common factor $\Delta(\lambda_1)$, there remains

$$\alpha_2(\mathrm{A}_{\alpha_1} + p\lambda_2\Delta_{\alpha_1}) + \gamma_2'(\mathrm{C}'_{\alpha_1} + r'\lambda_2\Delta_{\alpha_1}) + \beta_2'(\mathrm{B}'_{\alpha_1} + q'\lambda_2\Delta_{\alpha_1}) = \Delta_{\alpha_1}, \quad (97)$$

which reproduces the form of (81) with λ_2 for parameter and $\alpha_1,\ldots$ for base. Now use the first of (84), viz. $\Delta_a\Delta(\lambda) = \Delta_a^3$ or its equivalent the first of (85) $\Delta_a\mathrm{J}(a, \lambda) = \Delta_a$ in the three forms required to get α_2 and α_1 from the base a, and to get α_2 from the base α_1. They are $\Delta_{\alpha_1}\mathrm{J}(a, \lambda_1) = \Delta_a = \Delta_{\alpha_2}\mathrm{J}(a, \lambda_1+\lambda_2)$, and $\Delta_{\alpha_2}\mathrm{J}(\alpha_1, \lambda_2) = \Delta_{\alpha_1}$; and their product gives

$$\mathrm{J}(a, \lambda_1 + \lambda_2) = \mathrm{J}(a, \lambda_1) \times \mathrm{J}(\alpha_1, \lambda_2), \quad \cdot \quad \cdot \quad (98)$$

i. e. $J(a, \lambda_1 + \lambda_2)$ is the product of the discriminants required for the two steps. [In terms of $\Delta(\lambda) \dots$ the property is $\Delta_{\alpha_1}^2 \Delta(\lambda_1 + \lambda_2) = \Delta(\lambda_1) \Delta_{\alpha_1}(\lambda_2)$.]

Apply this to the potential of a conductor at an external point on λ_1, which is

$$\psi_{\lambda_1} = \frac{ke}{6\tau_0} \int_{\lambda_1}^{\infty} \frac{d\lambda_1}{\sqrt{J(a, \lambda_1)}} = \frac{ke}{6\tau_0} \int_0^{\infty} \frac{d\lambda_2}{\sqrt{J(a, \lambda_1 + \lambda_2)}}$$

$$= \frac{ke}{6\tau_0 \sqrt{J(a, \lambda_1)}} \int_0^{\infty} \frac{d\lambda_2}{\sqrt{J(\alpha_1, \lambda_2)}} = \frac{ke}{6\tau_1} \int_0^{\infty} \frac{d\lambda_2}{\sqrt{J(\alpha_1, \lambda_2)}}. \quad (99)$$

The potential is the same whether the charge is on the original surface, or on the æolian on which the point lies, or as at once follows on any intermediate æolian. In the application to volume distribution we make the same change from $\int_{\lambda_1}^{\infty}$ to $\int_0^{x}$ by writing α as dependent on $\lambda_1 + \lambda_2$ instead of λ_1, and integrating with regard to λ_2. Then α_2 is expressed in terms of α_1 and λ_2, and through $\sqrt{J}$ (the form for L being $\int \alpha d\lambda/\sqrt{J}$) we have the alteration from τ_0 to τ_1 as above. Thus for a volume distribution also, the potential at any point on an æolian surface is the same whether the charge is evenly distributed through the volume of the original ellipsoid, or through that of the æolian in question or any intermediate æolian, with one total charge for all cases.

For conductors the whole meaning is simple, viz. if a conductor is supposed to shrink following æolian forms and retaining its charge, the potential is not thereby altered at any point external to the forms considered; and each point in space as it passes from the inside to the outside by the shrinking of the conductor attains a potential which then remains unaltered. Thus the energy of any part of the field outside a given æolian is not altered by the shrinkage, and the whole addition to energy is the energy of the volume which becomes external in the shrinkage. The element added during a small shrinkage is

$$\int \frac{1}{2} \Sigma X X' . dn . dS = -\int \frac{1}{2} \Sigma X \frac{d\phi}{dx} . dn . dS$$

$$= -\int \frac{1}{2} \frac{d\phi}{dn} dn \, \Sigma l X \, dS = -\frac{e}{2} d\phi ;$$

dn is an element of the normal, $+d\phi$ is the increment in moving outwards. *i. e.* $-d\phi$ the positive increment in moving inwards.

§ 27. The special forms suited to the motional problem with the ellipsoid referred to principal axes are got by putting $a'=b'=c'=0$, $A'=B'=C'=0$; and writing as in (72) $1-p^2$ for p, $-qr$ for $p'\ldots$, which involve

$$P=1-q^2-r^2,\quad P'=qr,\quad \Delta_p=1-\Sigma p^2.$$

The equation to an æolian surface is then

$$1+\lambda\Sigma a(1-p^2)+\lambda^2\Sigma bc(1-q^2-r^2)+\lambda^3 abc(1-\Sigma p^2)$$
$$=\Sigma ax^2\{1+\lambda(b\,\overline{1-q^2}+c\,\overline{1-r^2})+\lambda^2 bc(1-q^2-r^2)\}$$
$$+2\Sigma bc\,qr\,yz\,\lambda(1+\lambda a),\ .\ .\ .\ (100)$$

the equation giving the value of J on the left hand, those of $J\alpha,\ldots J\alpha'\ldots$ on the right. Or if we write $1/a^2$ for a, $\ldots$ so that the new a's are principal semi-axes,

$$a^2b^2c^2+\lambda\Sigma b^2c^2(1-p^2)+\lambda^2\Sigma a^2(1-q^2-r^2)+\lambda^3(1-p^2-q^2-r^2)$$
$$=\Sigma x^2[b^2c^2+\lambda\{c^2(1-q^2)+b^2(1-r^2)\}+\lambda^2(1-q^2-r^2)]$$
$$+2\Sigma qr\,yz\,\lambda(a^2+\lambda)\ .\ .\ (101)$$

We may use J' for the expression on the left, and then in terms of J' we have for a conductor

$$\phi=\frac{e(1-\Sigma p^2)}{8\pi}\int_\lambda^\infty \frac{d\lambda}{\sqrt{J'}},\quad S=\frac{e^2(1-\Sigma p^2)}{16\pi}\int_0^\infty\frac{d\lambda}{\sqrt{J'}}=\frac{\rho e(1-\Sigma p^2)}{12}\phi_0,\quad(102)$$

and for a volume distribution

$$\psi=\frac{3e(1-\Sigma p^2)}{16\pi}\int_\lambda^\infty\frac{d\lambda}{\sqrt{J'}}\left[1-J'^{-1}\Sigma\{x^2(\ldots)+2qr\,yz\,\lambda(a^2+\lambda)\ldots\}\right]$$
$$S=\frac{3e^2(1-\Sigma p^2)}{40\pi}\int_0^\infty\frac{d\lambda}{\sqrt{J'}}=\frac{\rho e(1-\Sigma p^2)}{10}\phi_0,$$

$$\left.\right\} (103)$$

the term multiplying J'^{-1} in ψ being the right-hand member of (101).

§ 28. We may now consider the derivation of T and E from S, and the comparison of their values with the statical case. First notice the form of J', viz.

$$J'=(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)-p^2\lambda(b^2+\lambda)(c^2+\lambda)-q^2\lambda(c^2+\lambda)(a^2+\lambda)$$
$$-r^2\lambda(a^2+\lambda)(b^2+\lambda)$$
$$=(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\left[1-\frac{p^2\lambda}{a^2+\lambda}-\frac{q^2\lambda}{b^2+\lambda}-\frac{c^2\lambda}{r^2+\lambda}\right]$$
$$=J_s'\left[1-p^2-q^2-r^2+\frac{a^2p^2}{a^2+\lambda}+\frac{b^2q^2}{b^2+\lambda}+\frac{c^2r^2}{c^2+\lambda}\right]$$

$$\left.\right\}, (104)$$

J_s' being the statical value of J'. Write

$$S = C(1 - \Sigma p^2) \int_0^\infty \frac{d\lambda}{\sqrt{J'}},$$

C standing for $e^2/16\pi$ or $3e^2/40\pi$, with conductor or volume-distribution. Then

$$-\frac{1}{p}\frac{dS}{dp} = C \int_0^\infty \frac{d\lambda}{\sqrt{J'}} \left[2 - (1 - \Sigma p^2)\lambda(b^2 + \lambda)(c^2 + \lambda)/J' \right], \quad (105)$$

and since

$$2J' - (1 - \Sigma p^2)\lambda(b^2 + \lambda)(c^2 + \lambda)$$
$$= 2a^2b^2c^2 + \lambda\{b^2c^2(1 + \Sigma p^2) + 2a^2(c^2\overline{1 - q^2} + b^2\overline{1 - r^2})\}$$
$$+ \lambda^2\{b^2(1 + q^2) + c^2(1 + r^2) + 2a^2(1 - q^2 - r^2)\} + \lambda^3(1 - \Sigma p^2),$$

$-\dfrac{1}{p}\dfrac{dS}{dp}$ which is V^2P/u is essentially positive if $\Sigma p^2 < 1$,

i. e. ' electromagnetic mass ' is positive [*].

From (105) with (104) we get

$$T = \Sigma Pu = -\Sigma p \frac{dS}{dp} = C \int_0^\infty \frac{d\lambda}{\sqrt{J'^3}} \left[J'(1 + \Sigma p^2) - J_s'(1 - \Sigma p^2) \right], \quad (106)$$

In terms of $\phi_0 L_0, \dots$ the simplest expression of P is

$$VP = \frac{\rho e}{10}\left(p\phi_0 + pL_0a^2 + rM_0'c^2 + qN_0'b^2 \right), \quad . \quad (107)$$

and N_0' contains the factors pq explicitly. To obtain this form we make use of

$$pJ' = (pa^2\alpha + qb^2\gamma' + rc^2\beta')J' + p(1 - \Sigma p^2)\lambda(b^2 + \lambda)(c^2 + \lambda).$$

The relation

$$pq(La^2 - Mb^2) = N_0'\{a^2(1 - q^2 - r^2) - b^2(1 - r^2 - p^2)\} + c^2r(pL' - qM')$$

is also of service in working with (107).

How does S compare with the statical value S_s or E_s? Going back to first principles, in the element of integral $1/r$ is replaced by $(1 - \Sigma p^2)/R$, where

$$R^2 = (1 - \Sigma p^2)\Sigma(x - x')^2 + (\Sigma p(x - x'))^2.$$

---

[*] With $\Sigma p^2 > 1$, the bracket in (105) is positive as long as J' is positive. The integral is then curtailed at the upper limit (*v. infra*) and the condition is not necessary.

It is clear that S in comparison with S_s is not so much reduced as the factor $1 - \Sigma p^2$ would imply, and is more reduced than the factor $(1 - \Sigma p^2)^{\frac{1}{2}}$ would imply, *i.e.* $S > S_s(1 - \Sigma p^2)$ and $< S_s(1 - \Sigma p^2)^{\frac{1}{2}}$; a conclusion which is confirmed by the last form of J' in (104). Apply to (106) the inequalities $J' > J_s'(1 - \Sigma p^2)$ and $< J_s'$, and it appears that

$$T > S \Sigma p^2/(1 - \Sigma p^2), \text{ and } < 2S \Sigma p^2/(1 - \Sigma p^2),$$

or since

$$E = S + T, \; E > S/(1 - \Sigma p^2), \text{ and } < S(1 + \Sigma p^2)/(1 - \Sigma p^2);$$

$$\left. \begin{array}{c} \end{array} \right\} \quad (108)$$

and if we apply $S > S_s(1 - \Sigma p^2)$ and $< S_s(1 - \Sigma p^2)^{\frac{1}{2}}$ to these,

$$\left. \begin{array}{l} T > S_s \Sigma p^2, \text{ and } < 2 S_s \Sigma p^2 / \sqrt{1 - \Sigma p^2} \\ E > S_s, \quad \text{ and } < S_s(1 + \Sigma p^2)/\sqrt{1 - \Sigma p^2} \end{array} \right\} . \quad (109)$$

What orientation gives a minimum value to S for a given value of Σu^2 or Σp^2?

The general stationary condition

$$\frac{dS}{du} = ku\ldots, \quad \text{or} \quad \frac{1}{p}\frac{dS}{dp} = \frac{1}{q}\frac{dS}{dq} = \frac{1}{r}\frac{dS}{dr}$$

leads to $a = b = c$, which denotes the obvious indifference of the sphere to direction. Take then a change from r_0 along c with no p or q, to pqr with $p^2 + q^2 + r^2 = r_0^2$, p and q small. Then

$$J' = (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda \cdot \overline{1 - r_0^2})$$
$$- \lambda p^2(c^2 - a^2)(b^2 + \lambda) - \lambda q^2(c^2 - b^2)(a^2 + \lambda).$$

$$\therefore \int_0^{\infty} \frac{d\lambda}{\sqrt{J_1'}} = \int_0^{\infty} \frac{d\lambda}{\sqrt{J_1'}}\left[1 + \frac{\lambda}{2 J_1'}\{(c^2 - a^2)p^2(b^2 + \lambda) + (c^2 - b^2)q^2(a^2 + \lambda)\} + \ldots\right].$$

If c is the greatest axis, the value given by the use of J_1' with r_0, $p = q = 0$ is a minimum; if c is the least axis it is a maximum, and if c is the mean axis change depends on direction. Since S is a momental expression for energy we expect a minimum to correspond to stability, and that requires the greatest axis to be in the direction of the translation.

There is a maximum question relating to the amount of energy when different shapes of ellipsoid with the same volume are considered. The condition of no variation is $\Sigma \dfrac{dS}{da} da = 0$, subject to $\Sigma \dfrac{da}{a} = 0$, *i. e.* $a \dfrac{dS}{da} = b \dfrac{dS}{db} = c \dfrac{dS}{dc}$, or $a^2 \dfrac{dS}{d(a^2)} = b^2 \dfrac{dS}{d(b^2)} = c^2 \dfrac{dS}{d(c^2)}$, *i. e.* $L_0 a^2 = M_0 b^2 = N_0 c^2$. If it is assumed that the translation must be along a principal axis z say, the equation (95) makes $\dfrac{\phi_0}{3} = L_0 a^2 = M_0 b^2 = N_0 c^2$. The other relation of (95) gives $L_0 + M_0 + N_0(1 - r^2) = 2$, and when these are combined $6 = \phi_0 \{ a^{-2} + b^{-2} + c^{-2}(1 - r^2) \}$. Thus the transcendental character has disappeared from the integral $\int d\lambda / \sqrt{J'}$, and therefore J' has the form $(1 + \lambda)^3 (1 - r^2)$. The conditions are satisfied by $a = b$ and $c^2 = a^2(1 - r^2)$, and these conditions make $\phi_0 = 2a^2$, $L_0 = M_0 = N_0(1 - r^2) = \dfrac{2}{3}$. This is the spheroid which has in other respects the status of the sphere in statical potential.

Again, reference was made above § 25 to the fraction of the whole energy which, for a case of volume distribution, belongs to the field-integral within the ellipsoid. When the translation is along a principal axis this fraction is $L_0^2 a^2 + M_0^2 b^2 + N_0^2 c^2 (1 - r^2) : 4\phi_0$ [the quadric is more complicated in the general case]. This ratio is also a maximum for $a = b$ and $c^2 = a^2(1 - r^2)$, and has then the value $1/6$.

The condition for these two maximum questions is not consistent with that for stability as regards the effect of translation. They may be of moment when losses or accretions of charge are in view; or the conflict of conditions on the electrical and mechanical sides may lead to a volume distribution which is not uniform.

§ 29. We may now deal briefly with special cases, the simpler exact cases and the general approximation. J' is at once divisible into factors (1) for the sphere, viz.

$$J' = (a^2 + \lambda)^2 \{ a^2 + \lambda(1 - \Sigma p^2) \},$$

(2) for any ellipsoid with translation in the direction of a principal axis :—Thus for $p = q = 0$,

$$J' = (a^2 + \lambda)(b^2 + \lambda) \{ c^2 + \lambda(1 - r^2) \}.$$

The integrations for (2) can be effected when $a = b$, and as this case has been treated, it is sufficient to schedule the

results for reference. We have

$$S = \frac{\rho e}{10}(1-r^2)\phi_0, \qquad T = \frac{\rho e}{10}r^2(\phi_0 + N_0 c^2),$$

for volume distribution, denominator 12 for conductor;

$$\phi_0 = \int_0^\infty \frac{a^2 c\, d\lambda}{(a^2+\lambda)\{c^2+\lambda(1-r^2)\}^{\frac{1}{2}}}, \quad L_0 = M_0 = \int_0^\infty \frac{a^2 c\, d\lambda}{(a^2+\lambda)^2\{c^2+\lambda(1-r^2)\}^{\frac{1}{2}}},$$

$$N_0 = \int_0^\infty \frac{a^2 c}{(a^2+\lambda)\{c^2+\lambda(1-r^2)\}^{3/2}}.$$

(i.) For $a^2(1-r^2) > c^2$, and $\gamma^2 = a^2(1-r^2) - c^2$

$$\phi_0 = \frac{a^2 c}{\gamma}\tan^{-1}\frac{\gamma}{c}, \quad L_0 = M_0 = \frac{1}{\gamma^2}\left\{\frac{1}{2}\phi_0(1-r^2) - c^2\right\}, \quad N_0 = \frac{1}{\gamma^2}(2a^2 - \phi_0);$$

(ii.) for $a^2(1-r^2) < c^2$, and $\gamma^2 = c^2 - a^2(1-r^2)$

$$\phi_0 = \frac{a^2 c}{\gamma}\log\frac{c+\gamma}{c-\gamma}, \quad L_0 = M_0 = \frac{1}{\gamma^2}\left\{c^2 - \frac{1}{2}\phi_0(1-r^2)\right\}, \quad N_0 = \frac{1}{\gamma^2}(\phi_0 - 2a^2).$$

This includes the sphere for which

$$\phi_0 = \frac{a^2}{r}\log\frac{1+r}{1-r}, \quad L_0 = M_0 = -\frac{1-r^2}{2r^3}\log\frac{1+r}{1-r} + \frac{1}{r^2}, \quad N_0 = \frac{1}{r^3}\log\frac{1+r}{1-r} - \frac{2}{r^2},$$

with

$$S = S_s\frac{1-r^2}{2r}\log\frac{1+r}{1-r}, \qquad T = S_s\left\{\frac{1+r^2}{2r}\log\frac{1+r}{1-r} - 1\right\},$$

$$E = S_s\left\{\frac{1}{r}\log\frac{1+r}{1-r} - 1\right\}, \qquad S_s = \frac{\rho e a^2}{5}\ \text{or}\ \frac{\rho e a^2}{6}.$$

(iii.) For $c^2 = a^2(1-r^2)$, $\phi_0 = 2a^2$, $L_0 = M_0 = N_0(1-r^2) = \frac{2}{3}$.

(110)

It may also be mentioned that we get the case for two dimensions by putting $c=0$ in the original notation (100), viz.

$$1 + \lambda\{a(1-p^2) + b(1-q^2)\} + \lambda^2 ab(1-p^2-q^2)$$
$$= ax^2\{1 + \lambda b(1-q^2)\} + by^2\{1 + \lambda a(1-p^2)\} + 2abpq\lambda xy,$$

which is independent of r.

When (pqr) are all small quantities, the general case is much simplified. Using (104),

$$S = \frac{\rho e}{10}\int_0^\infty \frac{abcd\lambda}{\sqrt{J_s'}}\left[1 - \frac{1}{2}\Sigma p^2 - \frac{1}{2}\Sigma\frac{a^2 p^2}{a^2+\lambda}\right]$$

$$= \frac{\rho e}{10}\left[\phi_0\left(1 - \frac{1}{2}\Sigma p^2\right) - \frac{1}{2}\Sigma a^2 p^2 L_0\right],$$

and then

$$E = S - \Sigma p\frac{dS}{dp} = \frac{\rho e}{10}\left[\phi_0\left(1 + \frac{1}{2}\Sigma p^2\right) + \frac{1}{2}\Sigma a^2 p^2 L_0\right],$$

(111)

in which $J_s' = (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)$, and so $\phi_0 L_0 \ldots$ belong to the isotropic solution.

Thus E is as much above the statical value E_s or S_s, as S is below it : or the total energy associated with the nucleus is raised, while the part interpreted as radiant energy is diminished by an equal amount.

The z-component of momentum is

$$R = -\frac{dS}{dw} = -\frac{1}{V}\frac{dS}{dr} = wE_s(1 + c^2N_0/\phi_0)/V^2.$$

Since $\Sigma L_0 a^2 = \phi_0$, the possible range of such a fraction as $N_0 c^2 / \phi_0$ is 0 to 1 ; or R/w ranges from E_s/V^2 to $2E_s/V^2$, the latter form giving $E_s = \dfrac{mV^2}{2}$ with $R = mw$.

The special values for a much elongated form of ellipsoid and for a much flattened form, say the needle and disk limits, may be noted as extreme cases. If two axes are small compared with the third c,

$$N_0 c^2 = \phi_0, \quad S = E_s(1 - r^2 - \tfrac{1}{2}p^2 - \tfrac{1}{2}q^2) ;$$

and thus $P/u = Q/v = E_s/V^2$, and $R/w = 2E_s/V^2$ for the needle limit. If one axis c is small and the others equal

$$L_0 a^2 = M_0 b^2 = \phi_0/2, \quad S = E_s\left(1 - \frac{r^2}{2} - \frac{3}{4}p^2 - \frac{3}{4}q^2\right);$$

and $P/u = Q/v = 3E_s/2V^2$, $R/w = E_s/V^2$ for the disk limit. When r only exists but is finite $S = E_s(1 - r^2)$ for the needle limit, and $= E_s\sqrt{1 - r^2}$ for the disk limit.

§ 30. The solution when density is a linear function of xyz may be dealt with constructively. First we verify that

$$\chi_1 = \frac{\rho}{4}\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}}(ax + \gamma'y + \beta'z)(1 - u_a) \quad . \quad (112)$$

is the external potential for a density $\rho(ax + c'y + b'z)$, and that the internal potential is got by taking 0 for the lower limit of the integral (u_a as before being $\Sigma ax^2 + 2a'yz$).

We have

$$\frac{d\chi_1}{dx} = \frac{\rho}{4}\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}}\left\{ \alpha(1 - u_a) - (ax + \gamma'y + \beta'z)\frac{\partial u_a}{\partial x} \right\},$$

$$\frac{d\chi_1}{dy} = \frac{\rho}{4}\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}}\left\{ \gamma'(1 - u_a) - (ax + \gamma'y + \beta'z)\frac{\partial u_a}{\partial y} \right\},$$

while

$$\nabla_e^2 \chi_1 = \left(p\frac{d}{dx} + r'\frac{d}{dy} + q'\frac{d}{dz} \right)\frac{d\chi_1}{dx} + \left(r'\frac{d}{dx} + q\frac{d}{dy} + p'\frac{d}{dz} \right)\frac{d\chi_1}{dy}$$

$$+ \left(q'\frac{d}{dx} + p'\frac{d}{dy} + r\frac{d}{dz} \right)\frac{d\chi_1}{dz}.$$

The part of $\nabla_e^2 \chi_1$ which is due to implicit differentiation through λ is

$$\frac{\rho}{4\sqrt{J}}\,(\alpha x + \gamma' y + \beta' z)\left\{ \frac{\partial u_a}{\partial x}\left(p\frac{d\lambda}{dx} + r'\frac{d\lambda}{dy} + q'\frac{d\lambda}{dz} \right) \right.$$

$$\left. + \frac{\partial u_a}{\partial y}\left(r'\frac{d\lambda}{dx} + q\frac{d\lambda}{dy} + p'\frac{d\lambda}{dz} \right) + \frac{\partial u_a}{\partial z}\left(q'\frac{d\lambda}{dx} + p'\frac{d\lambda}{dy} + r\frac{d\lambda}{dz} \right) \right\},$$

which by (78) is $\rho(\alpha x + \gamma' y + \beta' z)/\sqrt{J}$. The part of $\nabla_e^2 \chi_1$ which is due to explicit differentiation with regard to xyz is

$$-\frac{\rho}{2}\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}}\left[\left\{ (p\alpha + r'\gamma' + q'\beta')\frac{\partial u_a}{\partial x} + (r'\alpha + q\gamma' + p'\beta')\frac{\partial u_a}{\partial y} \right.\right.$$

$$\left.\left. + (q'\alpha + p'\gamma' + r\beta')\frac{\partial u_a}{\partial z} \right\} + (\alpha x + \gamma' y + \beta' z)\Sigma(p\alpha + 2p'\alpha') \right].$$

The coefficient of x in $\{\ \}$ is

$$2\left\{ \alpha(p\alpha + r'\gamma' + q'\beta') + \gamma'(r'\alpha + q\gamma' + p'\beta') + \beta'(q'\alpha + p'\gamma' + r\beta') \right\}$$

which by (80) is $-2\dot\alpha$, i. e. $\{\ \}$ is $-2(\dot\alpha x + \dot\gamma' y + \dot\beta' z)$; also by (86) $\Sigma(p\alpha + 2p'\alpha') = \dot J/J$.

Hence the second part of $\nabla_e^2 \chi_1$ is

$$\rho\int_\lambda^\infty \frac{d\lambda}{\sqrt{J}}\left\{ \dot\alpha x + \dot\gamma' y + \dot\beta' z - \dot J(\alpha x + \gamma' y + \beta' z)/J \right\}$$

$$= \rho\int_\lambda^\infty \frac{d}{d\lambda}\frac{\alpha x + \gamma' y + \beta' z}{\sqrt{J}}\,d\lambda = -\frac{\rho}{\sqrt{J}}(\alpha x + \gamma' y + \beta' z).$$

Thus outside the ellipsoid, where both parts occur, the terms are cancelled and $\nabla_e^2 \chi_1 = 0$; while within, where only the latter part appears, $\nabla_e^2 \chi_1 + \rho(ax + c'y + b'z) = 0$, because the use of $\lambda = 0$ for the lower limit makes $J = 1$, $\alpha = a$, $\gamma' = c$ $\beta' = b'$.

The energy in this case is

$$\frac{\rho^2}{8}\int_0^\infty \frac{d\lambda}{\sqrt{J}}\int d\tau_i(ax + c'y + b'z)(\alpha x + \gamma' y + \beta' z)(1 - u_a),$$

viz.: the volume-integral of half the product of density and

potential. The integral of the section which is of the second order in xyz is

$$\frac{\rho^2\tau_0}{40\Delta_a}\int_0^\infty \frac{d\lambda}{\sqrt{J}}\left\{ \alpha(\mathrm{A}a+\mathrm{C}'c'+\mathrm{B}'b') \right.$$
$$\left. +\gamma'(\mathrm{C}'a+\mathrm{B}c'+\mathrm{A}'b')+\beta'(\mathrm{B}'a+\mathrm{A}'c'+\mathrm{C}b') \right\},$$

or
$$\frac{\rho^2\tau_0}{40}\int_0^\infty \frac{\alpha d\lambda}{\sqrt{J}}.$$

We shall show that the second part involving u_a is $\frac{3}{7}$ of the first part, so that the total energy is

$$\frac{\rho^2\tau_0}{70}\int_0^{\infty} \frac{\alpha d\lambda}{\sqrt{J}}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (113)$$

The constructive proof sets out from (87) which gives

$$3\Delta_a\int_0^\infty \frac{\alpha d\lambda}{\sqrt{J}}-2\int_0^\infty \frac{d\lambda}{\sqrt{J}}(\mathrm{A}\alpha^2+\mathrm{B}\gamma'^2+\mathrm{C}\beta'^2+2\mathrm{C}'\alpha\gamma'+2\mathrm{B}'\alpha\beta'+2\mathrm{A}'\beta'\gamma')$$

$$=\Delta_a\int_0^\infty \frac{d\lambda}{\sqrt{J}}\left\{ 3\alpha-2\frac{d}{d\lambda}(\alpha\lambda) \right\} =\Delta_a\int_0^\infty \frac{\alpha d\lambda}{\sqrt{J}}(3-\lambda\dot{J}/J),$$

integrating the second section by parts,

$$=\int_0^\infty \frac{\alpha d\lambda}{\sqrt{J}}\Sigma(\mathrm{A}\alpha+2\mathrm{A}'\alpha'),\text{ quoting (86).\quad Thus}$$

$$3\Delta_a\int_0^\infty \frac{\alpha d\lambda}{\sqrt{J}}=\int_0^\infty \frac{d\lambda}{\sqrt{J}}[\alpha\Sigma(\mathrm{A}\alpha+2\mathrm{A}'\alpha')+2(\mathrm{A}\alpha^2+\mathrm{B}\gamma'^2+\ldots)], \quad (114)$$

or the integral which is quadratic in $\alpha\ldots$ on the right-hand is reducible to a linear form. Now the integral

$$\int_0^\infty \frac{d\lambda}{\sqrt{J}}\int d\tau_i(ax+c'y+b'z)(\alpha x+\gamma'y+\beta'z)u_a$$

is of the second order in $\alpha\ldots$, of the fourth order in xyz. To effect the volume integrations we have *, with

$$\bar{x}^4\tau_0=\int x^4 d\tau_i, \quad \overline{x^3y}\tau_0=\int x^3y d\tau_i,\ldots$$
$$35\Delta_a^2\bar{x}^4=3\mathrm{A}^2, \quad 35\Delta_a^2\overline{x^2y^2}=\mathrm{AB}+2\mathrm{C}'^2, \quad 35\Delta_a^2\overline{x^3y}=3\mathrm{AC}',$$
$$35\Delta_a^2\overline{x^2yz}=2\mathrm{B}'\mathrm{C}'+\mathrm{AA}' \quad . \quad \cdot \quad \cdot \quad (115)$$

It will be found that the substitution of these means will

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* These statements are readily proved by transformation from the principal axes to other axes. I have not been able to find these, or indeed the simpler formulæ used before for $\bar{x}^2$ and $\overline{xy}$.

lead through (114) to the conclusion stated above. Thus for example the part involving α^2 is $\int \alpha^2 x^3 (ax + c'y + b'z) d\tau_i$

$$= \frac{\alpha^2 \tau_0}{35 \Delta_a^2} (3 A^2 a + 3 AC'c' + 3 AB'b') = \frac{\tau_0}{35 \Delta_a} (3 \bar{A} \alpha^2),$$

the term in the bracket being the same as that in the square bracket of (114).

With a view to other linear functions of xyz as density, we note that the energy-integral when χ_1 is multiplied by $\rho(c'x + by + a'z)$ as density $\left(\text{i. e. } \dfrac{\rho}{2} \dfrac{du}{dy} \text{ instead of } \dfrac{\rho}{2} \dfrac{du}{dx}\right)$, may be found by the same method to be $\dfrac{\rho^2 \tau_0}{70} \displaystyle\int_0^\infty \dfrac{\gamma' d\lambda}{\sqrt{J}}$; or with $\dfrac{\rho}{2} \dfrac{du}{dz}$ as density, to be $\dfrac{\rho^2 \tau_0}{70} \displaystyle\int_0^\infty \dfrac{\beta' d\lambda}{\sqrt{J}}$. The transformation corresponding to (114) is got by using $\int \gamma' d\lambda / \sqrt{J}$, and quoting the second relation of (87) as written for γ'.

In dealing with a density ρx, we see that x and χ_x are the same linear functions of $\dfrac{1}{2} \dfrac{du}{dx}, \dfrac{1}{2} \dfrac{du}{dy}, \dfrac{1}{2} \dfrac{du}{dz}$, and of the potentials due to densities $\dfrac{\rho}{2} \dfrac{du}{dx}, \dots$ say $\chi_1 \, \chi_2 \, \chi_3$; then $\chi_x = (A\chi_1 + C'\chi_2 + B'\chi_3)/\Delta_a$. The energy-term belonging to χ_1 with its proper density is $\dfrac{\rho^2 \tau_0}{70} \displaystyle\int_0^\infty \dfrac{\alpha d\lambda}{\sqrt{J}}$, but when it is associated with the densities proper to χ_2 and χ_3, α is replaced by γ' and β' respectively. Thus the energy belonging to ρx is got by writing the square of $(A\chi_1 + C'\chi_2 + B'\chi_3)/\Delta_x$, and putting α for χ_1^2, β for χ_2^2, γ' for $\chi_1 \chi_2 \dots$, and is

$$\frac{\rho^2 \tau_0}{70 \Delta_a^2} \int_0^\infty \frac{d\lambda}{\sqrt{J}} (A^2\alpha + C'^2\beta + B'^2\gamma + 2B'C'\alpha' + 2AB'\beta' + 2AC'\gamma') \dots \text{(116)}$$

For ρy the bracket is replaced by

$$C'^2\alpha + B^2\beta + A'^2\gamma + 2A'B\alpha' + 2A'C'\beta' + 2BC'\gamma';$$

and for the integral in which either ρx is associated with χ_y or ρy with χ_x, it is replaced by

$$C'(A\alpha + B\beta) + A'B'\gamma' + (BB' + C'A')\alpha' + (AA' + B'C')\beta' + (AB + C'^2)\gamma'.$$

If we apply these to the evaluation of $S' = \dfrac{1}{2V}\displaystyle\int d\tau_i \Sigma i_x F$ where $i_x = \omega_2 z - \omega_3 y, \ldots$ we have to deal with

$$\frac{\rho}{2V^2}\int d\tau_i \Sigma(\omega_2 z - \omega_3 y)(\omega_2 \chi_z - \omega_3 \chi_y),$$

or

$$\frac{\rho}{2V^2}\int d\tau_i [\Sigma \omega_3^2 (x\chi_z + y\chi_y) - \Sigma \omega_1 \omega_2 (x\chi_y + y\chi_x)].$$

The result when principal axes are used is

$$S' = \frac{\rho^2 \tau_0}{10 V^2}[\Sigma \omega_3^2 (L_0 a^4 + M_0 b^4) - 2\Sigma \omega_1 \omega_2 N_0' a^2 b^2]. \quad (117)$$

In this form it is not obvious, but it will appear later that S' is essentially positive. For a sphere without special choice of axes, if

$$\kappa^2 = \Sigma p^2 \quad \text{and} \quad f = \frac{1}{\kappa^2} - \frac{1-\kappa^2}{2\kappa^3}\log\frac{1+\kappa}{1-\kappa},$$

then

$$L_0 = f + \frac{\rho^2}{\kappa}\frac{df}{d\kappa}, \quad L_0' = \frac{qr}{\kappa}\frac{df}{d\kappa};$$

so that

$$S' = \frac{\rho^2 \tau_0 a^4}{35 V^2}\left[f\Sigma \omega^2 + \frac{1}{2\kappa}\frac{df}{d\kappa}\Sigma(p\omega_2 - q\omega_1)^2\right].$$

If the axis of translation is also that of rotation, viz. that of z,

$$S' = \frac{\rho^2 \tau_0 a^4 \omega_3^2}{35 V^2}\left[\frac{1}{r^2} - \frac{1-r^2}{2r^3}\log\frac{1+r}{1-r}\right];$$

and then

$$R = \frac{dS'}{dw} = \frac{1}{V}\frac{dS'}{dr} = \frac{\rho^2 \tau_0}{35}\left(\frac{\omega_3 a}{V}\right)^2 \frac{a^2}{V}\left/\left[\frac{3-r^2}{2r^4}\log\frac{1+r}{1-r} - \frac{3}{r^3}\right]\right.,$$

which may be compared with the value of R for the translation of charge, viz.

$$\frac{\rho^2 \tau_0 a^2}{5V}\left[\frac{1+r^2}{2r^2}\log\frac{1+r}{1-r} - \frac{1}{r}\right].$$

The ratio is $\dfrac{1}{35}\left(\dfrac{\omega_3 a}{V}\right)^2$ when r is small, and approaches $\dfrac{1}{7}\left(\dfrac{\omega_3 a}{V}\right)^2$ as r approaches 1.

The notation S' is used on the ground that magnetic force not induction is the quantity determined, *i. e.* a magnitude with the dynamical character of a velocity; and accordingly we write $R = \dfrac{dS'}{dw}$ cf. §§ 12 and 19. In this connexion the absence of a factor $1 - \Sigma p^2$ in the numerator of the vecto-

potentials is important. The positive character of $T' = \Sigma p \dfrac{dS'}{dp}$ will appear later (*cf.* end of § 34).

§ 31. The above method proceeds directly to the energy-integral without taking note of the values of FGH. The solution χ_1 attaches to the density $\rho(ax + c'y + b'z)$, or with principal axes to the density $\rho x/a^2$, and therefore $(\chi_1 a^2) \dfrac{\omega_3}{V}$ to $i_y/V = \rho \omega_3 x/V$; *i. e.* for ω_3, G is $(\chi_1 a^2) \dfrac{\omega_3}{V}$, while F is $-(\chi_2 b^2) \dfrac{\omega_3}{V}$. Hence for a general rotation,

$$\Sigma p F = \Sigma \frac{\omega_3}{V}(q a^2 \chi_1 - r b^2 \chi_2) = \Sigma \frac{\chi_1 a^2}{V}(q\omega_3 - r\omega_2). \quad (118)$$

Any influence of the vector terms on the scalar potential depends on $\Sigma p F$, and therefore vanishes when the axis of rotation is that of translation.

The forces electric and magnetic due to the scalar potential are odd functions of xyz (first order), those due to the vector potential are even functions (zero and second order). We shall consider the contributions which appear in the equations of energy and of mechanical force, referring to (56), (58), and (59), when integral values are taken.

First take the term $\int \Sigma X' i_x d\tau_i$, to which the scalar element, with $X' = \dfrac{k\rho}{2}(L_0 x + N_0' y + M_0' z)$, contributes an effective element. The term connected with ω_3 is

$$\int \Sigma X' i_x = \frac{k\rho^2 \omega_3}{2} \int d\tau_i \left[x(N_0' x + M_0 y + L_0' z) - y(L_0 x + N_0' y + M_0' z) \right]$$

$$= \frac{k\rho^2 \tau_0 \omega_3}{10 \Delta_a}(A - B) N_0' = \frac{k\rho^2 \tau_0}{10} \omega_3 (a^2 - b^2) N_0'$$

$$= \frac{k\rho^2 \tau_0}{10} \omega_3 (a^2 - b^2) p q \int_0^\infty \frac{abc \cdot \lambda(c^2 + \lambda) d\lambda}{J'^{3/2}}. \quad . \quad . \quad (119)$$

If all components are taken

$$\int \Sigma X' i_x d\tau_i = \frac{k\rho^2 \tau_0}{10} \Sigma \omega_3 (a^2 - b^2) N_0'.$$

When $L_0' M_0' N_0'$ are compared the factors which are peculiar to N_0' are $pq(c^2 + \lambda)$, from which follows $\Sigma r N_0'(a^2 - b^2) = 0$. Thus the above expression vanishes if $\omega_1 : \omega_2 : \omega_3 = p : q : r$, or if the axis of rotation is also that of translation.

The expression in (119) divided by ω_3 is a couple about the axis of z. As the components of a uniform translation taken in the varying directions of a rotating ellipsoid are periodic

functions of the time, these terms involve a departure from the category of stationary motions assumed at the outset. The terms involve an exchange of radiation with the outer field, and the period is half that of the rotation.

Equation (56) also contains the terms $\frac{dS}{dt} + \Sigma P \frac{du}{dt}$. If for a rotating ellipsoid we take $\frac{du}{dt} = \frac{\partial u}{\partial t} - \omega_3 v + \omega_2 w$, then if t is not explicitly contained $\frac{du}{dt} = -\omega_3 v + \omega_2 w$, and $\frac{dS}{dt}$ and $\Sigma P \frac{du}{dt}$ have values. But so far as S depends only on uvw, $P = -\frac{dS}{du}$ and the terms in question are cancelled. The contribution of S belonging to ω_3 is $\omega_3 \left(u \frac{dS}{dv} - v \frac{dS}{du} \right)$ or $\omega_3 \left(p \frac{dS}{dq} - q \frac{dS}{dp} \right)$; and $u \frac{dS}{dv} - v \frac{dS}{du}$ is the correct form for the couple due to dynamical asymmetry. Its value taken from (107) agrees with that given by (119). Thus if we choose we may regard $\Sigma P \frac{du}{dt} + \int \Sigma X' i_x d\tau_i$ as the cancelled terms, and these terms in $\frac{dS}{dt}$ as remaining; which simplifies the dynamical statement of (56).

The terms $\rho Z' + \frac{1}{V}(i_x \beta - i_y \alpha)$ occur in the equation for mechanical force (59). The Z' of the vector potentials and the $\alpha\beta$ of the scalar potential contribute to the integral of these terms. Attending to the first type $X' = \frac{d}{dx} \Sigma p F$, we first evaluate $\int \frac{d\chi_1}{dx} d\tau_i$ quoting the value (§ 30, p. 441), viz.:

$$\int \frac{d\chi_1}{dx} d\tau_i = \frac{\rho}{4} \int d\tau_i \int_0^\infty \frac{d\lambda}{\sqrt{J}} \left\{ \alpha(1 - u_a) - (\alpha x + \gamma' y + \beta' z) \frac{\partial u_a}{\partial x} \right\},$$

$$\left[\text{with } \frac{\partial u_a}{\partial y} \text{ for } \frac{d\chi_1}{dy} \cdots \right]$$

$$= \frac{\rho \tau_0}{4} \int_0^\infty \frac{d\lambda}{\sqrt{J}} \left\{ \alpha - \frac{\alpha}{5\Delta_a} \Sigma(A\alpha + 2A'\alpha') - \frac{2}{5\Delta_a}(A\alpha^2 + B\gamma'^2 \right.$$
$$\left. + C\beta'^2 + 2C'\alpha\gamma' + 2B'\alpha\beta' + 2A'\beta'\gamma') \right\},$$

$$= \frac{\rho \tau_0}{10} \int_0^\infty \frac{\alpha\, d\lambda}{\sqrt{J}}, \text{ quoting (114).}$$

That is,

$$\int \frac{d\chi_1}{dx} d\tau_i = \frac{eL_0}{10}, \quad \int \frac{d\chi_1}{dy} d\tau_i = \frac{eN_0'}{10}, \quad \int \frac{d\chi_1}{dz} d\tau_i = \frac{eM_0'}{10}, \quad (120)$$

the orders being $N_0'M_0'L_0'$ for χ_2, and $M_0'L_0'N_0$ for χ_3. With the value of ΣpF given by (118),

$$\int \rho Z' d\tau_i = \frac{\rho e}{10V} \left[\omega_1 (rb^2 L_0' - qc^2 N_0) + \omega_2 (pc^2 N_0 - ra^2 M_0') \right.$$
$$\left. + \omega_3 (qa^2 M_0' - pb^2 L_0') \right] . \quad (121)$$

For the term $(i_x\beta - i_y\alpha)/V$ we have $\alpha(1 - \Sigma p^2)$ or $kx = qZ' - rY'$, where $X'\ldots$ belong to the scalar solution, *i. e.* $X' = \frac{k\rho}{2} (L_0 x + N_0'y + M_0'z)$. Thus

$$\int \frac{1}{V} (i_x\beta - i_y\alpha) d\tau_i = \frac{\rho^2}{2V} \int d\tau_i \Big[(-\omega_3 y + \omega_2 z)(rX' - pZ')$$
$$- (-\omega_1 z + \omega_3 x)(qZ' - rY') \Big]$$
$$= \frac{\rho^2 \tau_0}{10V} \left[\omega_1 c^2 (qN_0 - rL_0') + \omega_2 c^2 (rM_0' - pN_0) \right.$$
$$\left. + \omega_3 (pb^2 L_0' - qa^2 M_0' + rN_0'\overline{a^2 - b^2}) \right].$$

Adding this to (121) we have

$$\int \big\{ \rho Z' + V^{-1}(i_x\beta - i_y\boldsymbol{\alpha}) \big\} d\tau_i = \frac{\rho e}{10V} r\Sigma(b^2 - c^2)\omega_1 L_0'. \quad (122)$$

The quantity under the sign of summation vanishes as stated above with the axis of rotation in the direction of translation.

The rate of change of momentum in (59) $\frac{dR}{dt}$ may be written as $\frac{\partial R}{\partial t} - \omega_2 P + \omega_1 Q$, or for steady motion $-\omega_2 P + \omega_1 Q$. This and similar terms vanish if $\omega_1 : \omega_2 : \omega_3 = P : Q : R$, or assuming $\omega_1 : \omega_2 : \omega_3 = p : q : r$ the general condition, if $Pq - Qp$ and two similar terms vanish, *i. e.* if $(a^2 - b^2)pqN_0'$ and two other terms vanish. These require the translation to be along a principal axis.

§ 32. We now consider the condition of things when the translation approaches the velocity of light, and when it exceeds it ; and for simplicity we first attend to the spheroid of revolution with translation along the axis, for which see schedule, p. 440 (110) (ii.).

As r approaches 1, ϕ_0 becomes infinite as $\log \frac{1 + r}{1 - r}$; S has

the multiplier $1-r^2$ (or k), and so vanishes, while T has not this multiplier and becomes infinite. Thus it may fairly be concluded that electrical action will not generate a relative velocity attaining that of light. The electric forces for this case are given by $X' = \frac{k\rho}{2} L x, \ldots$; for the interior in the limit $L_0 = M_0 = 1$, and N_0 becomes infinite as ϕ_0. Thus electric force vanishes with k, but the z-component is great in relation to the others. The magnetic force is given by $k\alpha = qZ' - rY'$, and for the limit $\alpha = -\frac{\rho}{2} y$, $\beta = \frac{\rho}{2} x$, $\gamma = 0$ in the interior, *i. e.* the force is finite.

As regards the character of the surfaces

$$\frac{x^2+y^2}{a^2+\lambda} + \frac{z^2}{c^2+\lambda(1-r^2)} = 1$$

when r approaches 1, the expansion of the c axis with increase of λ is reduced, and in the limit all the surfaces are crowded in contact with the ends of this axis, while expanding freely in the equatorial plane ; and this accords with the relatively great value of the component Z' of the force.

If $r > 1$, the integration for ϕ_0 starts from $\lambda = 0$ as before, this being necessary to determine the constant at the surface; but it proceeds only to a critical value $\lambda_0 = c^2/(r^2-1)$. Since when r is only slightly greater than 1, the above critical value is infinite, this course of integration establishes continuity with the case $r < 1$.

The value of ϕ_0 for the sphere then differs only from the case $r < 1$ in having $\log \frac{r+1}{r-1}$ in lieu of $\log \frac{1+r}{1-r}$; thus if the solution were applied beyond $r=1$, S would become negative through the factor $1-r^2$, but T would only suffer the change in the form of the logarithm.

But the case of volume-distribution breaks down through the breach of (91) and (93). In the latter, where an integration of energy through the volume was in question, $\lambda/\sqrt{J}$ was required to vanish at the upper limit, which with $r < 1$ was assured by the cube of λ contained in J; but when $r > 1$, one factor in J is $c^2 - \lambda(r^2-1)$, and the vanishing of this factor for the upper limit makes $\lambda/\sqrt{J}$ infinite. As regards (91) the details for the spheroid need consideration. The values of $\phi_0 L_0 M_0$ in (110) (ii.) are correct for this case if $\log \frac{\gamma+c}{\gamma-c}$ is written for $\log \frac{c+\gamma}{c-\gamma}$; but for N_0 the upper limit of integration introduces an infinite constant. N_0 has in

addition to the terms there given a term $\dfrac{2a^2c}{\gamma^2\sqrt{c^2-\lambda_0(r^2-1)}}$,

vanishing for λ_0 infinite and $r<1$, but infinite for the upper limit when $r>1$. The z-component of electric force, viz. $-\dfrac{\rho(r^2-1)}{2}\,\mathrm{N}_0 z$ is infinite with N_0. For the external solution this infinite constant also appears, viz.:

$$\mathrm{N}=\frac{\phi}{\gamma^2}-\frac{2a^2c}{\gamma^2\sqrt{c^2-\lambda(r^2-1)}}+\frac{2a^2c}{\gamma^2\sqrt{c^2-\lambda_0(r^2-1)}}.$$

In (91) or $\Sigma(p\mathrm{L}+2p'\mathrm{L}')+2\dot{\phi}=0$, which is the condition that $\nabla_e^2\psi=0$ should be satisfied, the variable terms disappear correctly, but the constant in N remains and invalidates the result.

Consider now the character of the surfaces

$$\frac{x^2+y^2}{a^2+\lambda}+\frac{z^2}{c^2-\lambda(r^2-1)}=1,$$

where $r>1$. The polar axis diminishes from c to 0, while the equatorial axis increases from a to $\sqrt{a^2+c^2(r^2-1)}$. As a consequence of this contrary movement each surface from $\lambda=0$ to $\lambda=c^2/(r^2-1)$ intersects the conductor for which $\lambda=0$, and there is mutual intersection of any pair of surfaces outside the conductor. For the solution of a conductor problem it is essential that there should be a series of equipotential surfaces not intersecting, and that one of these should be the surface of the conductor itself. This condition is broken, there are two potentials generally of finitely different values at points on or outside the conductor, and there is no field of constant potential within it.

The envelope of the surfaces λ consists of two equal cones having a common base $z=0$, $x^2+y^2=a^2+c^2/(r^2-1)$; the vertices are on opposite sides of this base and the angle of the cones is twice the angle arc sin $1/r$ or arc sin V/w. The cones also touch the spheroid $(x^2+y^2)/a^2+z^2/c^2=1$, and the contact lines separate the parts of the envelope belonging to values of λ greater and less than 0. These groups intersect each other within the spheroid.

Matters are not essentially different for the general case, but one or two points in connexion with the geometry may be of interest. In the isotropic case, as λ falls from $+\infty$ to $-\infty$ the order of types is ellipsoid, hyperboloid first of one sheet then of two sheets, imaginary surface ; with the points of division $\lambda=-c^2,\,-b^2,\,-a^2(a>b>c)$. In the motional case

with $\Sigma p^2 < 1$ the order is unchanged, but the points of division lie between $-c^2$ and $-b^2$, $-b^2$ and $-a^2$, $-a^2$ and $-\infty$. With $\Sigma p^2 > 1$ the discriminant is negative for large positive values of λ, and there is a positive root of $J'(\lambda) = 0$, say λ_0. As λ falls from $+\infty$ the order is now hyperboloid of one sheet changing at $\lambda = \lambda_0$ to ellipsoid, then to hyperboloid of one sheet, and hyperboloid of two sheets. When pqr all exist the two latter changes take place between $-c^2$ and $-b^2$, and between $-b^2$ and $-a^2$; since for $\lambda = -c^2$, $-b^2$, $-a^2$, the values of J' are $c^2 r^2 (a^2 - c^2)(b^2 - c^2)$, $-b^2 q^2 (a^2 - b^2)(b^2 - c^2)$, $a^2 p^2 (a^2 - b^2)(b^2 - c^2)$, i. e. $+$, $-$, $+$, in succession.

The change at the critical value λ_0 takes place through a disk form, and the normal to the disk has a direction given by $(\beta'\gamma' - \alpha\alpha')l = (\gamma'\alpha' - \beta\beta')m = (\alpha'\beta' - \gamma\gamma')n$. Quoting (85) with $-qr$ for p'... $A' = 0$,... this is $l/p = m/q = n/r$; or the disk faces the translation. The product of squares of its semi-axes is $(\alpha + \beta + \gamma)/\Delta_\alpha$ or $J(\alpha + \beta + \gamma)/\Delta_a$, i. e.

$$\Sigma \left[b^2 c^2 + \lambda_0 \{ c^2 (1 - q^2) + b^2 (1 - r^2) \} + \lambda_0^2 (1 - q^2 - r^2) \right]$$

or $$\Sigma \{ a^2 + \lambda_0 (1 - p^2) \} \{ b^2 + \lambda_0 (1 - q^2) \} - \lambda_0^2 \Sigma p^2 q^2.$$

The sum of squares of semi-axes is $\Sigma(\alpha\beta - \gamma'^2)/\Delta_\alpha$ or $(\Sigma A + \lambda \Delta_a \Sigma p)/\Delta_a$, or in the notation for principal axes it is $\Sigma a^2 + \lambda_0 (3 - \Sigma p^2)$. Both forms are confirmed by the case $p = q = 0$, $c^2 + \lambda_0 (1 - r^2) = 0$, the product then being $(a^2 + \lambda_0)$ $(b^2 + \lambda_0)$, the sum $a^2 + \lambda_0 + b^2 + \lambda_0$.

§ 33. In isotropic attraction the law of inverse distance is exact for the sphere, and the integral $\int \rho d\tau / 4\pi r$ is deduced for other forms. The integral makes the first differential coefficients continuous at the frontier and satisfies certain differential equations for external and internal spaces. Objections (1) that the law is not true exactly for any shape of element, and (2) that the volume of a body cannot be made up wholly of spheres, appear to have no force. There is just as good reason for considering the similar integral in æolotropic potential as a correct solution, the simple form of the law applying to a particular shape of ellipsoid; and it is proposed to give some consequences of equating the direct integral to the solution found above by the method of differential equations.

In (96) we found for the ellipsoid which has the status of the sphere in isotropic work, a potential $= k\rho\tau_0/4\pi(u_{\scriptscriptstyle P})^{\frac{1}{2}}$. In the element of a potential integral $d\tau$ is written for τ_0, and x in $u_{\scriptscriptstyle P}$ (which is $\Sigma P x^2 + 2P' yz$) is replaced by $x - x'$, xyz referring to the element, and $x'y'z'$ to the place at which the potential is reckoned. Consider first the potential at the

centre of an ellipsoid $u_a = 1$; the comparison of the direct integral and the solution gives

$$\frac{k\rho}{4\pi}\iiint \frac{dx\,dy\,dz}{\sqrt{u_{\mathrm{P}}}} = \frac{k\rho}{4}\int_0^\infty \frac{d\lambda}{\sqrt{\mathrm{J}}}, \quad \text{or} \quad \iiint \frac{dx\,dy\,dz}{\sqrt{u_{\mathrm{P}}}} = \pi\int_0^\infty \frac{d\lambda}{\sqrt{\mathrm{J}}} \quad (123)$$

and the volume-integral embraces all values for which $u_a \gtrless 1$. For various points on a line in the direction (lmn), the ratio $\sqrt{u_{\mathrm{P}}} : r$ is constant, and $= \sqrt{u_{\mathrm{P}}\,(l,\,m,\,n)}$. We may therefore integrate with regard to r for a cone of fine angle $d\omega$ and axis in the direction (lmn), up to the surface where $r_1^{-2} = u_a(l,\,m,\,n)$. Thus

$$\iiint \frac{dx\,dy\,dz}{\sqrt{u_{\mathrm{P}}(x,y,z)}} = \iint \frac{r^2\,dr\,d\omega}{r\,\sqrt{u_{\mathrm{P}}(l,m,n)}} = \int \frac{r_1^2\,d\omega}{2\,\sqrt{u_{\mathrm{P}}(l,m,n)}} = \int \frac{d\omega}{2u_a\sqrt{u_{\mathrm{P}}}},$$

and so

$$\int \frac{d\omega}{u_a\,\sqrt{u_{\mathrm{P}}}} = 2\pi\int_0^\infty \frac{d\lambda}{\sqrt{\mathrm{J}}}$$

is the equivalent of (123), when l, m, n are variables in u_a and u_{P}, which need not be written as the angular integral is always shown by $d\omega$. The value of J being

$$1 + \lambda\Sigma(pa + 2p'a') + \lambda^2\Sigma(\mathrm{PA} + 2\mathrm{P'A'}) + \lambda^3\Delta_p\Delta_a,$$

it is not altered if $a\ldots$ and $p\ldots$ are interchanged. Thus we have the double form

$$\int \frac{d\omega}{u_a\sqrt{u_{\mathrm{P}}}} = \int \frac{d\omega}{u_p\sqrt{u^{\mathrm{A}}}} = 2\pi\int_0^\infty \frac{d\lambda}{\sqrt{\mathrm{J}}} . \quad . \quad . \quad (124)$$

In (123) there is also an alternative form

$$\int \frac{d\tau}{(u_{\mathrm{A}})^{\frac{1}{2}}} \text{subject to } u_p \gtrless 1.$$

In the original problem of solving a differential equation, $a\ldots$ and $p\ldots$ appeared as primary constants, *i. e.*, $a\ldots$ defined the ellipsoid in reference to which a solution was sought, and $p\ldots$ defined the differential equation. In (124) a and P occur as primary constants in the first case, and p and A in the second or reciprocal case. To meet this we may return to the original determinant $\Delta(\lambda) = \Delta_a^2\mathrm{J}(\lambda)$, and at the same time use $\mu = \lambda\Delta_a$ for parameter; *i. e.*,

$$\left. \begin{array}{c} d\lambda/\sqrt{\mathrm{J}(\lambda)} = d\mu/\sqrt{\Delta(\mu)}, \\ \text{where } \Delta(\mu) \text{ is the determinant A} + p\mu,\ldots \end{array} \right\} \quad (125)$$

With this notation

$$\int \frac{d\omega}{u_p \sqrt{u_\Lambda}} = 2\pi \int_0^\infty \frac{d\mu}{\sqrt{\Delta(\mu)}} \quad . \quad . \quad . \quad (126)$$

expresses one of the relations (124) directly in terms of $p \ldots A \ldots$ which are primary constants in the statement of the analytical theorem.

This notation is also more convenient for the transition from ϕ_0 to $L_0 \ldots$ by differentiation, ϕ_0, L_0 having the meanings in (95). Thus

$$\alpha\Delta(\mu) = \Delta_a \Lambda(\mu) = \Delta_a \frac{d\Delta(\mu)}{dA}, \text{ and } 2\alpha'\Delta(\mu) = \Delta_a \frac{d\Delta(\mu)}{dA'}. \quad (127)$$

Therefore from $\phi_0 = \int_0^\infty \frac{d\mu}{\sqrt{\Delta(\mu)}}$, we derive

$$\left.\begin{array}{l} L_0 = \int_0^\infty \frac{\alpha d\mu}{\sqrt{\Delta(\mu)}} = \Delta_a \int_0^\infty \frac{d\mu}{\{\Delta(\mu)\}^{3/2}} \frac{d\Delta(\mu)}{dA} = -2\Delta_a \frac{d\phi_0}{dA}, \\[2mm] 2L_0' = -2\Delta_a \frac{d\phi_0}{dA'} \\[2mm] \text{and} \\[2mm] \phi_0 - \Sigma\left(L_0 x^2 + 2L_0' yz\right) = \phi_0 + 2\Delta_a\Sigma\left(x^2 \frac{d\phi_0}{dA} + yz \frac{d\phi_0}{dA'}\right) \end{array}\right\} \quad (128)$$

[The same results follow of course from differentiating the form in $J(\lambda)$, but the proof requires the elimination of an integral containing λ^3 in the numerator (*cf.* p. 437), the ultimate form proceeding only to λ^2: this has been fully verified.]

§ 34. If we set out from the equivalent form of ϕ_0 given by (124), viz., $\frac{1}{2\pi}\int \frac{d\omega}{u_p\sqrt{u_\Lambda}}$, then since $\frac{du_\Lambda}{dA} = l^2$, $\frac{du_\Lambda}{dA'} = 2mn$, we have

$$\phi_0 + 2\Sigma\left(x^2 \frac{d\phi_0}{dA} + yz \frac{d\phi_0}{dA'}\right) = \frac{1}{2\pi}\int \frac{d\omega}{u_p(u_\Lambda)^{\frac{1}{2}}}\left[1 - \frac{\Delta_a(lx + my + nz)^2}{u_\Lambda}\right]$$

Thus for the internal potential we have

$$\psi = \frac{k\rho}{8\pi}\int \frac{d\omega}{u_p(u_\Lambda)^{\frac{1}{2}}}\left[1 - \frac{\Delta_a(lx + my + nz)^2}{u_\Lambda}\right], \quad . \quad (129)$$

the variables in u_p, u_Λ being l, m, n. Now $lx + my + nz$ is the perpendicular ϖ_1 from the centre on a plane through xyz with its normal in the direction (lmn); and if ϖ_2 is the perpendicular from the centre on the parallel tangent plane, $\varpi_2^2 \Delta_a = u_\Lambda$; hence the bracket is $1 - \varpi_1^2/\varpi_2^2$. Again, if σ is

the area of the section of the ellipsoid by the plane through xyz, and σ_0 that by the parallel plane through the centre, $\sigma = \sigma_0(1 - \varpi_1^2/\varpi_2^2)$, and $\sigma_0 = \pi/\sqrt{u_\Lambda}$.   Using these values in (129) we have

$$\psi_i = \frac{k\rho}{8\pi^2}\int \frac{\sigma d\omega}{u_p}. \quad . \quad . \quad . \quad . \quad (130)$$

In the isotropic problem k (which is Δ_p in the general case and $1 - \Sigma p^2$ in the motional case) is 1, and also u_p which is $\Sigma p l^2 + 2p' mn$ becomes Σl^2 or 1.   Hence

$$\psi_i = \frac{\rho}{8\pi^2}\int \sigma d\omega = \frac{\rho}{2\pi}\bar{\sigma}, \quad . \quad . \quad . \quad (131)$$

$\bar{\sigma}$ being the mean area of all sections of the ellipsoid drawn through the point whose potential is considered.   At the centre the potential is $\frac{\rho\bar{\sigma}_0}{2\pi}$, and the potential of a conductor with charge $\rho\tau_0$ is $\frac{\rho\bar{\sigma}_0}{3\pi}$.   The expression for the energy is

$$\frac{\rho e \bar{\sigma}_0}{5\pi} \quad \text{or} \quad \frac{e^2 \bar{\sigma}_0}{5\pi\tau_0}$$

with volume-distribution ; for a conductor 6 replaces 5.   [On the general validity of (130) *vide infra*.]

In working by differential methods the use of $\rho = \Sigma \dfrac{d\mathrm{X}}{dx}$ for density, and $\frac{1}{2}\Sigma \mathrm{X}\mathrm{X}' + \frac{1}{2}\Sigma \alpha\alpha'$ for energy has been followed. The older method has $4\pi\rho_0$ and $\dfrac{1}{8\pi}(\Sigma \mathrm{X}\mathrm{X}' + \Sigma \alpha\alpha')$; the values in that notation are got by multiplying the expressions for potential and also for energy by 4π.   The older fashion gives here the simpler formulæ, and they stand thus:—

$\psi_i = 2\rho_0\bar{\sigma}$, with $2\rho_0\bar{\sigma}_0$ at centre, and $\dfrac{4\rho_0}{3}\bar{\sigma}_0$ for a conductor

Energy $\dfrac{4}{5}\rho_0 e_0 \bar{\sigma}_0$ or $\dfrac{4e_0^2}{5}\cdot\dfrac{\sigma_0}{\tau_0}$ for vol. case, $\dfrac{4e^2}{6}\cdot\dfrac{\bar{\sigma}_0}{\tau_0}$ for conductor $\Bigg\} .(132)$

For the general æolotropic case the area of each element is modified by the multiplier k/u_p or Δ_p/u_p depending on the exposure of the plane in relation to the æolotropic axes.   For the motional case the multiplier is

$$(1 - \Sigma p^2)/(1 - \mathrm{U}^2/\mathrm{V}^2) \quad \text{or} \quad \frac{1 - \Sigma p^2}{2}\left(\frac{1}{1 - \mathrm{U/V}} + \frac{1}{1 + \mathrm{U/V}}\right),$$

U being $lu + mv + nw$; *i.e.*, apparently it is the mean of

$(1-\Sigma p^2)/(1-U/V)$ for a given translation, and for that translation reversed. The factor can be traced, and proves to be the connecting link between plane-wave and source forms of the argument. The plane-wave form is $\Sigma lx-(V-U)t$, the source form $\sqrt{(1-\Sigma p^2)\Sigma x^2+(\Sigma px)^2}+\Sigma px-Vt(1-\Sigma p^2)$. If lmn of the first form is the normal to the wave-surface in the second form, we readily find with R for the radical

$$\begin{rcases} (l-p)(R+\Sigma px)=x(1-\Sigma p^2), \qquad \Sigma lx=R, \\ (1-U/V)(R+\Sigma px)=R(1-\Sigma p^2) \end{rcases}.(133)$$

leading to
$$R+\Sigma px-Vt(1-\Sigma p^2)=\{\Sigma lx-(V-U)t\}(1-\Sigma p^2)/(1-U/V)$$

It seems clear then that this form of the integral (for potential or for energy) is related to (I think we may say based on) plane wave forms, and that when translation exists a modifying factor is needed to pass to the source form. The essential difference between the two is that the first shows a period varying with direction if $\Sigma lx-(V-U)t$ multiplied by a constant is the argument; the second shows a fixed period if $R+\Sigma px-t(1-\Sigma p^2)$ multiplied by constant is the argument; the factor of transition has the function of allowing for change of period due to the motion.

The external potential is got by distributing an equal charge over the volume of the æolian on which the point xyz lies, *i. e.*, we must write for the above

$$\psi=\frac{ke}{8\pi\tau}\int\frac{d\omega}{u_p(u_{A_a})^{\frac{1}{2}}}\left[1-\frac{\Delta_a(lx+my+nz)^2}{u_{A_a}}\right]. \text{ But } \tau=\tau_0\sqrt{J(\lambda)},$$

$J\Delta_a=\Delta_a$, and $JA_a=A+p\lambda\Delta_a,\dots$; therefore

$$\psi=\frac{k\rho}{8\pi\tau_0}\int\frac{d\omega}{u_p(u_A+\lambda\Delta_a u_p)^{\frac{1}{2}}}\left[1-\frac{\Delta_a(lx+my+nz)^2}{u_A+\lambda\Delta_a u_p}\right]. \quad (134)$$

Interpretation with reference to the ellipsoid u_a in the first shape of the formula shows that the range of values of (lmn) is complete, for xyz being on u_a every plane cuts the ellipsoid.

The values given to $\phi_0 L_0 \dots$ by (129) are

$$\phi_0=\frac{1}{2\pi}\int\frac{d\omega}{u_p u_A^{\frac{3}{2}}}, \quad L_0=\frac{\Delta_a}{2\pi}\int\frac{l^2 d\omega}{u_p(u_A)^{\frac{3}{2}}}, L_0'=\frac{\Delta_a}{2\pi}\int\frac{mn\,d\omega}{u_p(u_A)^{\frac{3}{2}}}. \quad (135)$$

They satisfy the condition $\Sigma(A L_0+2A'L_0')=\Delta_a\phi_0$ in an obvious way, and $\Sigma(pL+2p'L')=2$ requires

$$2=\frac{\Delta_a}{2\pi}\int\frac{d\omega}{(u_A)^{\frac{3}{2}}}=\frac{\sqrt{\Delta_A}}{2\pi}\int\frac{d\omega}{(u_A)^{\frac{3}{2}}},$$

i. e. $\displaystyle\int \frac{d\omega}{(u_{\text{A}})^{\frac{3}{2}}} = \frac{4\pi}{\sqrt{\Delta_{\text{A}}}}$, or of course $\displaystyle\int \frac{d\omega}{(u_a)^{\frac{3}{2}}} = \frac{4\pi}{\sqrt{\Delta_a}}$. . (136)

In the middle integral of (124) write A for p in u_p, it is then $\int d\omega / u_{\text{A}}^{\frac{3}{2}}$; and in the third integral J, which is

$$1 + \lambda\Sigma(pa + 2p'a') + \lambda^2\Sigma(\text{PA} + 2\text{P}'\text{A}') + \lambda^3\Delta_p\Delta_a,$$

becomes $1 + \lambda(3\Delta_a) + \lambda^2(3\Delta_{\text{A}}) + \lambda^3\Delta_{\text{A}}\Delta_a$ or $(1 + \lambda\Delta_a)^3$.

Thus (124) yields in that case

$$\int \frac{d\omega}{(u_{\text{A}})^{\frac{3}{2}}} = 2\pi \int_0^\infty \frac{d\lambda}{(1 + \lambda\Delta_a)^{\frac{3}{2}}} = \frac{4\pi}{\Delta_a} = \frac{4\pi}{\sqrt{\Delta_{\text{A}}}}.$$

These expressions for $\text{L}_0 \dots$ may be used in the formula for S' (117). For principal axes $u_{\text{A}}/\Delta_a = \Sigma a^2 l^2$, and therefore

$$\text{S}' = \frac{\rho^2\tau_0}{70\text{V}^2} \cdot \frac{abc}{2\pi} \int \frac{d\omega}{u_p(\Sigma a^2 l^2)^{3/2}} \times \Sigma(a^2 l\omega_2 - b^2 m\omega_1)^2$$

which is essentially positive. Also, since for the motional case $u_p = 1 - (\Sigma lp)^2$, T' which is $\Sigma p \dfrac{d\text{S}'}{dp}$ shows the additional factor $2(\Sigma lp)^2/u_p$ under the sign of integration, and is essentially positive.

§ 35. The formula (134) may be shown to be a solution by a direct method. For that we set out from

$$\int \frac{d\omega}{(u_{\text{A}} + \lambda\Delta_a u_p)^{\frac{3}{2}}} = \frac{4\pi}{\sqrt{\Delta(\lambda)}} \text{ or } \int \frac{d\omega}{(u_{\text{A}} + \mu u_p)^{\frac{3}{2}}} = \frac{4\pi}{\sqrt{\Delta(\mu)}}, \quad (137)$$

which is a case of (136). Differentiate both sides with regard to A or A', using the formula in μ, with

$$\frac{d\Delta(\mu)}{d\text{A}} = \text{A}(\mu) \text{ or } \frac{\alpha\Delta(\mu)}{\Delta_a}, \quad \frac{d\Delta(\mu)}{d\text{A}'} = 2\text{A}'(\mu) \text{ ; then}$$

$$3\Delta_a \int \frac{l^2 d\omega}{(u_{\text{A}} + \mu u_p)^{\frac{5}{2}}} = \frac{4\pi x}{\{\Delta(\mu)\}^{\frac{1}{2}}}, \; 3\Delta_a \int \frac{mn\, d\omega}{(u_{\text{A}} + \mu u_p)^{\frac{5}{2}}} = \frac{4\pi\alpha'}{\{\Delta(\mu)\}^{\frac{3}{2}}}, \; (138)$$

and therefore

$$3\Delta_a \int \frac{(lx + my + nz)^2 d\omega}{(u_{\text{A}} + \mu u_p)^{\frac{5}{2}}} = \frac{4\pi}{(\Delta(\mu))^{\frac{3}{2}}}(\alpha x^2 + \dots + 2\alpha' yz)$$

$$= \frac{4\pi}{(\Delta(\mu))^{\frac{1}{2}}} = \int \frac{d\omega}{(u_{\text{A}} + \mu u_p)^{\frac{3}{2}}}.$$

This makes

$$\frac{\partial \psi}{\partial \mu} = 0, \text{ and gives } \frac{d\psi_e}{dx} = -\frac{k\rho\Delta_a}{4\pi}\int \frac{l(lx+my+nz)d\omega}{u_p(u_\lambda+\mu u_p)^{\frac{3}{2}}}. \quad (139)$$

The part of $\nabla_e^2\psi$ due to explicit differentiation with regard to xyz, $\nabla_e^2\psi$ being

$$\left(p\frac{d}{dx}+r'\frac{d}{dy}+q'\frac{d}{dz}\right)\frac{d\psi}{dx}+\left(r'\frac{d}{dx}+\ldots\right)\frac{d\psi_i}{dy}+\ldots, \text{ is}$$

$$-\frac{k\rho\Delta_a}{4\pi}\int\frac{d\omega}{(u_\lambda+\mu u_p)^{\frac{3}{2}}} \text{ or } -\frac{k\rho\Delta_a}{(\Delta(\mu))^{\frac{1}{2}}} \text{ or} -\frac{k\rho}{\sqrt{\mathbf{J}(\lambda)}}.$$

The part due to implicit differentiation through μ is

$$\frac{3k\rho\Delta_a}{8\pi}\int\frac{d\omega(lx+my+nz)}{(u_\lambda+\mu u_p)^{\frac{3}{2}}}\left\{l\left(p\frac{d\mu}{dx}+r'\frac{d\mu}{dy}+q'\frac{d\mu}{dz}\right)+m(\quad)\ldots\right\},$$

which by (138) is

$$\frac{k\rho}{2[\Delta(\mu)]^{\frac{1}{2}}}\left[(\alpha x+\gamma'y+\beta'z)\left(p\frac{d\mu}{dx}+r'\frac{d\mu}{dy}+q'\frac{d\mu}{dz}\right)+\ldots\right]$$

$$=\frac{k\rho\Delta_a}{(\Delta(\mu))^{\frac{1}{2}}}\text{by (78)}=\frac{k\rho}{\sqrt{\mathbf{J}(\lambda)}}.$$

Thus $\nabla^2\psi$ is zero when λ or μ is variable; when λ is constant only the first part exists, and this for $\lambda=0$ is reduced to $-k\rho$. The vanishing value of $\frac{\partial\psi}{\partial\lambda}$ when (138) is used has the form

$$-\frac{k\rho}{4}\cdot\frac{\Delta_a}{[\Delta(\lambda)]^{\frac{1}{2}}}(1-u_a) \text{ or } -\frac{k\rho}{4}\cdot\frac{1-u_\alpha}{\sqrt{\mathbf{J}(\lambda)}},$$

agreeing with the original solution.

The formula is of sufficient interest to justify a separate statement of the isotropic case for elementary work. Taking the usual notation as to density, we have to show that

$$\psi=\frac{\rho_0 abc}{2}\int\frac{d\omega}{(\Sigma a^2 l^2+\lambda)^{\frac{1}{2}}}\left[1-\frac{(lx+my+nz)^2}{\Sigma a^2 l^2+\lambda}\right]$$

satisfies the differential equation $\nabla^2\psi=0$, when λ is a variable given by $\Sigma\frac{x^2}{a^2+\lambda}=1$; and that if λ is omitted from the

formula, $\nabla^2\psi + 4\pi\rho_0 = 0$. We set out from

$$\int \frac{d\omega}{(\Sigma a^2 l^2)^{\frac{3}{2}}} = \frac{4\pi}{abc} \, ^*,$$

which is used in the form

$$\int \frac{d\omega}{(\Sigma a^2 l^2 + \lambda)^{\frac{3}{2}}} = \frac{4\pi}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad \cdot \quad \cdot \ (140)$$

and differentiate with regard to a^2, giving

$$\int \frac{3l^2 d\omega}{(\Sigma a^2 l^2 + \lambda)^{\frac{5}{2}}} = \frac{4\pi}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \cdot \quad (141)$$

Since $\int lm\, d\omega/(\Sigma a^2 l^2 + \lambda)^{\frac{5}{2}}$ and other products vanish, we get by multiplying (141) by $x^2 y^2 z^2$,

$$\int \frac{3(lx+my+nz)^2 d\omega}{(\Sigma a^2 l^2 + \lambda)^{\frac{5}{2}}} = \frac{4\pi}{\sqrt{(\ )(\ )(\ )}}\left(\frac{x^2}{a^2+\lambda} + \frac{y}{b^2+\lambda} + \frac{z^2}{c^2+\lambda}\right)$$

$$= \frac{4\pi}{\sqrt{(\ )(\ )(\ )}} = \int \frac{d\omega}{(\Sigma a^2 l^2 + \lambda)^{\frac{3}{2}}}.$$

In virtue of this equation $\dfrac{\partial\psi}{\partial\lambda} = 0$, and therefore

$$\frac{d\psi}{dx} = -\rho_0 abc \int \frac{l\, d\omega\,(lx+my+nz)}{(\Sigma a^2 l^2 + \lambda)^{\frac{3}{2}}} = -\rho_0 abc \int \frac{l^2 x\, d\omega}{(\Sigma a^2 l^2 + \lambda)^{\frac{3}{2}}}.$$

The part of $\nabla^2\psi$ due to explicit differentiation is

$$-\rho_0 abc \int \frac{d\omega}{(\Sigma a^2 l^2 + \lambda)^{\frac{3}{2}}} = -\frac{4\pi\rho_0 abc}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \, ;$$

the part due to implicit differentiation through λ is

$$\frac{3\rho_0 abc}{2} \int \frac{(lx+my+nz)d\omega}{(\Sigma a^2 l^2 + \lambda)^{\frac{5}{2}}}\left(l\frac{d\lambda}{dx} + m\frac{d\lambda}{dy} + n\frac{d\lambda}{dz}\right)$$

$$= \frac{2\pi\rho_0 abc}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \Sigma\frac{x}{a^2+\lambda}\frac{d\lambda}{dx} \text{ by (141)}$$

$$= \frac{4\pi\rho_0 abc}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \text{ by a simple property of } \lambda.$$

* Independently proved thus:—

$$= \int \frac{d\omega}{(\Sigma a^2 l^2)^{\frac{3}{2}}} = \int_0^{2\pi} d\phi \int_{-1}^{+1} \frac{dn}{[a^2\cos^2\phi + b^2\sin^2\phi + n^2(c^2 - a^2\cos^2\phi - b^2\sin^2\phi)]^{\frac{3}{2}}}$$

$$= \int_0^{2\pi} \frac{d\phi}{a^2\cos^2\phi + b^2\sin^2\phi}\left[\frac{n}{\{a^2\cos^2\phi + b^2\sin^2\phi + n^2(c^2 - a^2\cos^2\phi - b^2\sin^2\phi)\}^{\frac{1}{2}}}\right]_{-1}^{+1}$$

$$= \int_0^{2\pi} \frac{2d\phi}{c(a^2\cos^2\phi + b^2\sin^2\phi)} = \frac{4\pi}{abc}.$$

The value of $\dfrac{\partial \psi}{\partial \lambda}$ given directly is

$$-\frac{\rho_0 abc}{4} \int \frac{d\omega}{(\Sigma a^2 l^2 + \lambda)^{\frac{3}{2}}} \left[1 - \frac{3(lx + my + nz)^2}{\Sigma a^2 l^2 + \lambda} \right]$$

or using (137) and (138) it is

$$- \frac{\pi \rho_0 abc}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \left[1 - \Sigma \frac{x^2}{a^2 + \lambda} \right];$$

and this gives immediately the usual form

$$\psi = \pi \rho_0 abc \int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \left[1 - \Sigma \frac{x^2}{a^2 + \lambda} \right].$$

The potential of a conductor is $\frac{2}{3}$ of the first term of this formula and

$$= \frac{e_0}{2} \int_\lambda^\infty \frac{d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}.$$

The formula for χ_1 the potential for a density $\rho(a x + c' y + b' z)$ is

$$\chi_1 = \frac{\rho \Delta_a}{8\pi} \int \frac{l(lx + my + nz) d\omega}{u_p (u_\Lambda + \mu u_p)^{\frac{3}{2}}} \left[1 - \frac{\Delta_a (lx + my + nz)^2}{u_\Lambda + \mu u_p} \right]. \quad (142)$$

The verification will be merely sketched for this case. Setting out from a member of (138) we derive by differentiation

$$5 \Delta_a^2 \int \frac{d\omega}{(u_\Lambda + \mu u_p)^{\frac{7}{2}}} (l^4, \ l^2 m^2, \ l^3 m, \text{ or } l^2 mn)$$

$$= \frac{4\pi}{3 \sqrt{\Delta(\mu)}} [3\alpha^2, \ \alpha\beta + 2\gamma'^2, \ 3\alpha\gamma', \text{ or } \alpha\alpha' + 2\beta'\gamma'] \text{ respectively,}$$

and thence

$$5 \Delta_a^2 \int \frac{(l^2 \text{ or } lm)(lx + my + nz)^2 d\omega}{(u_\Lambda + \mu u_p)^{\frac{7}{2}}} = \frac{4\pi}{3 \sqrt{\Delta(\mu)}} \times$$

$$[\alpha u_a + 2(\alpha x + \gamma' y + \beta' z)^2, \text{ or } \gamma' u_a + 2(\alpha x + \gamma' y + \beta' z)(\gamma' x + \beta y + \alpha' z)]$$

and

$$5 \Delta_a^2 \int \frac{l(lx + my + nz)^3 d\omega}{(u_\Lambda + \mu u_p)^{\frac{7}{2}}} = \frac{4\pi}{\sqrt{\Delta(\mu)}} (\alpha x + \gamma' y + \beta' z) u_a,$$

$\left. \right\} (143)$

where of course 1 may be written for u_a. These values make $\dfrac{\partial \chi}{\partial \lambda} = 0$, and $\dfrac{d\chi_1}{dx}$, $\dfrac{d\chi_1}{dy}$, $\dfrac{d\chi_1}{dz}$ have the values due to explicit

differentiation; also the value of $\dfrac{\partial \chi}{\partial \lambda}$ when (143) is applied

has the shape belonging to the original formula of § 30.

§ 36. It will be observed that the results of the last few pages all attach to the use of the reciprocal form or the second integral in (124). The comparison of a direct integral under an æolotropic law of potential with the general solution has only been made for one position, the centre. A method of derivation well known in isotropic potential is applied to extend the result, and the new forms of the potential integrals have been verified directly. In Part II. of Thomson and Tait's 'Natural Philosophy,' p. 44, § 494 *k*, a direct integration for the potential at any point within an ellipsoid is given. A similar process for æolotropic potential requires a modification of the argument of § 33. Since u_{P} has $x-x', \ldots$ for variables, when a fine cone is drawn from $x'\, y'\, z'$ in the direction (lmn), u_{P} still becomes $ru_{\mathrm{P}}(lmn)$, but r is given by the quadratic

$$r^2 u_a(lmn) + r\left(x'\frac{\partial u}{\partial l} + y'\frac{\partial u}{\partial m} + z'\frac{\partial u}{\partial n}\right) = 1 - u_a(x',\, y',\, z').$$

If instead of using $r_1{}^2$ for one end of the cone and $r_2{}^2$ for the other, we take $\frac{1}{2}(r_1{}^2 + r_2{}^2)$ for each, the internal potential is given by

$$\psi_i = \frac{k\rho}{8\pi}\int \frac{d\omega}{u_a u_{\mathrm{P}}^{-\frac{1}{2}}}\left[1 - u_a(x,y,z) + \left(x\frac{\partial u}{\partial l} + y\frac{\partial u}{\partial m} + z\frac{\partial u}{\partial n}\right)^2 / 2u_a,\right] \quad (144)$$

the dashes now dropped because only one x occurs: also u and u_{P} have *lmn* for variables except in the term where $u_a(xyz)$ is written. Thus the form for the coefficients is dictated by the quadratic determining r. If we set out from the value of ϕ_0 implied here, viz..

$$\frac{1}{2\pi}\int \frac{d\omega}{u_a u_{\mathrm{P}}^{\frac{1}{2}}} \quad \text{which is equal to} \quad \int_0^\infty \frac{d\mu}{\sqrt{\Delta(\mu)}},$$

and obtain $L_0 \ldots$ by differentiation as before, we reach the same result. By the two methods we get

$$\left.\begin{aligned}
&L_0 = a\phi_0 - \frac{1}{\pi}\int \frac{d\omega(al + c'm + b'n)^2}{u_a{}^2 u_{\mathrm{P}}^{\frac{1}{2}}}, \\[4pt]
&L_0{}' = a'\phi_0 - \frac{1}{\pi}\int \frac{d\omega(c'l + bm + a'n)(b'l + a'm + cn)}{u_a{}^2 u_{\mathrm{P}}^{\frac{3}{2}}} \\[2pt]
&\text{and} \\[2pt]
&L_0 + a\phi_0 = \frac{1}{\pi}\int \frac{d\omega(\mathrm{C}m^2 + \mathrm{B}n^2 - 2\mathrm{A}'mn)}{u_a{}^2 u_{\mathrm{P}}^{\frac{1}{2}}} \\[4pt]
&L_0{}' + a'\phi_0 = \frac{1}{\pi}\int \frac{d\omega(\mathrm{B}'lm + \mathrm{C}'lm - \mathrm{A}mn - \mathrm{A}'l^2)}{u_a{}^2 u_{\mathrm{P}}^{\frac{1}{2}}}
\end{aligned}\right\} \quad (145)$$

respectively. For the differentiation of u_a we have

$$\Delta_a \frac{du_a}{dA} = -\frac{a}{2} u_a(lmn) + Cm^2 + Bn^2 - 2A'mn,$$

and $\Delta_a \frac{du_a}{dA'} = -a'u_a(lmn) + 2(B'lm + C'ln - Amn - A'l^2)$;

and the identity of results depends on

$$au_a(lmn) = Cm^2 + Bn^2 - 2A'mn + (la + mc' + nb')^2, \text{ and}$$

$$a'u_a(lmn) = B'lm + C'ln - Amn - A'l^2 + (lc' + mb + na')(lb' + ma' + nc).$$

In the latter shape of (145), the relation

$$\Sigma(AL_0 + 2A'L_0') = \Delta_a \phi_0$$

is at once verified, while $\Sigma p L_0 + 2p' L_0' = 2$ involves a formula which, if the form given above for $\frac{du_a}{dA}$ is used, amounts to

$$\frac{2\pi}{\Delta_a} + \Sigma p \frac{d}{dA} \int \frac{d\omega}{u_a u_{P^{\frac{1}{2}}}} = 0, \quad \text{or} \quad \frac{1}{\Delta_a} + \Sigma p \frac{d}{dA} \int_0^\infty \frac{d\mu}{\sqrt{\Delta(\mu)}} = 0.$$

But

$$\frac{d}{dA} \Delta(\mu) = A(\mu) = \frac{a\Delta(\mu)}{\Delta_a}, \quad \text{and} \quad \therefore \Sigma p \frac{d}{dA} \Delta(\mu) = \frac{\Delta(\mu)}{\Delta_a} \Sigma p a = \frac{d\Delta(\mu)}{d\mu}.$$

Thus

$$\Sigma p \frac{d}{dA} \int_0^\infty \frac{d\mu}{\sqrt{\Delta(\mu)}} = \int_0^\infty \frac{d}{d\mu} \left(\frac{1}{\sqrt{\Delta(\mu)}}\right) d\mu = -\frac{1}{\sqrt{\Delta(0)}} = \frac{-1}{\sqrt{\Delta_A}} = -\frac{1}{\Delta_a}.$$

In fact the two relations (95) turn on the relations (86), which were requisite in the original proof.

When the motional case is compared with the statical one the extra factor $k/u_{P^{\frac{1}{2}}}$ or $(1 - \Sigma p^2)/\sqrt{1 - \Sigma p^2 + (\Sigma p l)^2}$ appears in the forms for potential or for energy. The radical is a form which occurs with ray-cosines.

The connexion with radiation-formulæ is less clear than in the reciprocal case. For a moving standpoint ϖ replaces p, and $\sqrt{V^2 - \Sigma u^2 + U_r^2}$ replaces V in the energy-formula p^4/V^3. Now $V^2 - \Sigma u^2 + U_r^2 = V^2 u_P$, and we have for an ellipsoid $u_a = 1$, the energy-expression $\frac{\rho e(1 - \Sigma p^2)}{10\pi} \int \frac{d\omega}{u_a u_{P^{\frac{1}{4}}}}$. If the special spheroid $u_P(xyz) = $ constant is used as base, then $u_{P^{\frac{3}{2}}}$ occurs here, but in the general case a factor u_P is replaced

by u_a belonging to the topography of the ellipsoid which figures as source. Further

$$\int \frac{d\omega}{u_{\mathrm{P}}^{\frac{3}{2}}} = \frac{4\pi}{\sqrt{\Delta_{\mathrm{P}}}} = \frac{4\pi}{\Delta_p} = \frac{4\pi}{1 - \Sigma p^2},$$

and the factor $1 - \Sigma p^2$ is therefore cancelled in the special case where u_{P} replaces u_a.

§ 37. If the various processes used in this theory of æolotropic potential are examined, it will be found that the arguments are stated in a form suited to any number of dimensions. It was not considered advisable to write out the work as for n dimensions, and set down the results for three dimensions as a particular case, because the latter has the wider interest and appeals to physicists as well as to pure mathematicians. Analytically a general scheme would be very little more difficult or laborious. It is proposed to indicate some of the points where a difference in constants appears, and to give attention mainly to the later section, which has analytical applications.

For the scheme of equations, § 21, a more general notation is needed, say for the p's $p_{11}, p_{12} \ldots p_{nn}$ with $p_{rs} = p_{sr}$, and the like for a's and α's. There are then n groups of n equations (not all independent) for the α's, and their determinant solution is expressed as before. The potential for a volume distribution is given as before by

$$\psi = \frac{k\rho}{4} \int_\lambda^\infty \frac{d\lambda}{\sqrt{J}} (1 - u_a),$$

that for a conductor by

$$\frac{k\rho}{2n} \int_\lambda^\infty \frac{d\lambda}{\sqrt{J}}.$$

If we use for base form the special ellipsoidal homaloid which has the status of the sphere, viz., $\Sigma(\mathrm{P}_{rr} x_r^2 + 2\mathrm{P}_{rs} x_r x_s) = \epsilon \Delta_p$, the æolian surfaces are similar surfaces with $\epsilon + \lambda$ for ϵ, and $J = \left(1 + \dfrac{\lambda}{\epsilon}\right)^n$. Then comparing with § 25, (96),

$$\psi = \frac{k\rho\epsilon^{\frac{n}{2}}}{4} \int_\lambda^\infty \frac{d\lambda}{(\epsilon + \lambda)^{\frac{n}{2}}} \left[1 - \frac{\Sigma(\mathrm{P}_{rr} x_r^2 + 2\mathrm{P}_{rs} x_r x_s)}{(\epsilon + \lambda)\Delta_p} \right]$$

$$= \frac{k\rho}{4} \cdot \frac{4}{n(n-2)} \cdot \frac{\epsilon^{\frac{n}{2}} \Delta_p^{\frac{n}{2}-1}}{\{\Sigma(\mathrm{P}_{rr} x_r^2 + 2\mathrm{P}_{rs} x_r x_s)\}^{\frac{n}{2}-1}} \quad . \quad . \quad (96)_n$$

where to facilitate comparison the results are marked with

their original numbers, and a subscript n appended. The volume of a sphere radius c in n dimensions is

$$c^n \{\Gamma(\tfrac{1}{2})\}^n / \Gamma\left(\frac{n}{2} + 1\right);$$ for an ellipsoid c^n is replaced by $1/\Delta_a^{\frac{1}{2}},$

and here the elements "a" being $\mathrm{P}/\epsilon\Delta_p$, $\Delta_a = 1/\epsilon^n \Delta_p$. Introducing the volume

$$\psi = \frac{k\rho}{4} \cdot \frac{4\Delta_p^{\frac{n-3}{2}}}{n(n-2)} \frac{\Gamma\left(\dfrac{n}{2}+1\right)}{\{\Gamma(\tfrac{1}{2})\}^n} \cdot \frac{\tau}{\{u_p(x_1 \ldots x_n)\}^{\frac{n}{2}-1}} \quad . \quad . \quad (96)_n$$

and this represents with $d\tau$ for τ the law of potential. If this is integrated to give the value at the centre and the result compared with the solution, we have

$$\frac{4}{n(n-2)} \frac{\Gamma\left(\dfrac{n}{2}+1\right)}{\left\{\Gamma\left(\dfrac{1}{2}\right)\right\}^n} \Delta_p^{\frac{n-3}{2}} \int \frac{dx_1 \ldots dx_n}{\{u_{\mathrm{P}}(x_1 \ldots x_n)\}^{\frac{n}{2}-1}} = \int_0^\infty \frac{d\lambda}{\sqrt{\mathrm{J}}}.$$

i. e., if we write

$$c_n = 2\left\{\Gamma\left(\frac{1}{2}\right)\right\}^n \Big/ \Gamma\left(\frac{n}{2}-1\right),$$ we have

$$\Delta_p^{\frac{n-3}{2}} \int \frac{dx_1 \ldots dx_n}{[u_{\mathrm{P}}(x_1 \ldots x_n)]^{\frac{n}{2}-1}} = \frac{c_n}{2} \int_0^\infty \frac{d\lambda}{\sqrt{\mathrm{J}}} \quad . \quad . \quad (123)_n$$

the integration on the left embracing all values for which $u_a(x_1 \ldots x_n) \not> 1$, i. e. $\Sigma(a_{rr}x_r^2 + 2a_{rs}x_r x_s) \not> 1$. We then introduce $x_1 = l_1 r, \ldots x_n = l_n r$, with $\Sigma l^2 = 1$, and use ω as before for the angular element: the result is

$$\Delta_p^{\frac{n-3}{2}} \int \frac{d\omega}{u_a u_{\mathrm{P}}^{\frac{n}{2}-1}} = c_n \int_0^\infty \frac{d\lambda}{\sqrt{\mathrm{J}}} = \Delta_a^{\frac{n-3}{2}} \int \frac{d\omega}{u_p u_{\mathrm{A}}^{\frac{n}{2}-1}}. \quad . \quad . (121)_n$$

J being symmetrical with regard to a and p. [The angular element can be expressed by an extension of the usual spherical co-ordinate system and probably in other ways.] The connexion between J and the determinant is

$$\Delta(\lambda) = \Delta_a^{n-1} \mathrm{J}(\lambda),$$

where the members of $\Delta(\lambda)$ are $A_{11} + p_{11}\lambda\Delta_a, \ldots$; or if we use μ for $\lambda\Delta_a$, and

$$d\lambda / \sqrt{\mathrm{J}} = d\lambda \cdot \Delta_a^{\frac{n-1}{2}} / \sqrt{\Delta(\lambda)} = d\mu \cdot \Delta_a^{\frac{n-3}{2}} / \sqrt{\Delta(\mu)},$$

$\Delta(\mu)$ being the same determinant, viz., $A_{11}+p_{11}\mu, \ldots$, the latter part of $(124)n$ is written

$$\int \frac{d\omega}{u_p u_{\text{A}}^{\frac{n}{2}-1}} = c_n \int_0^\infty \frac{d\mu}{\sqrt{\Delta(\mu)}}. \quad . \quad . \quad (124)_n$$

the variables in $u_p\, u_{\text{A}}$ being $l_1 \ldots l_n$.

The particular case in which $p = \text{A}$ gives

$$\Delta(\mu) = \Delta_{\text{A}}(1+\mu)^n = \Delta_a{}^{n-1}(1+\mu)^n, \text{ and therefore}$$

$$\int \frac{d\omega}{(u_{\text{A}})^{\frac{n}{2}}} = \frac{2c_n}{n-2} \cdot \frac{1}{\sqrt{\Delta_{\text{A}}}}. \quad . \quad . \quad . \quad (136)_n$$

and

$$\int \frac{d\omega}{(u_{\text{A}} + \mu u_p)^{\frac{n}{2}}} = \frac{2c_n}{n-2} \cdot \frac{1}{\sqrt{\Delta(\mu)}}. \quad . \quad (137)_n$$

The expression for internal potential deduced as for three dimensions from the value at the centre is

$$\psi = \frac{k\rho}{4c_n} \Delta_a{}^{\frac{n-3}{2}} \int \frac{d\omega}{u_p(u_{\text{A}})^{\frac{n}{2}-1}} \left[1 - \frac{(n-2)(\Sigma l_r x_r)^2}{u_{\text{A}}} \right], \quad . \quad (129)_n$$

The expression for external potential is

$$\psi = \frac{k\rho}{4c_n} \Delta_a{}^{\frac{n-3}{2}} \int \frac{d\omega}{u_p(u_{\text{A}} + \mu u_p)^{\frac{n}{2}-1}} \left[1 - \frac{(n-2)(\Sigma l_r x_r)^2}{u_{\text{A}} + \mu u_p} \right]. \quad (134)_n$$

The direct proof of this formula is got by using $(137)n$, and deducing by differentiation

$$\int \frac{l_r l_s d\omega}{(u_{\text{A}} + \mu u_p)^{\frac{n}{2}+1}} = \frac{2c_n}{n(n-2)\Delta_a} \frac{\alpha_{rs}}{\sqrt{\Delta(\mu)}}. \quad . \quad (138)_n$$

It should also be stated that the method by which the theorem akin to Maclaurin's was proved is applicable in n dimensions, and in particular the theorem (98) for the addition of parameters. In the statement of that theorem the simpler form belongs to the variable λ. For the purpose of differentiation of formulæ with regard to $A \ldots$, the use of μ is more convenient; what is wanted is the simpler result got by taking $\lambda\Delta_a$ on both sides of the equations as independent of A. The dimensions of λ are more convenient for the final physical formulæ.

§ 38. In conclusion it is proposed to demonstrate that the formula established in (130) for the interior of an ellipsoid, is valid for all uniform volume distributions, and applies to external as well as internal positions. We take the simpler

form of statement, in which density ρ_0 is so defined that $4\pi\rho_0$ occurs in the flux equation, and make $\rho_0 = 1$. For isotropic potential we take the area σ of each section of the body by a plane passing through the point at which potential is reckoned : it is a function of the angular position of the normal N to the plane. If $d\omega_N$ is an element of area about the position of N on a sphere of unit radius, the statement is that $\int \sigma d\omega_N$ is 2π times the potential, the integration covering the sphere.

When the attracted point O is taken as the centre of a polar system, the element of area is $\frac{1}{2}r^2 d\phi$, that of potential is $\frac{1}{2}r^2 d\omega_P$ (*cf.* § 6) ; there may be one section r^2 or $r_2^2 - r_1^2$, or several sections according to the position of O and the shape of the body, but the integrand is alike in the two cases. The first integral has three angular integrations, one in the plane and two comprised in $d\omega_N$, and the mode of integration must be altered so as to bring together the elements which are plane-sections of the cone about OP. Thus N makes a circuit about P, while the plane element to which it is normal is a section of the cone through its axis in various azimuths, and we may denote by χ an angle defining the position of N in the circuit. When P is taken as pole $d\omega_N$ is represented by $d\chi d . \widehat{PN}$, since $\widehat{PN}$ differs by an infinitesimal from a right-angle ; and in a like manner $d\omega_P$ is represented by $d\phi d . \widehat{PN}$. Thus $d\omega_N$ and $d\omega_P$ have a common element $d . \widehat{PN}$, the angle $d\phi$ of the plane section is then used to complete the specification of $d\omega_P$, with the result that $d\chi d\omega_P$ represents the element of the triple integral. The position of N in the circuit about P is indifferent, because the various plane sections of the cone are not distinguished ; thus the integrand does not depend on χ, and the circuit integral is $2\pi d\omega_P$.

The special cases so far verified are :—The frustum of a cone for the vertex, sphere for all points, spheroid for external points on the axis of revolution, and a ring of finite circular section for all points on the axis of symmetry. The ring is the only non-quadric surface which has been examined ; the work for the cone elucidates the general proof.

The theorem appears to be a general theorem of integration on a sphere. Thus if $f(P)$ is a function of position on a sphere, and its mean for a great circle, pole N, viz. $\dfrac{1}{2\pi}\int f(P)d\phi$

is taken to define a function of the position of N, viz. F(N) ; then the means of $f(\mathrm{P})$ and F(N) over the sphere are equal.

In æolotropic potential each integral form has a factor Δ_p or $1-\Sigma p^2$ in the numerator ; the first is also modified by a divisor $u_p(l, m, n)$, and the second by a divisor $\sqrt{u_\mathrm{P}(l', m', n')}$, (lmn) defining the direction of the normal N to each plane section, $(l'm'n')$ the direction of a radius OP. Let (lmn) correspond to $(\theta\phi)$, $(l'm'n')$ to $(\theta'\phi')$ in a polar system with OZ for axis, and let χ be the angle between the planes OPZ and OPN, NP being a quadrant. As above, the triple integral is represented by $d\chi d\omega'$, but the divisor u_p is different for each plane section of the cone about OP, and u_p is a function of χ, while the other part of the integrand is independent of χ. From the spherical triangle ZPN, we have (1) $\cos\theta = \sin\theta'\cos\chi$, (2) $\sin\chi = \sin\theta\sin(\phi'-\phi)$, and (3) $\cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\phi'-\phi)=0$, or by (1) $\cos\theta'\cos\chi + \sin\theta\cos(\phi'-\phi) = 0$. With $l = \sin\theta\cos\phi$, $m = \sin\theta\sin\phi$, $n = \cos\theta$, these give

$$l = \sin\chi\sin\phi' - \cos\chi\cos\phi'\cos\theta',$$
$$-m = \sin\chi\cos\phi' + \cos\chi\sin\phi'\cos\theta', \quad n = \sin\theta'\cos\chi.$$

Thus u_p assumes the form $\alpha\sin^2\chi + \beta\cos^2\chi + 2\gamma\sin\chi\cos\chi$, and $\int_0^{2\pi} d\chi/u_\mathrm{P} = 2\pi/\sqrt{\alpha\beta-\gamma^2}$. It will be found that

$$\alpha\beta-\gamma^2 = \mathrm{P}l'^2 + \mathrm{Q}m'^2 + \mathrm{R}n'^2 + 2\mathrm{P}'m'n' + 2\mathrm{Q}'n'l' + 2\mathrm{R}'l'm'$$
$$= u_\mathrm{P}(l', m', n').$$

Thus the two modifications exactly correspond. On the one view of potential the element is a plane area, which in æolotropic potential is modified by a divisor depending on the exposure of the plane in relation to the æolotropic axes, viz. $u_p(lmn)$ in the general case, $1-(\Sigma l_\mu)^2$ in the motional case. On the other view the element of volume is divided by distance from the point attracted ; in æolotropic potential this is modified by a divisor which changes

$$\sqrt{\Sigma(x-x')^2} \quad \text{to} \quad \sqrt{u_\mathrm{P}(x-x', y-y', z-z')},$$

and in the motional case to

$$\sqrt{(1-\Sigma p^2)\Sigma(x-x')^2 + (\Sigma p(x-x'))^2}.$$

We may characterize these two forms of the expression for potential as the *wave* and the *ray* forms.

XLIII. *On the Curvature Method of Teaching Optics.*
By C. V. Drysdale, *D.Sc.*[*]

NOTWITHSTANDING the fact that the wave theory of light has been employed to demonstrate some of the more simple problems in the domain of what is generally termed geometrical optics, and with manifest simplicity and convenience, this appears to have been done rather with the object of verifying the wave theory than of showing how the subject of optics can be completely dealt with from this standpoint : and few men of science or teachers of optics appear to have realized the advantages of physical methods both for practical work and teaching, and that they should entirely supersede the geometrical or ray methods. This is unquestionably due to a very large extent to unfamiliarity, and to the cramping effect of our university curricula and text-books ; but a possible factor in the question is the impression which seems to be prevalent, even among optical specialists, that the physical methods have to be abandoned at a certain stage, and that the more complex problems relating to lens systems and aberrations must be treated by geometrical methods. Owing to the fact that British men of science have been actively engaged in extending the wave theory towards penetrating the more fascinating mysteries of interference, polarization, and electromagnetic theory, the practical applications of optics have passed for the last half century into the hands of the Germans, who took from us the geometrical methods then in vogue and have since extended them with such marked success as to give the impression that these geometrical methods are the most suitable for the purpose. At the present time a strong attempt is being made to revive the study of technical optics in this country, unfortunately with great difficulties, owing to the lack of satisfactory teaching ; and the object of this paper is to show that not only are physical methods the most suitable at the outset, but that they are capable of being employed with the same increased simplicity in the whole domain of " geometrical optics."

A few words are desirable at the outset as to what has been done in the application of physical problems to reflexion and refraction. The first step in this direction appears to have been made by Herschel in 1827 [†], who seems to have dealt with ordinary lens problems fairly completely and to have devised a very satisfactory curvature notation. According

to Mr. Cheshire [*], in an article just published, Porro, in 1857 [†], gave a very complete exposition of lenses and thin lens combinations from the physical standpoint, including suggestions for the study of spherical aberration and of thick lenses. In Preston's 'Light' [‡] a short account of the elementary treatment of lenses by this method appears ; and Lord Rayleigh [§] has also devoted attention to the subject. It is, however, to Prof. S. P. Thompson [||] that we owe nearly everything that has been done in recent years towards the rational teaching of optics, and in 1889 he wrote a paper on the curvature method of treating lens problems which he had rediscovered and had used in teaching for about eight years ; and he has followed this up by several valuable papers, and his two important works—' Optical Tables and Memoranda,' and the translation of Lummer's ' Photographic Optics.'

It will perhaps be well to mention here that the present writer has unfortunately, until the last few days, been absolutely unconscious of practically all that has been done in the application of wave methods by others. Although a student under Prof. Thompson in 1888–90, for some reason none of this work was done, and the only thing that directed his attention to the subject was a few lectures given by Dr. Sumpner at the Central Technical College in 1892, dealing with the matter as in Preston's 'Light.' The interest roused by the manifest superiority of the wave method, however, led him afterwards to make some simple applications in practical work, and later to start a course of lectures on technical optics at the Northampton Institute, entirely based on physical optics. It is a singular fact that although we owe to scientific men the wave theory of light and its applications to optical theory, the optical trade have, apparently quite independently, adopted a system of lens and prism nomenclature which harmonizes completely with it, and makes its application to reflexion and refraction problems simple and logical. It was due to the writer becoming acquainted with this notation that he was led to take up the subject so fully, and he ventures to think that few teachers of optical science would find a knowledge of optical trade methods detrimental.

Before dealing with the curvature method, it may be interesting to note that there are several alternative methods of

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[*] F. J. Cheshire, British Optical Journal, Nov. 1904.

[†] Porro, *Société Française de Photographie*, vol. iii. pp. 211–222, 1857.

[‡] Preston, 'Light,' pp. 99–106, Third edition, 1901.

[§] Lord Rayleigh, *Encyclopedia Britannica*, 1884. Article on "Optics."

[||] S. P. Thompson, Phil. Mag. 1889, vol. xxxiii., "Notes on Geometrical Optics," Part i. pp. 232–248.

attacking lens problems, of which one only, and that probably
the least suitable, has been given a fair trial in this country.
They may perhaps be enumerated as follows :—

(a) The method of reckoning in conjugate distances—
Geometrical or Gauss method.
(b) The physical or curvature method.
(c) The method of deviations—Von Seidel. Finster-
walter.
(d) The use of the characteristic function, or principle
of least time—Hamilton, Thiesen, and Chalmers.
(e) $\begin{cases} \text{Thermodynamic method—Clausius.} \\ \text{Employment of the "eikonal"—Bruns.} \end{cases}$
(f) Vector or Quaternion treatment.

Of these methods (b), (c), and (d) are the ones which the
writer believes will be found most suitable, and it will be
shown, as is almost self-evident, that they are essentially
similar. Of the eikonal treatment he has no experience, but
is inclined to think that its use is confined to problems of an
advanced nature, and that it is unsuitable for a general ele-
mentary treatment. As to (f) it seems curious that no one
has proposed the application of the modern vector calculus
of Prof. Henrici, Mr. Oliver Heaviside, and Prof. Gibbs, to
geometrical optics, as it should be capable of effecting con-
siderable simplifications. For example, if $\hat{a}$ is a unit vector
representing the direction of an incident ray, $\hat{b}$ the corre-
sponding refracted ray, and $\hat{n}$ the vector normal to the
surface, both of the ordinary laws of refraction are summed
up in the simple vector relation

$$\mu_1 \, [\hat{a} \, \hat{n}] \; = \; \mu_2 \, [\hat{b} \, \hat{n}],$$

or in Heaviside's notation

$$\mu_1 \, \mathrm{V}\hat{a} \, \hat{n} \; = \; \mu_2 \, \mathrm{V}\hat{b} \, \hat{n}.$$

The writer has deduced a few interesting consequences of
this fundamental expression, but has not yet had an oppor-
tunity of following it up completely ; moreover, as the method
is a purely geometrical one, it could only have advantages
in a possible simplification of ordinary procedure, and would
not have any other physical signification.

Practical Optical Units.—The simple device of opticians
for spectacle-lens notation was to adopt two units which are
now fairly generally known. The first of these may be used
to express either the curvature of a surface or the power of

a lens or mirror, and is termed a Dioptre *. This unit is a curvature corresponding to a radius of one metre ; and we consequently say that a lens has a convergence of one dioptre when it has a focal length of one metre, or that a wave-front or surface has a curvature of one dioptre when it has a radius of curvature of one metre. The other unit adopted is one of angle or deviation and is termed the Prism Dioptre : it is defined as a deviation of one centimetre on a tangent line at a distance of one metre, and one prism dioptre therefore corresponds approximately to ·01 of a radian. The prism dioptre and the radian thus serve as two units of angular measurement, and the former is very conveniently related to the dioptre, as the deviation in prism dioptres produced by any thin lens at any zone is simply obtained by multiplying the convergence of the lens in dioptres by the radius of the zone in centimetres.

In the prism dioptre and the radian we have two units of angular measurement which cover most requirements ; but the curvature dioptre, although an exceedingly convenient unit for many purposes, is inconvenient to deal with when we are concerned with microscope lenses and others of short focal length. The writer has therefore recently proposed † the adoption of multiples and sub-multiples of this unit, using the ordinary prefixes as follows :—

Radius of curvature	Curvature.
Kilometre	Millidioptre.
Metre	Dioptre.
Centimetre	Hectodioptre.
Millimetre	Kilodioptre.

The great advantages of such a system is, that to any particular length measurement there is a corresponding curvature measurement, which enables one to have a physical idea of the magnitudes involved and avoids the use of unwieldy fractions which frequently occur otherwise. The range of curvatures may be very great, the radius varying from a wave-length to infinity.

Notation.—The writer has also found it convenient to adopt a standard notation in optical work, using small letters for the distances, radii of curvature, focal lengths, and

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* Variously spelt, Dioptrie, Dioptre, and Diopter. The author has hitherto preferred the spelling Dioptre, harmonizing with metre, but modern usage favours meter and diopter. It appears, however, that Dioptrie was proposed by Monoyer in 1872, and adopted by the Brussels International Congress in 1875.

† "On some Points in the Design of Optical Instruments." Proc. Optical Soc., December 18th, 1902.

thicknesses ; large letters for the corresponding curvatures, and Greek letters for the angles *. The conjugate distances and focal lengths being denoted by u, v, and f respectively ; we have U, V, and F as the corresponding curvatures or convergences. If the distances u, v, &c. are expressed in metres, we have $U = \dfrac{1}{u}$, $V = \dfrac{1}{v}$, &c., the curvatures in dioptres, while if u and v are in mm. U and V are in kilodioptres.

In like manner, if we denote the radii of the surfaces of a lens by r_1 and r_2, we should have R_1 and R_2 as the curvatures of the surfaces, and $C = R_1 - R_2$ the total curvature in the case of a thin lens. Curvatures of wave-fronts may be reckoned positive when they are convergent, and surfaces are said to have positive curvature when they are curved in the same direction as convergent emergent light. This harmonizes the formulæ for reflexion and refraction at curved surfaces.

Part I.—Elementary Optics.

In dealing with this subject it was the writer's first intention to give a fairly complete exposition of the methods he has adopted, but a subsequent perusal of Dr. Thompson's article in the Phil. Mag. has shown him that the procedure he has followed is so closely identical with that advocated by Dr. Thompson as to render this unnecessary. For the sake of completeness, however, the various steps in the development of an elementary optical course may be briefly given here.

Nature and mode of Propagation of Light.—The first important step is to familiarize the student with the notion of waves and their propagation. A beginner is easily convinced by a few simple illustrations that light must be either propagated by projectiles or undulations. The ripple tank can then be used to demonstrate the propagation of waves, and the effects of reflexion, &c. shown simultaneously by optical projection, and in the tank. The justification for the undulatory theory can be well shown by interference, ripples being excited from two simultaneously vibrating points, and the results compared with interference from a bi-prism. Even elementary students can quite appreciate that the wave theory is the only one which can satisfactorily account for the dark interference-bands. The finite rate of propagation of waves is then pointed out with references to determinations of the speed of light, and it is also shown that ripples of different frequencies can be excited and that the wave-lengths differ.

* This appears to be almost identical with the notation employed by Herschel and by Dr. Thompson.

Finally it is shown by analogy with the ripples that the wave-fronts diverge in expanding spheres, becoming less curved as they recede from the source, and ultimately plane, and that the direction of propagation is perpendicular to the wave-front; and the difference between plane, divergent, and convergent waves is illustrated. At the same time, it may be pointed out with advantage that if the medium were not isotropic, the speed of propagation would be different in different directions, the wave-front being ellipsoidal and not perpendicular to the lines of propagation. The writer has never found that elementary students have any difficulty over this, and it prepares the way for subsequent work in polarization, &c.

The next stage is to explain the formation of shadows and of an image by a pinhole, referring at the same time to diffraction. The formation of the penumbra may also be well illustrated by waves, and photometry explained by showing that any portion of the wave-front carrying a certain amount of light-energy expands proportionally to the square of the distance from the source.

The study of reflexion at plane surfaces comes next, first with plane and then with spherical waves, proving the laws of reflexion, application to various instruments, measurement of angles of a prism, &c., and formation of images by one or more plane mirrors.

Curvature and its Measurement.—At this stage the ideas and units of curvature are introduced, explaining the necessity for the measurement of the curvatures both of waves and of reflecting surfaces. Students will easily realize that a satisfactory definition of curvature is the reciprocal of the radius, but it may be proved from the ordinary definition of curvature if preferred. The dioptre with its multiples and submultiples may then be introduced and the relations:

$$\text{curvature in dioptres} = \frac{1}{r \text{ (metres)}} = \frac{100}{r \text{ (cm.)}} = \frac{39 \cdot 37}{r''},$$

and the corresponding conversions from curvatures to radii.

Practical curvature measurement may then be taken up, the equality of the products of the segments of any two chords of a circle being demonstrated either by geometry or by simple measurement, and from this the ordinary spherometer formula

$$r = \frac{c^2 + h^2}{2h}, \text{ or } R = \frac{2h}{c^2 + h^2} = \frac{2}{c^2} h$$

when the curvature is small. When c and h are in millimetres

the resultant R must evidently be in kilodioptres, and the curvature in dioptres is then $\dfrac{2000\,h}{c^2+h^2}=\dfrac{2000}{c^2}h$ for small curvatures. It is then pointed out that if $c^2=2000$ or $c=44{\cdot}7$ mm. the sagitta or sag of the curve denotes directly the curvature in dioptres, and Prof. Thompson's " dioptric " spherometer and other forms are explained. At the same time attention is called to the facts, first that the curvatures of all circular curves having the same chord are proportional to their sagittas, and secondly, that this is only true for chords of small lengths compared with the radius. This assists the explanation of spherical aberration later.

The properties of spherical mirrors follow immediately. From the fact that light striking the surface travels back the same distance as it would have gone forward in the absence of the mirror, it follows immediately that the curvature of the mirror is the mean between the curvatures of the incident and reflected wave-fronts. Hence $R=\dfrac{V-U}{2}$ or $V-U=2R$, where V and U are the convergences of the incident and reflected light and $2R=F$ is shown to be the convergence of the mirror. By considering the image of an object formed by the mirror when the latter is stopped down to a pinhole at the vertex, we at once find the magnification $m=\dfrac{v}{u}=\dfrac{U}{V}$, and the whole of the properties of mirrors are deduced from the two equations

$$V-U = 2R = F$$
$$m = \frac{v}{u} = \frac{U}{V}.$$

The error of considering the sagitta proportional to curvature for large apertures is again pointed out, with reference to parabolic mirrors, caustics, &c.

Refraction and Dispersion.—It is at this point that the value of the wave method begins to show itself most strikingly. Refraction at a simple plane surface may be shown by Huyghens' method. The writer has found it of the greatest assistance to students to illustrate the wave-fronts by parallel lines of men marching towards a river which they can only ford at some fraction of their marching speed. The swinging round of the line is easily realized ; and if at the same time the idea is introduced of the lines being made up of men of different heights, with uniforms ranging from violet to red

according to size and strength, the reason for dispersion is at once grasped. It is at the same time explained that the more rapid violet light vibrations or shorter waves would naturally be more encumbered by dense or gross matter than the larger red waves. At the same time the general effects of refraction by parallel plates, prisms and lenses are illustrated by the crossing of rivers with parallel, oblique or curved banks. This paves the way for a complete study of refraction and of the properties of prisms and lenses. As it would take up too much space to give a detailed course here, it is proposed simply to give one or two examples as showing either original methods or ones which are less generally known and which the writer considers should be adopted.

It should be here mentioned that it has been found convenient when dealing with thin prisms and lenses, to consider the alteration of the wave-front by the prism or wave as a whole, and only to introduce the formulæ for single surfaces when thick lenses and prisms or refracting systems in general are considered.

Retardation by a Parallel Plate.—Since the refractive index $\mu = \dfrac{\text{velocity in air}}{\text{velocity in medium}}$, it follows that in passing through a thickness t of glass, the wave would have travelled forward a distance μt in air, and the retardation caused by the plate is $\mu t - t = (\mu - 1)t$.

Refraction by a Thin Prism.—In the case of a thin prism (fig. 1) having a thickness t at its base and of zero at its

Fig. 1.
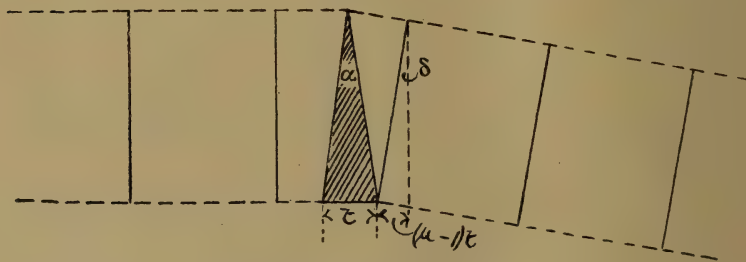
Refraction by Thin Prism.

apex, the retardation of the wave at the base being $(\mu - 1)t$, we have evidently $\delta = (\mu - 1)\alpha$. It is pointed out that this cannot hold for thick prisms, as in that case the distance traversed by the light depends on the direction of entry, and it is not justifiable to take the angles as being proportional to the bases.

Thick Prisms.—Here we have only to take a section of the prism ABC (fig. 2), and indicate the direction of the

Fig. 2.
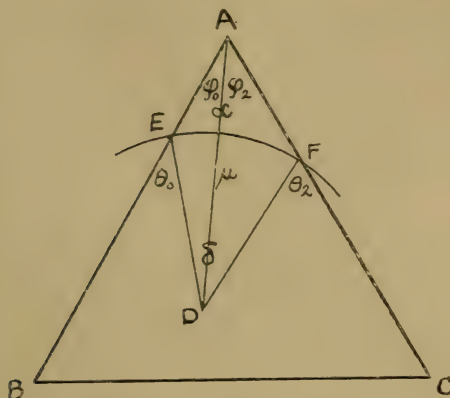
Refraction by Thick Prism.

three wave-fronts before incidence, in the glass, and after emergence by DE, DA, and DF respectively. The angles θ_0 and ϕ_0 are obviously the angles of incidence and of refraction with reference to the left face of the prism, while θ_2 and ϕ_2 are those relating to the right face. We thus have immediately $\alpha = \phi_0 + \phi_2$ and $\delta + \alpha = \theta_0 + \theta_2$, and since $\sin \theta_0 = \mu \sin \phi_0$ and $\sin \theta_2 = \mu \sin \phi_2$ we have the distances DE, DF, and DA

Fig. 3.
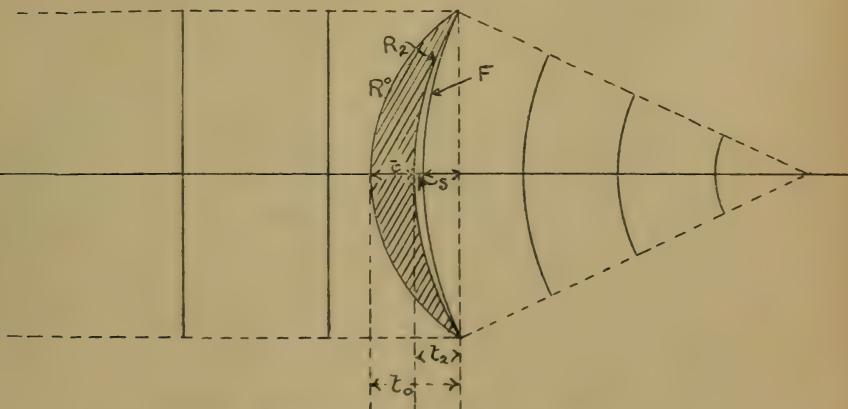
Refraction by Thin Lens.

in the ratio of $1 : 1 : \mu$, and we are immediately led to the ordinary construction for, and properties of thick prisms.

Thin Spherical Lenses.—In fig. 3 we have a thin lens, the

thickness at its centre being t, which is the sag corresponding to the total curvature of the lens. A wave passing through it will have its centre retarded by an amount $s = (\mu - 1)t$ as with a thin prism. If the wave is initially plane it receives a curvature corresponding to the sag S, which we term the convergence F of the lens, and we therefore have immediately

$F = (\mu - 1)C$ corresponding to $\delta = (\mu - 1)\alpha$ for a thin prism.

If the wave is initially curved or convergent by an amount U, the retardation increases its convergence by the amount

$$V - U = (\mu - 1)C = F.$$

The thickness t of the lens can be made up of the two sags t_0 and t_2 to the curvatures R_0 and R_2 of the faces.

Hence $V - U = F = (\mu - 1)C = (\mu - 1)(R_0 - R_2).$

By considering a pinhole stop in contact with the lens we immediately have $m = \dfrac{v}{u} = \dfrac{U}{V}$ as with the mirror; and the properties of lenses and formation of images follow immediately.

Thin Cylindrical Lenses.—For a principal section of a cylindrical lens perpendicular to its axis, we have obviously the same relation as for a spherical lens. For an oblique section the retardation of the wave-front is obviously the same, but the breadth of the beam is increased (fig. 4). From the spherometer formula $R = \dfrac{2h}{c^2}$ we have

$$R' = R\left(\frac{c}{c'}\right)^2 = R \sin^2 \theta,$$

and hence also $F' = F \sin^2 \theta$. In a meridian perpendicular to the former we must similarly have $F'' = F \cos^2 \theta$, hence $F' + F'' = F$, or the sum of the convergences in two meridians at right angles is constant and equal to the convergence of the lens, analogous to a well-known theorem for curved surfaces.

Deviation and Decentring of Lenses.—By multiplying the equation $V - U = F$ by the height of intersection of the pencil with the surface we have immediately

$$\tan \sigma_2 - \tan \sigma_1 = Fh \quad \text{or} \quad \sigma = \sigma_2 - \sigma_1 = Fh$$

if the angles are small. This gives us the valuable result that the deviation at any zone of a thin lens in prism-dioptres is given by the product of the convergence of the lens in dioptres

and the radius of the zone or decentration in cm. The more exact equation is also a valuable introduction to Von Seidel's

Fig. 4.
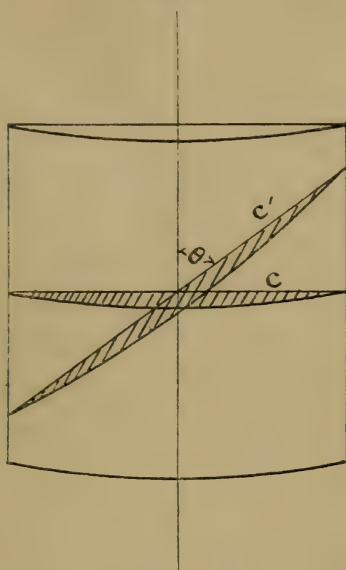
Cylindrical Lens.

method of deviations. From the equation $m = \dfrac{U}{V}$ we have immediately $m = \dfrac{\tan \sigma_1}{\tan \sigma_2}$ or Helmholtz's expression for the magnification.

Combinations of Thin Lenses in contact.—It is obvious from the summation of the thicknesses that the resultant convergence $F = \Sigma F_1$ for any number of thin lenses in contact, and that in the case of cylindrical or sphero-cylindrical lenses, the powers in any meridian may be added. It has been already shown that the convergence of a cylindrical lens in a meridian making an angle of θ with the axis is

$$F \sin^2 \theta = \frac{F}{2} (1 - \cos 2\theta).$$

Hence if we have any number of cylindrical lenses in contact of convergences F_1, F_2, &c., and whose axes make angles α_1, α_2, &c., with a reference meridian, we have

$$F = \tfrac{1}{2}\Sigma F_1 - \tfrac{1}{2}\Sigma F_1 \cos 2(\theta - \alpha_1)$$

in any meridian making an angle θ with the reference meridian.   On expanding we have

$$F = \tfrac{1}{2}\Sigma F_1 - \tfrac{1}{2}\{\cos 2\theta \Sigma F_1 \cos 2\alpha_1 - \sin 2\theta \Sigma F_1 \sin 2\alpha_1\},$$

and this is a maximum or minimum for

$$\tan 2\theta = \frac{\Sigma F_1 \sin 2\alpha_1}{\Sigma F_1 \cos 2\alpha_1},$$

giving the direction of the axes of the combination, while the maximum and minimum values are given by the expression

$$\tfrac{1}{2}\Sigma F_1 \pm \tfrac{1}{2}\sqrt{(\Sigma F_1 \sin 2\alpha_1)^2 + (\Sigma F_1 \cos 2\alpha_1)^2}$$

corresponding to the convergences in the two principal meridians.   Fig. 5 shows curves exhibiting the relation

Fig. 5.
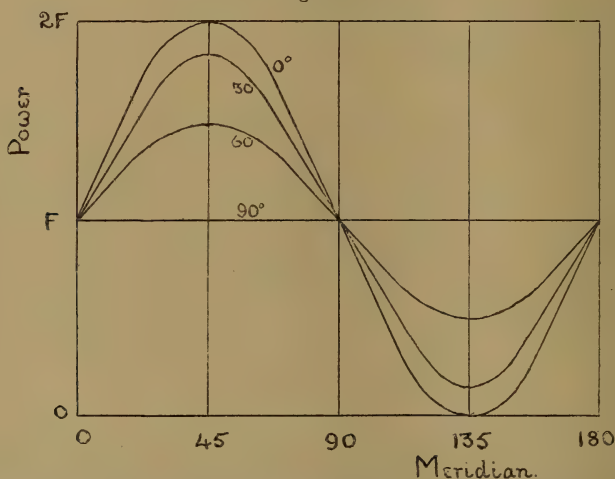
Power of Combination of two equal cylinders crossed at various angles.

between the convergence and angle for two cylinders crossed at various angles.   The properties of such combinations have already been brought before this Society by Dr. Thompson [*], but the writer had come to the same conclusions earlier.   It is interesting to note that while the displacement of a prism corresponds to an ordinary vector, the curvature of a cylinder is what is called by Steinmetz a double frequency vector, and this can be well illustrated experimentally.   In general the distinction which Prof. Thompson has drawn between unipolar

* S. P. Thompson, Proc. Phys. Soc.

and dipolar quantities has its origin in the distinction between simple and double frequency vectors.

Combinations of Two Thin Lenses.—These may be mentioned here before considering thick lenses, as they do not imply a knowledge of refraction at a single surface. Considering two thin lenses of convergences F_0 and F_2 separated by an interval d_1, parallel light falling on the first emerges from it with a convergence F_0 which becomes $\dfrac{1}{1/F_0 - d}$ or $\dfrac{F_0}{1 - F_0 d_1}$ on reaching the second. To this is added the convergence of the second lens, giving us the emergent convergence as

$$\frac{F_0}{1 - F_0 d_1} + F_2 = \frac{F_0 + F_2 - F_0 F_2 d_1}{1 - F_0 d_1}.$$

This may be termed the "back" or "emergent" convergence if desired, corresponding to the "back focus," and the other emergent convergence will evidently be $\dfrac{F_0 + F_2 - F_0 F_2 d_1}{1 - F_2 d_1}$.

Equivalent Convergence.—Introducing the equivalent lens as that giving the same size of image as the combination, we have $x = \dfrac{\tan \alpha}{F}$ or $F = \dfrac{\tan \alpha}{x}$, where α is the angular magnitude of the object and x the magnitude of the image. For the first lens we have $x_0 = \dfrac{\tan \alpha}{F_0}$, and the magnification by the second lens is

$$\frac{U_2}{V_2} = \frac{F_0}{1 - F_0 d_1} \frac{1 - F_0 d_1}{F_0 + F_2 - F_0 F_1 d_1} = \frac{F_0}{F_0 + F_2 - F_0 F_2 d_1}.$$

Hence

$$x = m x_0 = \frac{\tan \alpha}{x} \frac{F_0}{F_0 + F_2 - F_0 F_2 d_1} = \frac{\cdot \tan \alpha}{F_0 + F_2 - F_0 F_2 d_1},$$

and $$F = \frac{\tan \alpha}{x} = F_0 + F_2 - F_0 F_2 d_1.$$

Nodal or Principal Points.—The Nodal points being defined as those points through which light passes undeviated, we have, since the deviation at any zone of a lens $= Fh$,

$$F_0 h_0 + F_2 h_2 = 0,$$

or $$F_0 d_0 + F_2 d_2 + 0 \quad \text{(see fig. 6)},$$

but $$d_0 - d_2 = d \quad \text{the distance between the lenses.}$$

Hence $d_0 = \dfrac{F_2 d}{F_0 + F_2}$ giving the optical centre,

or $D_0 = \dfrac{F_0 + F_2}{F_2 d}$ the convergence of the light
                                     focussing at O.

Fig. 6.
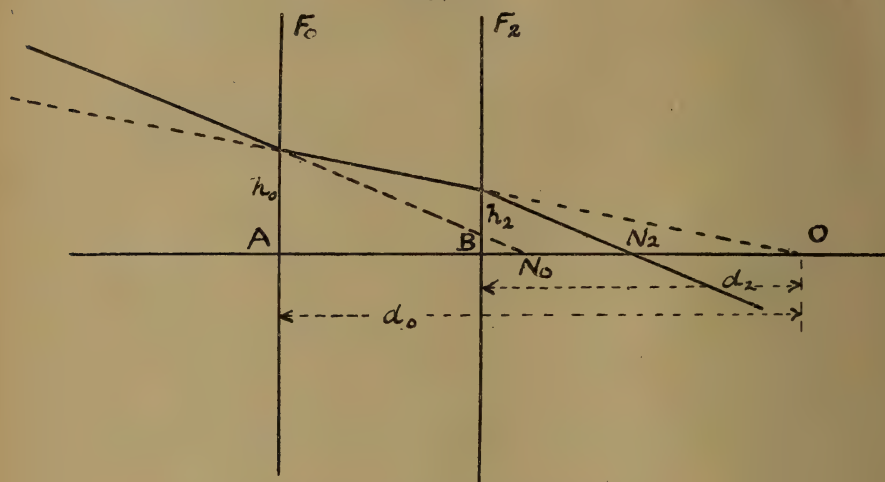
Nodal points of a Two-lens Combination.

Similarly $D_2 = -\dfrac{F_0 + F_2}{F_0 d}.$

Hence $D_0' = D_0 - F_0 = \dfrac{F_0 + F_2 - F_0 F_2 d}{F_2 d},$

the convergence of the light proceeding to N_0,

and $d_0' = \dfrac{F_2 d}{F_0 + F_2 - F_0 F_2 d}$ the distance AN_0,

while $d_2' = -\dfrac{F_0 d}{F_0 + F_2 - F_0 F_2 d}$ the distance BN_2.

We have seen above that the emergent convergence at the
second lens $F' = \dfrac{F_0 + F_2 - F_0 F_2 d_1}{1 - F_0 d_1}$, and now that the con-
vergence at any distance $d_2 = \dfrac{F'}{1 - F' d_2}.$

But $d_2 = -\dfrac{F_1 d}{F_1 + F_2 - F_1 F_2 d}$; $\therefore$ $1 + F' d_2 = \dfrac{1}{1 + F_1 d}$,

and therefore convergence at Nodal point $=$

$\dfrac{F_0 + F_2 - F_0 F_2 d_1}{1 - F_0 d_1} (1 - F_0 d_1) = F_0 + F_2 - F_0 F_2 d_1 = F$;

or the convergence in the plane through the nodal point of emergence is the equivalent convergence ; or the distance between the nodal and focal points is the equivalent focal length. This latter might of course have been directly obtained by taking the back focal length and distance of nodal point.

Thick Lenses and Refracting Systems.

Refraction at Single Surface.—Up to the present we have not needed to consider refraction at a single curved surface, but it is necessary to do so before studying thick lenses and lens systems. If light passes through a surface (fig. 7)

Fig. 1.
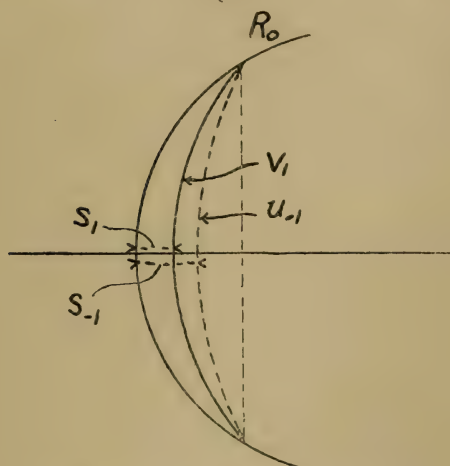
Refraction at a Single Spherical Surface.

separating two media of refractive indices μ_{-1} and μ_1, we have always as our fundamental equation $\mu_{-1}s_{-1} = \mu_1 s_1$, where s_{-1} and s_1 are the departures or sags of corresponding parts of the incident and refracted wave-fronts from the surface.

In the case of a normal homocentric pencil of convergence U and a spherical surface of curvature R_0 this gives us immediately

$$\mu_{-1}(R_0 - U_{-1}) = \mu_1(R_0 - V_1)$$

or

$$\mu_1 V_1 - \mu_{-1}U_{-1} = (\mu_1 - \mu_{-1})R_0 = F_0.$$

In the interspace between two surfaces we have

$$U_1 = \frac{1}{\frac{1}{V_1} - t_1} = \frac{V_1}{1 - V_1 t_1},$$

where U_1 is the convergence on meeting the second surface and t is the thickness.

This equation can, however, be written

$$\mu_1 U_1 = \frac{1}{\dfrac{1}{\mu_1 V_1} - \dfrac{t_1}{\mu_1}}.$$

Nomenclature for Thick Lenses and Lens Systems.—We can greatly reduce the labour and simplify the working of problems relating to lens systems by adopting a suitable nomenclature and notation. This was done by Gauss [*], who introduced the idea of "absolute" and "reduced" distances and thicknesses. Following this we may introduce the term "reduced convergence" of a pencil as the product of its actual or "absolute" convergence and the refractive index of the material; while the "reduced thickness," as with Gauss, is defined as the negative of the "absolute thickness" divided by the refractive index of the medium.

Denoting these quantities by accented letters, we have

$$U' = \mu U, \qquad V' = \mu V, \qquad t' = -\frac{t}{\mu};$$

and the equations for the surface and interspace are

$$V_1' - U'_{-1} = (\mu_1 - \mu_{-1}) R_0 = F_0,$$

and

$$U_1' = \frac{1}{t_1' + \dfrac{1}{V_1'}} = \frac{V_1'}{V_1' t_1' + 1}.$$

It is worthy of note that each of the reduced quantities has a simple physical meaning. For in the first equation, if the surface is plane

$$V_1' - U'_{-1} = 0, \quad \text{or} \quad V_1' = U'_{-1}.$$

Consequently the reduced convergence of a pencil does not change in passing through a plane surface separating any two media. But if the second medium is air $\mu_1 = 1$ and $V = U'_{-1}$. Hence the reduced convergence of a wave-front in any medium is the convergence it would have if it emerged from that medium into air through a plane surface. The term "equivalent" instead of reduced convergence may therefore be employed if preferred, but is preferably kept for the equivalent convergence of the combination. Similarly $t_1' = -\dfrac{t}{\mu}$ is the "apparent thickness" of a parallel plate of the medium when viewed in air.

* Pendlebury, 'Lenses and Systems of Lenses.'

The magnification produced by a single surface is readily seen by the same device of a pinhole stop at the vertex, fig. 8.

Fig. 8.
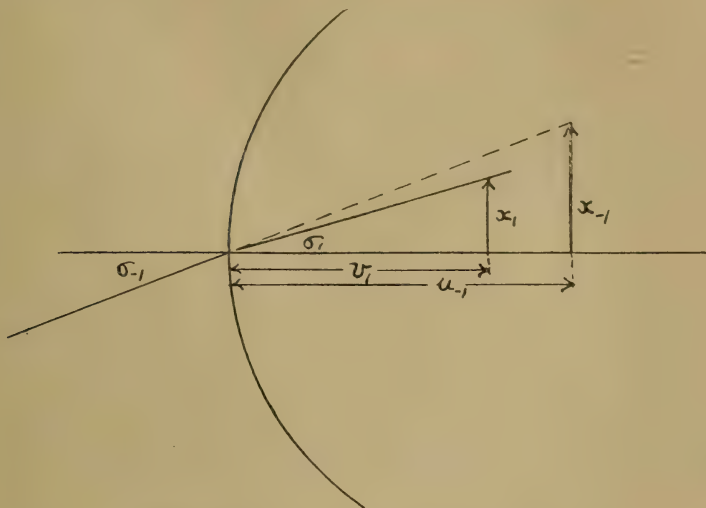
Magnification by Single Spherical Surface.

We then have

$$x_{-1} = \mu_1 \tan \sigma_{-1} = \frac{\tan \sigma_{-1}}{U_{-1}}$$

and

$$x_1 = \frac{\tan \sigma_1}{V_1}.$$

Hence

$$m = \frac{x_1}{x_{-1}} = \frac{U_{-1}}{V_1} \frac{\tan \sigma_1}{\tan \sigma_{-1}}$$

$$= \frac{U_{-1}}{V_1} \frac{\sin \sigma_1}{\sin \sigma_{-1}} \text{ approx.} = \frac{U'_{-1}}{V'_1}.$$

It will be unnecessary to go here more fully into the properties of the single surface, its two principal convergences of the object and image space, principal and nodal points, &c. We will immediately pass to the case of the thick lens.

Simple Thick Lens.—Denoting the curvatures of the surfaces by R_0 and R_2 respectively, and the refractive indices of the media by μ_{-1}, μ_1, and μ_3, we have :—

At first surface, $V_1' - U_{-1}' = F_0$ or $V_1' = E_0 + U'_{-1}$

Interspace . . . . $U_1' = \dfrac{1}{t_1' + \dfrac{1}{V_1'}} = \dfrac{V_1'}{V_1' t_1' + 1}$

Second surface . . . . . . $V_3' = F_2 + U_1'.$

Combining these equations we have immediately

$$V_3' = F_2 + \frac{F_0 + U'_{-1}}{(F_0 + U'_{-1})t_1' + 1} = \frac{F_0 + F_2 + F_0 F_2 F_1' + (F_2 t_1' + 1)U'_{-1}}{F_0 t_1' + 1 + t_1' U'_{-1}}$$

$$= \frac{A + BU'_{-1}}{C + DU'_{-1}},$$

where $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = -1.$

To find the size of the image produced we have

$$\frac{1}{m} = \frac{1}{m_0}\frac{1}{m_2},$$

where m_0 and m_2 are the successive magnifications at the surfaces.

But $\dfrac{1}{m_0} = \dfrac{V_1'}{U'_{-1}}$ and $\dfrac{1}{m_2} = \dfrac{V_3'}{U_1'} = \dfrac{A + BU'_{-1}}{C + DU'_{-1}} \dfrac{V_1' t_1' + 1}{V_1'}$

and $C + DU'_{-1} = V_1 t_1' + 1$ by the above.

Hence $\dfrac{1}{m_2} = \dfrac{A + BU'_{-1}}{V_1'}$ and $\dfrac{1}{m} = \dfrac{A + BU'_{-1}}{U'_{-1}}.$

This re-establishes the three Gauss relations in the convergence form. It should be noted that the quantity A is the reduced equivalent convergence of the lens.

It will not be necessary to show how the properties of thick lenses may be deduced from these equations as methods similar to those of Gauss can be followed. We may therefore pass on to the consideration of a complete system, but before doing so reference may be made to the deviation or Von Seidel method.

If we take our equation for a single surface

$$\mu_1 V_1 - \mu_{-1} U_{-1} = F_0$$

and multiply it by h_0, the lateral distance of intersection of a pencil, we have

$$\mu_1 \tan \sigma_1 - \mu_{-1} \tan \sigma_2 = F_0 h_0.$$

If we now denote the product of the tangent of an angle by the refractive index of the medium, by the term "reduced angle" and represent these reduced angles by accented letters, we have

$$\sigma_1' - \sigma'_{-1} = F_0 h_0.$$

In the interspace between two refracting surfaces we have clearly

$$h_2 - h_0 = -t_1 \tan \sigma_1' = -\frac{t_1}{\mu_1}\mu_1 \tan \sigma_1',$$

or

$$h_2 - h_0 = \sigma_1' t_1'.$$

Hence, for tracing the course of a pencil through any system, we have simply

Surface 0. $\sigma_1' - \sigma'_{-1} = F_0 h_1$, or $\sigma_1' = F_0 h_0 + \sigma'_{-1}$;

Interval 1. $h_2 - h_0 = \sigma_1' t_1'$ or $h_2 = t_1' \sigma_1' + h_0$; .

Surface 2. $\sigma_3' - \sigma'_{-1} = F_2 h_2$ or $\sigma_3' = F_2 h_2 + \sigma_1'$;

and this may be extended to any number of surfaces by putting down the convergences and reduced thicknesses, and the corresponding lateral intersections and reduced angles in the order

$$F_0 \quad t_1' \quad F_2 \quad t_3' \quad \&c.$$
$$\sigma'_{-1} \quad h_0 \quad \sigma_1' \quad h_2 \quad \sigma_3' \quad \&c.$$

each member of the lower series being derived by multiplying the penultimate member by the corresponding value above it, and adding the antepenultimate. This is the extremely valuable method due to Von Seidel *.

For a single thick lens we have to take simply the three equations given, and we find

$$h_2 = t_1' \sigma_1' + h_0 = (F_0 t_1' + 1) h_0 + t_1' \sigma'_{-1} = C h_0 + D \sigma'_{-1}$$

and

$$\sigma_3' = F_2 h_2 + \sigma_1' = (F_0 + F_2 + F_0 F_2 t_1') h_0 + (F_2 t_1' + 1) \sigma'_{-1} = A h_0 + B \sigma'_{-1}.$$

We may express this either in the ordinary Gauss or in the convergence form by taking

$$v_3' = \frac{h_2}{\sigma_3'}, \quad \text{or} \quad V_3' = \frac{\sigma_3'}{h_2}.$$

In the latter case

$$V_3 = \frac{A h_0 + B \sigma'_{-1}}{C h_0 + D \sigma'_{-1}} = \frac{A + B U'_{-1}}{C + D U'_{-1}} \text{ as before.}$$

Extension of Convergence Theory to any System.

Our equations for the convergence method were

$$V_1' - U'_{-1} = F_0 ;$$
$$U_1' = \cfrac{1}{t_1' + \cfrac{1}{V_1'}} ;$$
$$V_3' - U_1' = F_3, \ \&c.$$

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* Dr. S. P. Thompson's translation of Lummer's Photographic Optics, Appendix II.

Combining these equations we have for any system

$$V'_{2n+1} = F_{2n} + \frac{1}{t_{2n-1}} + \frac{1}{F_{2n-2}} + \cdots\cdots + \frac{1}{t_1'} + \frac{1}{F_0 + U'_{-1}}$$

corresponding to the ordinary continued fraction of Gauss.

We could derive the properties of complete lens systems by the theory of continued fractions as was done by Gauss; but it is much more simple and satisfactory to have recourse to a proof by induction, as the writer believes was first done by Dr. Sumpner, for the ordinary Gauss method.

Let it be granted that for a given system the equations

$$V'_{2n-1} = \frac{A + BU'_{-1}}{C + DU'_{-1}} \quad \frac{1}{m_{2n-1}} = \frac{A' + BU'_{-1}}{U'} \quad \text{and} \quad \begin{vmatrix} A & C \\ B & D \end{vmatrix} = -1$$

hold.

Now let the light pass through a space t'_{2n+1} and another surface of convergence F_{2n}. We shall have

$$V'_{2n-1} = \frac{A + BU'_{-1}}{C + DU'_{-1}};$$

$$U_{2n-1} = \frac{V'_{2n-1}}{V'_{2n-1}t'_{2n-1} + 1};$$

$$V_{2n+1} = U'_{2n-1} + F_{2n}.$$

We then have

$$V_{2n+1} = \frac{A(F_{2n}t'_{2n-1} + 1) + CF_{2n} + \{B(F_{2n}t'_{2n-1} + 1) + DF_{2n}\}V'_{-1}}{At'_{2n-1} + (Bt'_{2n-1} + D)U'_{-1}}$$

$$= \frac{A' + B'U'_{-1}}{C' + D'U'_{-1}},$$

where

$$\begin{vmatrix} A' & C' \\ B' & D' \end{vmatrix} = \begin{vmatrix} A(F_{2n}t'_{2n-1} + 1) + CF_{2n} & At'_{2n-1} + C \\ B(F_{2n}t'_{2n-1} + 1) + DF_{2n} & Bt'_{2n-1} + D \end{vmatrix} = \begin{vmatrix} A & C \\ B & D \end{vmatrix} = -1$$

by the ordinary properties of determinants.

Also

$$\frac{1}{m_{2n+1}} = \frac{1}{m_{2n}} \frac{1}{m_{2n-1}}$$

$$\frac{1}{m_{2n-1}} = \frac{A + BU'_{-1}}{U'_{-1}};$$

$$\frac{1}{m_{2n}} = \frac{V'_{2n+1}}{U'_{2n-1}} = \frac{A' + B'U'_{-1}}{C' + D'U'_{-1}} \frac{At'_{2n-1} + C + (Bt'_{2n-1} + D)U'_{-1}}{A + BU'_{-1}}$$

$$= \frac{A' + B'U'_{-1}}{A + BU'_{-1}}.$$

Hence

$$\frac{1}{m_{2n+1}} = \frac{A' + B'U'_{-1}}{U'_{-1}}.$$

Consequently, since these equations were proved to hold for two surfaces, and have now been shown to hold for an additional surface, they must hold generally.

It may be of interest as a conclusion to this section to exhibit the three methods in a comparative form.

Comparative Table of Formulæ for Refracting Systems.

	GAUSS METHOD.	DEVIATION METHOD.	CURVATURE METHOD.
Surface 0 ...	$\dfrac{1}{v_1'} - \dfrac{1}{u'_{-1}} = F_0$	$\sigma_1' - \sigma'_{-1} = F_0 h_0$	$V_1' - U'_{-1} = F_0$
Interval 1 ...	$u_1' = v_1' + t_1'$	$h_2 - h_0 = t_1' \sigma_1'$	$U_1' = \dfrac{V_1'}{t_1 V_1' + 1}$
Surface 2 ...	$\dfrac{1}{v_3'} - \dfrac{1}{u_1'} = F_2$	$\sigma_3' - \sigma_1' = F_2 h_2$	$V_3' - U_1' = F_2$
Interval 3 ...	$u_3' = v_3' + t_3'$	$h_4 - h_2 = t_3' \sigma_3'$	$U_3' \dfrac{V_3'}{t_3' V_3' + 1}$
&c.	&c.	&c.	&c.
Results referred to first and last surfaces.	$v'_{2n+1} = \dfrac{Cu+D}{Au+B}$ $\dfrac{1}{m} = Au + B$ $AD - BC = -1$	$h_{2n} = Ch_0 + D\sigma'_{-1}$ $\sigma_{2n+1} = Ah_0 + B\sigma'_{-1}$ $\dfrac{1}{m} = \dfrac{Ah_0 + B\sigma'_{-1}}{\sigma'_{-1}}$ $AD - BC = -1$	$V_{2n+1} = \dfrac{A + BU'_{-1}}{C + DU'_{-1}}$ $\dfrac{1}{m} = \dfrac{A + BU'_{-1}}{U'_{-1}}$ $AD - BC = -1$
Results referred to principal planes.	$f' = \dfrac{1}{A}$ $\dfrac{1}{v'} - \dfrac{1}{u'} = \dfrac{1}{A}$ $m = \dfrac{v'}{u'}$	$\sigma'_{2n+1} - \sigma'_{-1} = Ah*$ $m = \dfrac{\sigma'_{-1}}{\sigma'_{2n+1}}$	$F' = A$ $V' - U' = A$ $m = \dfrac{U'}{V'}$

* h is here the height of intersection with the principal planes.

Aberrations.

The subject of aberrations will have to be treated in a separate paper; but to show how the curvature method lends itself to aberration problems, we will here take the case of oblique refraction in thin lenses. Just as we started the ordinary theory of thin lenses by considering the retardation produced by a plate, we may here commence with oblique refraction by a similar plate.

Retardation by Oblique Parallel Plate.—In fig. 9, PABQ is the course of a parallel beam of light, AB being the

Fig. 9.
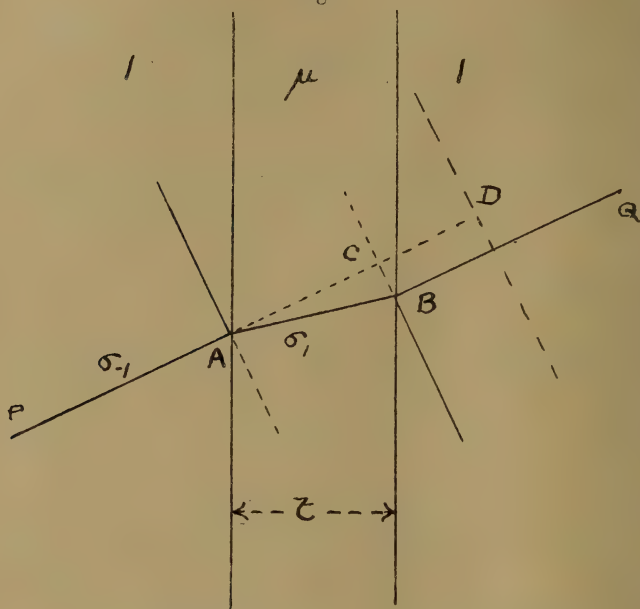
Oblique Refraction by a Parallel Plate.

distance travelled through the plate ; we have D as the position the wave-front would have reached in the absence of the plate such that

$$AD = \mu' \cdot AB$$

and retardation

$$s = \mu \cdot AB - AC.$$

But

$$AB = \frac{t}{\cos \sigma_1} \quad \text{and} \quad AC = AB \cos(\sigma_{-1} - \sigma_1).$$

Hence

$$s = \frac{\mu - \cos(\sigma_{-1} - \sigma_1)}{\cos \sigma_1} t.$$

Putting

$$\sin \sigma_{-1} = \mu \sin \sigma_1$$

and simplifying we have

$$s = \{ \sqrt{\mu^2 - \sin^2 \sigma_{-1}} - \cos \sigma_1 \} t$$

accurately, or to a second approximation

$$s = \left(\mu - 1 \right)\left(1 + \frac{\sigma_{-1}^2}{2\mu} \right) t.$$

Oblique Refraction through Thin Lens.—In the case of a thin lens we may consider the centre of the lens as a parallel plate, and the retardation of the centre of the wave-front is therefore as given above. Hence, if we consider two meridians of the lens, the primary or meridional plane containing the axis of the lens and beam, and the secondary or sagittal plane lying in the axis of the beam perpendicular to the other plane,—

We have in the sagittal plane therefore if ϕ is the obliquity

$$\mathscr{S}_2 = \mathrm{F}\left(1 + \frac{\phi^2}{2\mu}\right).$$

In the meridional plane the retardation of the centre of the beam is the same, but the effective aperture of the lens is reduced in the ratio of $\cos\phi$, and the curvature therefore increased in the ratio $\dfrac{1}{\cos^2\phi} = 1 + \phi^2$ approx. and hence the meridional convergence

$$\mathscr{S}_1 = \mathrm{F}\left(1 + \frac{\phi^2}{2\mu}\right)\left(1 + \phi^2\right).$$

The lens therefore acts as a sphero cylinder of spherical power $\mathrm{F}\left(1 + \dfrac{\phi^2}{2\mu}\right)$ and cylindrical power $\mathrm{F}\phi^2$, or when ϕ is in degrees

$$\text{Spherical power} = \mathrm{F}\left(1 + \frac{\phi^2}{5622\mu}\right).$$

$$\text{Cylindrical ,, } = \frac{\mathrm{F}\phi^2}{2861}.$$

The axis of the cylinder is of course in the sagittal plane.

The wave-front also evidently reaches one edge of the lens before the other, giving rise to asymmetrical refraction or coma, but this will not be further considered here.

Primary and Secondary Focal Lines.—We have evidently

$$\text{Primary convergence} = \mathscr{S}_1 = \mathrm{F}\left\{1 + (2\mu + 1)\frac{\phi^2}{2\mu}\right\}$$

$$\text{Mean ,, } = \mathscr{S}_0 = \mathrm{F}\left\{1 + (\mu + 1)\frac{\phi^2}{2\mu}\right\}$$

$$\text{Secondary ,, } = \mathscr{S}_2 = \mathrm{F}\left\{1 + \frac{\phi^2}{2\mu}\right\}.$$

The mean convergence corresponds to the circle of least

confusion.   Or in focal lengths we have

$$\text{Primary focal length } f_1 = f\left\{1 - (2\mu + 1)\,\frac{\phi^2}{2\mu}\right\}.$$

$$\text{Mean ,, ,, } f_0 = f\left\{1 - (\mu + 1)\,\frac{\phi^2}{2\mu}\right\}$$

$$\text{Secondary ,, ,, } f_2 = f\left\{1 - \frac{\phi^2}{2\mu}\right\}$$

Curvature of Focal Surfaces.—In order that the oblique foci should lie on a plane through the principal focus we have

$$\mathscr{G} = F\cos\phi = F\left(1 - \frac{\phi^2}{2}\right)\text{ approx.}$$

If $\Delta\mathscr{G}$ is the excess of the actual convergence over that necessary to focus on the plane, we have

$$\text{sag } \Delta f = \frac{\Delta\mathscr{G}}{F^2},$$

and lateral distance of focus from principal focus

$$x = \frac{\tan\phi}{F} = \frac{\phi}{F}\text{ approx.}$$

Hence curvature

$$= \frac{2\Delta f}{x^2} = \frac{2\Delta\mathscr{G}}{\phi^2}\text{ approx.}$$

Applying this to the above we have :

Curvature of primary focal surface, $R_1 = \dfrac{3\mu + 1}{\mu}\,F = 3F + \dfrac{F}{\mu}$

,,   of surface of least confusion, $R_0 = \dfrac{2\mu + 1}{\mu}\,F = 2F + \dfrac{F}{\mu}$

,,   of secondary focal surface, $R_2 = \dfrac{\mu + 1}{\mu}\,F = F + \dfrac{F}{\mu}.$

The corresponding radii of curvature can be at once written down and are in agreement with those given by Mr. Dennis Taylor and others [*].   It is also obvious from the method of proof that the curvature of the image surfaces is independent of the position of the object plane.

Focal Surfaces for Thin Lenses in contact.—Since the curvature is given by $\dfrac{\Delta\mathscr{G}}{\phi^2}$, and the convergence of any

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[*] H. Dennis Taylor, " Design of Telescope Object Glasses," Proceedings Royal Astron. Soc.

number of thin lenses in contact is the sum of their individual convergences, we immediately have the interesting result that for combinations of thin lenses in contact, without diaphragm separated from the combination, the curvatures of the fields may be added together just as with convergences. We thus have

$$R_1 = 3\mathbf{F} + \Sigma \frac{F}{\mu}$$

$$R_0 = 2\mathbf{F} + \Sigma \frac{F}{\mu}$$

$$R_2 = \mathbf{F} + \Sigma \frac{F}{\mu}$$

where $\mathbf{F} = \Sigma F.$

For such a combination the mean curvature therefore is $2\mathbf{F} + \Sigma \dfrac{F}{\mu}$, or $2\mathbf{F}$ if the Petzval condition is satisfied, and under these circumstances R_1, R_2, and $R_3 = F$, $2F$, and $3F$ respectively, while the curvature corresponding to the " astigmatic difference " is always $2\mathbf{F}$ independently of the materials of the lenses.

This example will serve to show how easily and directly aberration problems may be solved by physical methods, and the writer proposes to show in another paper that the whole theory of aberrations in refracting systems may be similarly treated with advantage. He ventures to hope, however, that enough has been said in this paper to convince everyone that not only is there no necessity for the abandonment of curvature methods at any stage in optical work, but that there is every advantage in retaining them throughout. It will also be obvious that they lend themselves to combining the study of diffraction and image formation, which should lead to valuable new results.

XLIV. *On* Maxwell's *Stress Theory.* *By* V. BJERKNES [*].

MAXWELL considered his theory of the stress in the dielectric medium as very important. But, on the other hand, he did not regard it as complete. His own words plainly prove both assertions [†] :—

" It must be carefully borne in mind that we have made only one step in the theory of the action of the medium.

We have supposed it to be in a state of stress, but we have not in any way accounted for this stress, or explained how it is maintained. This step, however, seems to me to be an important one, as it explains, by the action of the consecutive parts of the medium, phenomena which were formerly supposed to be only explicable by direct action at a distance.

"I have not been able to make the next step, namely, to account by mechanical considerations for these stresses in the dielectric. I therefore leave the theory at this point, merely stating etc."

After Maxwell nobody has yet been able to make this *next step*. Some formal improvements have been made, but from the point of view of principle the stress theory has not advanced beyond the point where Maxwell left it. But still we have at least now the advantage of possessing a remarkably complete mechanical image of the electrostatic and the magnetic phenomena, worked out by the late Professor C. A. Bjerknes of Christiania[*].

It may therefore be worth while to see how Maxwell's stress theory works in the case of these *electroidic* phenomena, where we are able to account for everything by mechanical considerations.

Let us imagine a man, who sees the experiment with the attractions and repulsions of pulsating bodies, but who is not able to see the small pulsations, nor the liquid which propagates the action from the one body to the other. This observer will then be in exactly the same limited condition as to knowledge of the hydrodynamic phenomena before him as the electrician is relatively to the electrical phenomena. The attractions and repulsions of these bodies will make upon him exactly the same impression as the attraction and repulsion of electrified bodies or of magnetic poles. His measurements will bring him to the result, that they follow the law of Coulomb, only with the sign of the force reversed.

Suppose now this observer to be a Maxwell. He will then suspect, that these actions are not real actions at a distance. He will suspect the presence of a medium, and try to explain the apparent actions at a distance by some stress in this medium. And up to this point he will be perfectly right in his conclusions.

But this will no longer be the case if he follows Maxwell further, in attempting to give an explicit solution of the stress problem. He will then consequently arrive at the expression of the Maxwell stress, only with the sign reversed for each of

the six stress components. These stress formulæ will enable
him to calculate the forces acting on the pulsating bodies.
But if he draws the further conclusion from this result, that
this stress really exists and really produces the forces, he will
be wrong. For the cause of the attractions and repulsions
of the pulsating bodies is not the Maxwell stress, but a stress
of much simpler nature, namely, the isotropic stress, or the
pressure, in the liquid. Still, in the limited state of his
knowledge, he will have exactly the same right as the elec-
trician to believe that the Maxwell stress is the real cause of
the phenomena.

It is easy to see where the error comes in when Maxwell's
developments are applied to these electroidic phenomena of
hydrodynamics. Maxwell considers his problem as one of
pure statics. The stress has therefore only to produce the
required mechanical forces. In all points of space where no
such force is required, the stress is self-equilibrating. The
hydrodynamic phenomena on the other hand are not statical,
but kinetic, even if the motions be too small to be observed.
The stress or pressure is therefore nowhere self-equilibrating.
It has a double duty, first to maintain the hidden motions,
which constitute the field, and only secondly to produce the
required forces that give rise to the visible motions.

The question whether the Maxwell stress may or may not
represent a real stress in the dielectric will therefore be closely
related to the question, whether the electric or magnetic
phenomena are ultimately of statical or kinetic nature. In
the latter case, if they depend upon hidden motions, the
stress will have to maintain these motions, and it cannot be
self-equilibrating like the Maxwell stress. Now Maxwell
himself considers the magnetic phenomena as kinetic, as his
application of the Lagrange equations to electromagnetism
plainly shows. And if this view be right, his explanation at
least of the magnetic actions at a distance will not be legitimate.

In the meanwhile it will therefore be safest to consider
the Maxwell stresses as only *fictitious* stresses that might have
produced the required forces, and not as the real stresses
that do produce them. Other authors have also termed them
fictitious stresses, especially Lorentz *, who also considers the
stress formulæ only as useful analytical formulæ, but not
as representing any physical reality. The reason, however,
why he has come to this opinion is quite different from the
reason brought forward here. His view is that the stress-

* H. A. Lorentz, "Versuch eines Theorie der elektrischen und optischen
Erscheinungen in bewegten Körpern," p. 28. "Elektronentheorie,"
Encyklopädie der mathematischen Wissenschaften, vol. v. 2. p. 163.

problem in itself has no physical meaning. For, according to the doctrine of the immobility of the æther, it is an absurdity to speak of forces acting upon or stresses existing in the æther.

Let us now consider this theory in its application to the electroidic phenomena of hydrodynamics. Just as well as an observer of these phenomena might arrive at Maxwell's views, he might, with his limited knowledge of the things before him, arrive at the views of Lorentz. But, as we see at once, they do not hold. For the hydrodynamic actions at a distance do depend upon a stress, although a much simpler one than that found by Maxwell.

The question to what extent it may be allowable to draw, from the electroidic phenomena of hydrodynamics, positive conclusions as to real electric phenomena may be open to discussion. The intimate nature of the analogy makes it very probable that useful suggestions may be obtained from this analogy. But of course the method must be tried with the greatest caution. The results produced here may perhaps encourage electricians to take up the stress-problem upon a wider base than Maxwell himself was able to do at his first attack upon this, *the* fundamental problem of the theory of electricity. The solution may be simpler than we expect. For the hydrodynamic results plainly show that even the simple isotropic stress may produce actions of the kind, which Maxwell supposed explicable only as the effect of a stress of the anisotropic type.

XLV. *On the Pressure of Gases and the Equation of Virial.* By Lord RAYLEIGH, *O.M., F.R.S.*[*]

IF m be the mass of a particle, V its velocity, p the pressure and v the volume of the body composed of the particles, the virial equation is

$$\tfrac{1}{2}\Sigma m V^2 = \tfrac{3}{2}pv + \tfrac{1}{2}\Sigma \rho \phi(\rho), \quad . \quad . \quad . \quad . \quad (1)$$

where further ρ denotes the distance between two particles at the moment under consideration, and $\phi(\rho)$ the mutual force, assumed to depend upon ρ only. If the mutual forces can be neglected, either because they are non-existent or for some other reason, (1) coincides with Boyle's law, since the kinetic energy is supposed to represent temperature (T).

According to some experimenters, among whom may be especially mentioned Ramsay and Young, the relation between

pressure and temperature at constant volume is in fact linear, or

$$p = T\psi(v) + \chi(v) ; \quad \cdots \quad (2)$$

and it is of interest to inquire whether such a form is to be expected on theoretical grounds, when $\phi(\rho)$ can no longer be neglected. It has indeed been maintained[*] that (2) is a rigorous consequence of the general laws of thermodynamics and of the hypothesis that the forces between molecules are functions of the distance only. The argument proceeded upon the assumption that the distances of the particles, and therefore the mutual forces between them, remain constant when the temperature changes, provided only that the volume of the body is maintained unaltered. According to this the virial term in (1) is a function of volume only, so that (1) reduces to (2), with $\psi(v)$ proportional to v^{-1}. But, as Boltzmann pointed out, the assumption is unfounded, and in fact inconsistent with the fundamental principles of the molecular theory. The molecules are not at rest but in motion ; and when the temperature varies there is nothing to hinder the virial from varying with it.

The readiest proof of this assertion is by reference to the case where the molecules are treated as " hard elastic spheres," that is where the force is zero so long as ρ exceeds a certain value (the diameter of the spheres) and then becomes infinite. From the researches of Van der Waals, Lorentz, and Tait it is known that in that case

$$\tfrac{1}{2}\Sigma\rho\,\phi(\rho) = -\tfrac{1}{2}\Sigma m V^2 . \frac{b}{v}. \quad \cdots \quad (3)$$

where b, denoting four times the total volume of the spheres, is supposed to be small in relation to v. So far from the virial being necessarily independent of temperature, it is here directly proportional to temperature. The introduction of the special value (3) into (1) gives the well-known form

$$p(v-b) = \tfrac{1}{3}\Sigma m V^2 = RT, \quad \cdots \quad (4)$$

in which b is still regarded as small in comparison with v. It is worthy of note that this particular case, although of course sufficient to upset the general argument that the virial is independent of temperature, nevertheless itself conforms to (2), proportionality to T being for this purpose as good as independence of T.

Not only is the linear relation maintained in spite of the forces of collision of elastic spheres when no other forces

[*] M. Levy, *C. R.* t. lxxxvii. pp. 449, 488, 554, 649, 676, 826 (1878).

operate, but it remains undisturbed even when we introduce such forces, provided that they be of the character considered in the theory of capillarity, that is extending to a range which is a large multiple of molecular distances and not increasing so fast with diminishing distance as to make the total effect sensibly dependent upon the positions occupied by neighbours. Under these restrictions symmetry ensures that the resultant force upon a sphere, situated in the interior and not undergoing collision, is zero; and the whole effect of such forces is represented (Young, Laplace, Van der Waals) by an addition to the pressure of a quantity independent of the temperature and inversely proportional to the square of the volume. In Van der Waals' well-known form

$$\left(p + \frac{a}{v^2}\right)(v - b) = \mathrm{RT}, \quad . \quad . \quad . \quad . \quad (5)$$

the relation between p and T is still linear. Even if the particles depart from the spherical form, the virial of collisional and cohesive forces remains a linear function of the temperature [*].

The forces above considered are partly repulsive and partly attractive. Repulsion at a certain degree of proximity seems to be demanded in order to preserve the individuality of molecules and to prevent infinite condensation. It will be remembered that Maxwell proposed a repulsion inversely as the *fifth* power of the distance, partly as the consequence of some faulty experiments upon the relation of viscosity to temperature and partly no doubt on account of a special facility of calculation upon the basis of this law. So far as viscosity (η) is concerned, its relation to temperature (T) when the force of repulsion varies as ρ^{-n} is readily obtained by the method of dimensions[†]. It appears that

$$\eta \ \propto \ \mathrm{T}^{\frac{n+3}{2n-2}}. \quad . \quad . \quad . \quad . \quad (6)$$

The case of sudden collisions may be represented by taking $n = \infty$, so that

$$\eta \ \propto \ \mathrm{T}^{\frac{1}{2}}: \quad . \quad . \quad . \quad . \quad . \quad (7)$$

while if $n = 5$

$$\eta \ \propto \ \mathrm{T}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

According to experiments on the more permanent gases n

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[*] "On the Virial of a System of Hard Colliding Bodies," 'Nature,' xlv. pp. 80–82 (1891); Scientific Papers, iii. p. 469.

[†] Proceedings Royal Society, lxvi. p. 68 (1900); Scientific Papers, iv. p. 453.

would vary from ·68 for hydrogen to ·81 for argon ; but Sutherland's law*

$$\eta \; \alpha \; \frac{T^{\frac{1}{2}}}{1 + C/T} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9)$$

probably represents the facts better than (6), whatever value may be assigned to n. According to the theory of corresponding states, C should be proportional to the critical temperature when we pass from one gas to another.

A similar application of the method of dimensions will give interesting information respecting the virial, when the force of repulsion is

$$\phi(\rho) = -\mu\rho^{-n}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (10)$$

The virial is a definite function of N the number of molecules, m the mass of each molecule, V the velocity of mean square on which the temperature depends, μ the force at unit distance, and v the volume of the containing vessel. Of these quantities the virial is of the dimensions of energy, N has none, m is a mass simply, V is a velocity, v a volume, while μ has the dimensions

$$\text{mass} \times (\text{length})^{n+1} \times (\text{time})^{-2}.$$

Hence if we suppose that the virial varies as v^{-s}, we find that it must be proportional to

$$(mV^2)^{\frac{n-3s-1}{n-1}} \cdot \mu^{\frac{3s}{n-1}} \cdot v^{-s}; \quad \cdot \quad \cdot \quad \cdot \quad (11)$$

or since mV^2 represents temperature,

$$T^{\frac{n-3s-1}{n-1}} \cdot \mu^{\frac{3s}{n-1}} \cdot v^{-s}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (12)$$

For example, if $s = 0$,

$$\Sigma\rho\phi(\rho) \; \alpha \; T, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (13)$$

whatever n may be. Hence a term in the virial equation independent of volume must be proportional to temperature, as in (1). Again, if $s = 1$,

$$\Sigma\rho\phi(\rho) \; \alpha \; v^{-1} \cdot T^{\frac{n-4}{n-1}}. \quad \cdot \quad \cdot \quad \cdot \quad (14)$$

Of this we have already had examples, both the virial terms in Van der Waals' equation being proportional to v^{-1}. The first, representing the virial of collisional forces, corresponds in (14) to $n = \infty$, giving proportionality to T. The second is independent of T and can be reconciled with (14) only by

* Phil. Mag. vol. **xxxvi.** p. 513 (1893).

supposing $n=4$.   It might seem that in a rare gas, whenever the virial depends sensibly upon what occurs during the encounters of simple pairs of molecules, there must be proportionality to v^{-1}, so that (14) would apply.   If, as Maxwell supposed, $n=5$,

$$\Sigma\rho\,\phi(\rho)\ \propto\ v^{-1}\,.\,T^{\frac{1}{4}},\ .\ \ .\ \ .\ \ .\ \ (15)$$

in agreement with a result obtained by Boltzmann for this case.   If we retain $n=5$, but leave the relation to v open, we get from (12)

$$\Sigma\rho\,\phi(\rho)\ \propto\ v^{-s}\,.\,T^{1-\frac{3}{4}s}.\ \ .\ \ .\ \ .\ \ (16)$$

If we now discard the supposition that the dependence upon v follows the law of v^{-s}, we may interpret (16) to mean that considered as a function of v and T, the virial is limited to the form

$$\Sigma\rho\,\phi(\rho)=T\,.\,F(vT^{\frac{3}{4}}),\ .\ \ .\ \ .\ \ (17)$$

F denoting an arbitrary function of the *single* variable $vT^{\frac{3}{4}}$.

   And more generally, whatever n may be, we find from (12) that the virial is limited to the form

$$\Sigma\rho\,\phi(\rho)=T\,.\,F\!\left(\frac{v\,T^{\frac{3}{n-1}}}{\mu^{\frac{3}{n-1}}}\right).\ \ .\ \ .\ \ .\ \ (18)$$

   A further generalization may be made by discarding altogether the supposition that $\phi(\rho)$ is represented by any power of ρ.   In this case it is convenient to write

$$\phi(\rho)=-\mu'f(\rho/a),\ .\ \ .\ \ .\ \ .\ \ (19)$$

where a is a linear quantity.   Here f itself may be supposed to be of no dimensions, while μ' has the dimensions of a force.   The virial is a function of μ', a, m, V, v; and since its dimensions are those of energy, *i. e.* of mV^2 or T, we may write

$$\Sigma\rho\,\phi(\rho)=T\,.\,F(\mu',\,a,\,m,\,V,\,v),$$

where F is of no dimensions.   It is easy to see that μ', m, and V^2 can occur only in the combination μ'/mV^2 or μ'/T.   To make this of no dimensions, we introduce the factor a.   Thus F becomes a function of a, v, and $\mu'a/T$, in which again v can occur only in the form a^3/v.   Accordingly

$$\Sigma\rho\,\phi(\rho)=T\,.\,F\left(\frac{a^3}{v},\,\frac{\mu'a}{T}\right),\ \ .\ \ .\ \ .\ \ (20)$$

F being in general an arbitrary function of *two* variables.

From (20) we may fall back on (18) by the consideration that in accordance with (10) μ' and a can occur only in the combination $\mu' a^n$.

It may be well to remark that the method of dimensions does not tell us whether or no an available solution can be deduced from particular assumptions. What it teaches us is the form which an available solution must assume. For example, equation (14) gives the form of the term in the virial proportional to r^{-1}, under the law of force (10); and nothing has been said as to any restriction upon the value of n. But it is easy to see that n must in fact be greater than 4. Otherwise the integral representing the virial relating to a given particle would not be convergent. We have to consider

$$\int \rho \phi(\rho) \, \rho^2 \, d\rho$$

with infinity for the upper limit, and this diverges unless n exceed 4.

It is not to be expected that any law included under (10) could represent with completeness the mutual action of the particles of a gas. Under it no provision can be made for repulsion at small distances and attraction at greater ones. And when $n > 4$, the aggregate virial depends too much upon the encounters which take place at exceedingly small distances.

If, as for both the virial terms in Van der Waals' formula, there be proportionality to v^{-1}, (20) becomes

$$\Sigma \rho \phi(\rho) = \frac{a^3 \mathrm{T}}{r} \mathrm{F}\left(\frac{\mu' a}{\mathrm{T}}\right), \quad . \quad . \quad . \quad . \quad (21)$$

or, if we prefer it,

$$\Sigma \rho \phi(\rho) = \frac{\mu' a^4}{r} \mathrm{F}\left(\frac{\mu' a}{\mathrm{T}}\right), \quad . \quad . \quad . \quad . \quad (22)$$

F in both cases denoting an arbitrary function. According to Van der Waals F in (21) is a linear function, the constant part giving the collisional virial and the second term the cohesional virial which is independent of T. Except for one consideration to be mentioned presently, there would appear to be good reason for supposing the virial of a rare gas to be proportional to r^{-1}; but on the other hand it is doubtful whether the cohesional forces are altogether of the kind supposed by Laplace and Van der Waals. We should expect the cohesional virial to be more directly influenced by the approaches of molecules during an encounter; and on the experimental side D. Berthelot has shown cause for preferring to that of Van der Waals the Rankine and Clausius

form, in which a factor T is introduced in the denominator. The most natural extension of the formula would be by substituting a quadratic for a linear form of F in (21). We should then write

$$\tfrac{1}{2}\Sigma\,\rho\,\phi\,(\rho) = \frac{3a^3}{2v}\Big(-\text{AT} + \text{B}\mu'a + \text{C}\frac{\mu'^2a^2}{\text{T}}\Big), \quad . \quad (23)$$

A, B, C being arbitrary constants ; and the pressure equation, when written after Van der Waals' manner with neglect of v^{-2}, becomes

$$\Big\{p + \frac{a^3}{v^2}\Big(\text{B}\mu'a + \frac{\text{C}\mu'^2a^2}{\text{T}}\Big)\Big\}\Big\{v - \frac{a^3\text{A}}{\text{R}}\Big\} = \text{RT}. \quad . \quad (24)$$

As has already been said, Van der Waals' form corresponds to C=0. On the other hand, the Rankine and Clausius form requires that B=0, while C remains finite. It will be evident that the two alternatives differ fundamentally. According to the latter the cohesional terms tend to vanish when T is sufficiently increased.

If the cohesional terms are to vanish when T is infinite, the forces concerned must be of an entirely different character from that contemplated in Laplace's and Van der Waals' theory. It has been suggested by Sutherland[*] that the forces may be of electric origin and in themselves (except during actual collision) as much repulsive as attractive. This is not inconsistent with the preponderance of attraction in the final result. "There is this fundamental distinction in the effects of attractive and repulsive forces whose strength decreases with increasing distance, that the attractive forces by their own operation tend to increase themselves, while the repulsive forces tend to decrease themselves." The forces contemplated by Sutherland are such as are due to electric or magnetic doublets, but a rather simpler illustration may be arrived at by retaining the single character of the centres of force, and supposing them to be as much positive as negative, under the usual electrical law that similars repel while opposites attract one another. When T is infinite, so that the paths are not influenced by the forces, the cohesional virial will disappear, but it may become finite as the temperature falls and room is given for the attractive forces to assert their advantage. There is nothing in the argument upon which (21) was founded which is interfered with by the occurrence of the two kinds of particles, and it would

[*] Phil. Mag. vol. iv. p. 625 (1902).

seem that F must then become an *even* function of μ', so that in (23) B = 0.

As stated. the above argument is probably not quite legitimate. inasmuch as according to (19) a reversal of μ' would imply a reversal of the collisional forces as well as of those which operate at greater distances. The introduction of the two sorts of particles is not supposed to alter the repulsive forces called into play during actual collision. I believe, however, that the instantaneous collisional forces may be omitted from (19). The effect of the collisions may be defined without reference to any datum having dimensions other than a, representing the radius of a sphere. The collisions being thus, as it were, already provided for, the argument remains that the virial must be a definite function of N. m. V, μ', a, v, of which N need not be regarded, the force (outside actual collision) being given by (19). Equation (21) then follows as before with its approximate form (23). If we now suppose that the particles are repellent as much as attractive, (19) may be written

$$\phi(\rho) = \pm \mu' f(\rho/a) ; \quad . \quad . \quad . \quad . \quad . \quad (25)$$

and, since odd powers of μ' are now excluded, B = 0 in (23), (24).

We have thus discovered a possible theoretical foundation for the empirical conclusion that T should be introduced into the denominator of the cohesional virial, and it would seem to follow conversely that, if the empirical conclusion is correct, the forces must be intrinsically as much repellent as attractive. This argument may be regarded as a strong confirmation of Sutherland's idea, though a question remains as to how the attraction asserts its superiority over repulsion.

In the above argument the particles are regarded as simple centres of force, half of them being " positive " and half " negative." The advantage is that the form may still be treated as spherical, so that the collisions may be assimilated to those of " elastic spheres." But a polar constitution, such that the positive and negative elements are combined in every particle, is certainly more probable. This will introduce, as another linear datum, the distance between the poles, and the collisions will admit of greater variety. Moreover, there is now kinetic energy of rotation as well as of translation. However, since the kinetic energies are proportional, the argument remains unaffected, so far as it relates to the dependence of the virial of a given gas upon volume and temperature, and the Rankine-Clausius form (24) with B = 0 still obtains.

As to the preponderance of attractive over repulsive virial, I think that the conclusion is correct, although Sutherland's argument, quoted above, omits reference to the essential consideration of the *time* for which any particular value of the virial prevails. If we fix our attention upon a pair of particles, acting as simple centres of force, which encounter one another, the corresponding virial varies from moment to moment, but the mean contribution to the total may be represented by

$$\int \phi(\rho) \rho \, dt,$$

the integration being taken over the whole range for which $\rho \phi(\rho)$ is sensible. Since only relative motion is in question, the centre of gravity of the two particles may be supposed to be at rest and the problem becomes one of "central forces." In the usual notation we have

$$\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = \mathrm{P}, \quad r^2 \frac{d\theta}{dt} = h, \quad . \quad . \quad . \quad (26)$$

so that

$$\int \mathrm{P}.r.dt = \left[r \frac{dr}{dt} \right] - \int \left(\frac{dr}{dt} \right)^2 dt - \int r^2 \left(\frac{d\theta}{dt} \right)^2 dt$$

$$= \left[r \frac{dr}{dt} \right] - \int v^2 dt = \left[r \frac{dr}{dt} \right] - \int v \, ds, \; . \; (27)$$

v denoting the resultant velocity. At the upper limit dr/dt is equal to the velocity at ∞, say V, and at the lower limit $dr/dt = -\mathrm{V}$. Hence

$$\int \mathrm{P}.r.dt = 2r\mathrm{V} - \int v \, ds, \quad . \quad . \quad . \quad . \quad (28)$$

so that the mean virial is closely connected with the "action" in the orbit.

For a simple illustration it will be more convenient to make θ the independent variable. Thus by (26)

$$\int \mathrm{P}. \, r.dt = \frac{1}{h} \int \mathrm{P}.r^3.d\theta. \quad . \quad . \quad . \quad (29)$$

Suppose for example that

$$\mathrm{P} = \mu r^{-3} . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Then

$$\int \mathrm{P}.r.dt = \frac{\mu}{h} \int d\theta = \frac{\mu}{h} \theta, \quad . \quad . \quad . \quad (31)$$

where θ represents twice the vectorial angle between the initial asymptote and the apse. If h be given, a comparison between repellent and attractive forces (μ given in magnitude but variable in sign) shows that (31) is greater in the case

of attraction (fig. 1), so that if attractive and repellent forces occur indifferently the average effect corresponds to attraction.

Fig. 1.
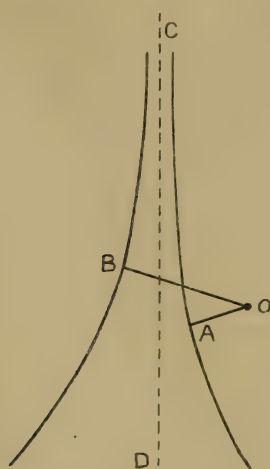
O, centre of force ; C D, asymptote ; A, B, apses.

In the case of the particular law (30) we can carry out the calculation.   If, as usual, $u = r^{-1}$, the equation of the orbit is

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2} u, \quad . \quad . \quad . \quad . \quad . \quad (32)$$

μ being positive in the case of attraction; whence, if μ be small,
$$u = U \sin \sqrt{(1 - \mu h^{-2})}\theta. \quad . \quad . \quad . \quad (33)$$

In (33) $u = 0$, or $r = \infty$, when $\theta = 0$ and when

$$\theta = \pi \div \sqrt{(1 - \mu h^{-2})} ; \quad . \quad . \quad . \quad (34)$$

so that from (31)

$$\int \mathrm{P}.r.dt = \frac{\mu \pi}{\sqrt{(h^2 - \mu)}} \quad . \quad . \quad . \quad . \quad (35)$$

The solutions (33), (35) hold if μ be numerically less than h^2, and (35) shows that when μ changes sign the virial of attraction preponderates.   This conclusion is accentuated by the consideration of what occurs if μ exceed h^2 numerically. Equations (33), (35) still hold if μ be negative, *i. e.* if the force be repulsive.   But when μ is positive, the form changes.   Thus if $\mu = h^2$, we have

$$u = U\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and neither θ in (31) nor the virial has a finite value. The like remains true when $\mu > h^2$.

In the above example $\mathrm{P}r^3$ remains constant, and the preponderance of attraction over repulsion depends upon the greater vectorial angle in the former case. If $\mathrm{P}r^3$, instead of remaining constant, continually increases with diminishing r, the preponderance of attraction follows *a fortiori*.

A particular case of (32) which arises when $\mu = h^2$ should be singled out for especial notice, *i. e.* the case of circular motion for which $u = \text{constant}$. The attracting particles then revolve round one another in perpetuity, and the virial is infinite in comparison with that of an ordinary encounter. It is this possible occurrence of re-entrant orbits which causes hesitation as to the accuracy with which we may assume the virial of a rare gas to be inversely as the volume. It seems to be generally supposed (see, for example, Meyer's 'Kinetic Theory of Gases,' § 4) that if a gas be rare enough no appreciable pairing can occur. But the question is not as to the frequency with which new pairs may form, but as to the relative number of them in existence at any time. It is easy to recognize that the coupling or the severance of a pair of particles cannot occur of itself, but requires always the cooperation of a third particle. If the gas is very rare, no doubt there are few opportunities for the formation of fresh pairs, but for the same reason those already formed have a higher degree of permanence. On the whole it would appear that the number of pairs in existence at any moment is independent of the volume v of a rare gas, and the same would be true of the corresponding virial. At this rate we should have terms in the virial which by (20) come under the form

$$\mathrm{T} \cdot \mathrm{F}\left(\frac{\mu' a}{\mathrm{T}}\right). \quad . \quad . \quad . \quad . \quad . \quad (37)$$

It will be remarked that if these terms in the virial, independent of v, are sensible, the density of the gas will depart from Avogadro's rule, however greatly it may be rarefied. In the case of elastic spheres, which come into collision when their centres approach to a certain distance, there is naturally a limit to the magnitude of the attraction, and then pairing becomes impossible if the velocity be sufficiently great. Any departure from Avogadro's rule at high rarefactions would thus tend to disappear as the temperature rises.

The behaviour of mere centres of force, which may approach one another without limit, appears to follow a different course. Taking for example the power law of (10),

we see from (18) that for any part of the virial which is independent of v, the function F must be constant, so that the virial is proportional to T and independent of μ.

To return to the question with which we started, there seems good reason to doubt that the relation of pressure to temperature with volume constant is accurately linear, even at high rarefactions. On the other hand, it is clear that this relation is approximately satisfied; and the natural course would be to take it as a foundation, determining the functions χ and ψ in (2), as well as the function of v and T jointly which may be required in supplement. As regards the latter part of the question, a differential arrangement in which two gases, say CO_2 and H_2, are balanced against one another at the same temperature, would appear to offer advantages. This is shown diagrammatically in fig. 2, where the two gas-reservoirs are connected

Fig. 2.
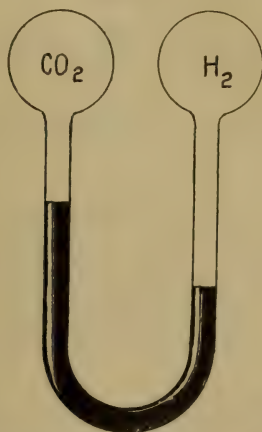
by a U-tube containing mercury. According to Boyle's law, even as modified by the introduction of a co-volume, the mercury may stand in the U-tube at fixed marks at the *same level*, in spite of variations of temperature affecting both bulbs alike. And under the more general law (2) the same fixity of the mercury thread can be attained, though now with the extremities at different levels. With such an arrangement the departure from (2) becomes a matter of direct observation, and so long as *uniformity* of temperature is secured, a precise measurement of it, or of the *total* pressure, is of secondary importance. Useful results would probably require a total pressure of four or five atmospheres.

XLVI. *A Reciprocal Relation in Diffraction.*
By A. A. Michelson [*].

SUPPOSE the vibration at the surface A of a sphere whose centre is at O to be a known function of x and y, the origin being a point P on the sphere,

$$V = \phi \cos nt + \psi \sin nt. \quad . \quad . \quad . \quad . \quad (1)$$

Then the vibration [†] on a sphere B passing through O whose centre is at P will be

$$W = -\frac{1}{\lambda f}\iint \phi\, dx\, dy \sin n(t-\tau) - \frac{1}{\lambda f}\iint \psi\, dx\, dy \cos n(t-\tau) \quad (2)$$

Putting

$$\tau = \frac{f}{a}\left(1 - \frac{x\xi + y\eta}{f^2}\right), \quad t - \frac{f}{a} = t_1, \quad \frac{n\xi}{fa} = u, \quad \frac{n\eta}{fa} = v,$$

$$
\left.
\begin{aligned}
W = &-\frac{1}{\lambda f}\left[\iint \phi\, dx\, dy \cos(ux+vy) - \iint \psi\, dx\, dy \sin(ux+vy)\right]\sin nt \\
&-\frac{1}{\lambda f}\left[\iint \phi\, dx\, dy \sin(ux+vy) + \iint \psi\, dx\, dy \cos(ux+vy)\right]\cos nt
\end{aligned}
\right\} \quad (3)
$$

or

$$W = P \sin nt_1 + Q \cos nt_1. \quad . \quad . \quad . \quad (4)$$

If now a spherical mirror be made to coincide with the sphere B an image of the source will be formed at A.

This image may also be considered as the resultant of the vibrations at B. Hence, if we designate by DW the

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[*] Communicated by the Author.

[†] Scientific Papers of Lord Rayleigh, vol. iii. p. 80. The results given by Lord Rayleigh for the *intensity* of the diffraction figure in the focal plane do not apply to the phase of the vibration. This restriction is removed if the surface considered be the sphere B; for the distance between two points on the spheres is

$$\rho^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2,$$

or if $f^2 = x^2 + y^2 + z^2 = \xi^2 + \eta^2 + (\zeta - f)^2$

$$\rho^2 = f^2 - 2x\xi - 2y\eta - 2z\zeta + 2f\zeta.$$

If ξ and η are small, ζ will be of the second order, and so is $f-z$, so that $(z-f)\zeta$ is of the fourth order and may be neglected. Hence

$$\rho^2 = f^2 - 2x\xi - 2y\eta$$

or approximately $\qquad \rho = f - \dfrac{x\xi + y\eta}{f}.$

operation by which W is obtained from V,

$$\left. \begin{array}{c} W = DV \\ V = DW \end{array} \right\} \quad \cdots \quad \cdots \quad \cdots \quad (5)$$

or $\qquad V = DDV. \quad \cdots \quad \cdots \quad \cdots \quad (6)$

These equations hold for light of any colour, and—with corresponding extension of the definition of D—to any combination of colours.

The analogy with Fourier's formula is apparent. In fact, the second of equations (5) is, putting $n/fa = p$,

$$V = -\frac{1}{\lambda f}\left[\iint Q\,d\xi\,d\eta \cos p(x_1\xi + y_1\eta) - \iint P\,d\xi\,d\eta \sin p(x_1\xi + y_1\eta)\right]\cos nt_2$$
$$-\frac{1}{\lambda f}\left[\iint Q\,d\xi\,d\eta \sin p(x_1\xi + y_1\eta) + \iint P\,d\xi\,d\eta \cos p(x_1\xi + y_1\eta)\right]\sin nt_2$$

Substituting the values of P and Q from (4),

$$\begin{aligned}
V &= \frac{1}{\lambda^2 f^2}\iiiint dx\,dy\,d\xi\,d\eta\,\psi(x,y)\sin\left[(x_1-x)p\xi + (y_1-y)p\eta\right] \\
&+ \frac{1}{\lambda^2 f^2}\iiiint dx\,dy\,d\xi\,d\eta\,\phi(x,y)\cos\left[(x_1-x)p\xi + (y_1-y)p\eta\right]
\end{aligned} \right\}\cos nt_2$$

$$\begin{aligned}
&+ \frac{1}{\lambda^2 f^2}\iiiint dx\,dy\,d\xi\,d\eta\,\phi(x,y)\sin\left[(x_1-x)p\xi + (y_1-y)p\eta\right] \\
&+ \frac{1}{\lambda^2 f^2}\iiiint dx\,dy\,d\xi\,d\eta\,\psi(x,y)\cos\left[(x_1-x)p\xi + (y_1-y)p\eta\right]
\end{aligned} \right\}\sin nt_2$$

The first and third integrals are identically zero, so that putting

$$u = \frac{n\xi}{fa} \quad \text{and} \quad v = \frac{n\eta}{fa}$$

and disregarding the phase difference between t and t_2, we have by (1)

$$\frac{4\pi^2}{f^2}\phi(x_1, y_1) = \iiiint du\,dv\,dx\,dy\,\phi(x,y)\cos\left[(x_1-x)u + (y_1-y)v\right]$$

This, disregarding the intensity factor f^2, is the Fourier formula extended to two dimensions.

Formulæ (5) express the fact that if

W is the diffraction image of V, then
V is the diffraction image of W.

In applying the formulæ it must be remembered that V and W represent the vibration—not merely the intensity.

2 L 2

XLVII. *Escape of Gases from Atmospheres.*

To the Editors of the Philosophical Magazine.

Gentlemen,

A LETTER under the above heading, by Dr. G. Johnstone Stoney, in reply to my note in ' Nature ' of the 24th of March, published in the June number of the Philosophical Magazine, has just come to my notice, and I am sorry to observe that it places some of the results of my former researches on the above subject in a false light. I therefore beg the hospitality of the Philosophical Magazine for a very brief note.

In the investigations on the escape of gases from atmospheres by Dr. Stoney, Dr. Bryan, and the writer, two widely different conclusions have been reached by two distinctly different methods. Dr. Stoney takes the position that helium is escaping into the atmosphere through springs and other natural sources, at a greater rate than it is by any means whatever being returned to the earth ; and, since it is chemically a very inert gas, it is not uniting with any atmospheric gas, and hence the only way to account for the removal of so large a portion of the helium from the atmosphere is by assuming that it escapes into outer space[*]. The very great importance of this assumption, with its many and very interesting applications to the kinetics of atmospheres, led me to attempt to verify this assumption by applying the laws of the kinetic theory to the escape of molecules from the atmosphere. After giving the most liberal interpretation to the laws, and after taking into consideration everything that could consistently be employed, in the light of our present knowledge of the kinetic theory, that would assist the escape of the molecules in the highly attenuated atmosphere, it was found that only 13×10^8 c.c. of hydrogen and only 10×10^{-11} c.c. of helium would escape in a year.

The real point at issue between Dr. Stoney and the writer is the validity of the method of attacking the problem. Dr. Stoney's first memoirs on the subject appeared before helium had been shown to be a measurable constituent of the atmosphere ; and the fact that helium had been poured into the atmosphere for ages and was not found in the atmosphere, formed the basis of Dr. Stoney's theory that helium was escaping from the outer limits of the atmosphere.

[*] " On Atmospheres upon Planets and Satellites," Trans. of the Royal Dublin Society, vol. vi. p. 305 (Oct. 1897) ; Astrophysical Journal, vol. vii. p. 25 (January 1898).

The writer[*] attacked the problem from the standpoint of Maxwell's equation for the distribution of velocities in a gas, and showed that, if the law of the distribution of velocities holds for a gas at normal temperature and pressure, helium would not escape from an atmosphere of helium whose outer limits were at 5° C. and in other conditions similar to the earth's atmosphere at normal pressure, with nothing but gravitation to prevent the molecules from flying off into space. Dr. Stoney criticises the use of Maxwell's formula for the distribution of velocities, as applied to the highly attenuated gas in the limits of the atmosphere[†]. He maintains that there are causes, other than those comprehended within the Maxwell or the Boltzmann-Maxwell law, which would cause the molecules in the outer atmosphere to assume velocities much greater than under normal conditions. So that if π is Maxwell's velocity function, and δ a function made up of all other factors that would cause the molecules to attain to a critical velocity : then the total number of molecules that would attain to the critical velocity would be

$$n(\pi + \delta).$$

Allowing the validity of this equation, I have shown[‡], that even though δ may attain a value of 100π, the permanency of an atmosphere of helium on the earth would not be materially affected.

One point that I wish especially to call attention to in Dr. Stoney's letter, which puts my data in a false light, is on page 691, where he says : " What convinced me several decades ago that the conclusion at which I arrived, and at which Mr. Cook has arrived, is false, is that it represents the moon as incompetent to get rid of the atmosphere which it originally shared with the earth, and of the gases which it has since evolved in abundance from its own interior." In answer to this I only need to cite the results of my investigation, which explain fully the escape of the atmosphere from the moon. Quoting from my first paper (Astrophysical Journal, Jan. 1900, p. 41) : " This table (a table showing the temperature which the moon, the earth, and three of the planets, would necessarily attain in order that an atmosphere of air should escape) shows that an atmosphere of hydrogen would escape from the moon with its outer layer at −256° C., an atmosphere of air at −10° C., and an atmosphere of carbon dioxide at 270° C.''

Whether, as Dr. Stoney believes, helium is really escaping

[*] Astrophysical Journal, vol. xi. p. 36 (January 1900).
[†] Astrophysical Journal, vol. xi. pp. 251 & 358 (1900).
[‡] " The Permanency of Planetary Atmospheres," Monthly Weather Review, August 1902.

from the atmosphere by the molecules gaining intermolecular energy from the æther, to a much greater extent than molecules do under normal conditions of temperature and pressure, which intermolecular energy is transformed into energy of motion during collision, or whether, as I would be more inclined to believe from the results of my own investigations and those of Dr. Bryan, the earth is able to retain its helium, is still in my mind a question for further more careful investigation. The Maxwell-Boltzmann equation should be made to include all the variables discussed by Dr. Stoney, which will in any way have an influence on the speed of the molecules, under the most attenuated conditions of the gas, and by the application of this more comprehensive equation to an atmosphere of helium at the possible ultra-atmospheric temperature, a much more probable result would be reached.

I am sure that no one will be more ready to accept the results of such an investigation than the writer, and with the sincere hope that the present discussion may induce some of the mathematicians to enter this field of scientific inquiry,

<div align="center">I am, dear Sirs, most faithfully yours, ·</div>

<div align="right">S. R. Cook.</div>

Cornell University,
Ithaca, New York, U.S.A.

XLVIII. Dr. G. Johnstone Stoney's *Reply.*

Through the courtesy of the Editors of the Philosophical Magazine I am given the opportunity of replying to Mr. Cook's letter in the same number of the Magazine in which his letter appears.

Mr. Cook in this letter repeats his belief that he had " attacked the problem from the standpoint of Maxwell's equation for the distribution of velocities in a gas." This seems to overlook the material circumstance that no such equation exists. On the contrary, Maxwell is careful to point out again and again that his law only gives the distribution of velocities in an artificial kinetic system consisting of hard elastic particles of equal mass, and in this model only " after a great number of collisions " have taken place. He shows that the law for viscosity within such a kinetic system and also the law for the diffusion of one such artificial system into another, are nearly the same as the corresponding laws for at least some of the gases of nature, when they are under the conditions which prevail at the bottom of our atmosphere. He was also the first to show that these conditions are such that the number of encounters met with by each molecule of the air at the bottom of our atmosphere is about eight

thousand millions per second ; and it is an easy inference from what he says that it is only within limits which can be defined that it is permissible to treat the distribution of velocities within his model as sufficiently approximating to the distribution within a gas ; as, for example, when investigating phenomena, like wind or viscosity or diffusion, in which the succession of events to be explained *is enormously slower than* the above rapidity of molecular encounters. Moreover, he is careful to point out that there are other respects in which his model fails to represent actual gases, and speaking of one such shortcoming Maxwell says—"This result seems decisive against the unqualified acceptation of the hypothesis that gases are such systems of hard elastic particles" (see Maxwell's Scientific Papers, vol. i. p. 409).

I am anxious to emphasise this because I knew Clerk Maxwell, with whom no one could be brought into contact without being impressed by the marvellous grasp and accuracy of his physical insight ; and I always feel regret when I see mistakes of subsequent writers represented as having been his. It is precisely by 'the unqualified acceptation' of Maxwell's hypothesis which Maxwell deprecates, and by misapplying it to gas under the conditions which prevail in the penultimate stratum of an atmosphere, that the results come out which Mr. Cook has given.

The present writer arrived at what were practically the same numerical results either 36 or 37 years ago, at an early stage of his investigation, by applying Maxwell's formula in the same way. But there was this marked difference between Mr. Cook's treatment of the subject and mine. I was all through aware that I was not dealing with gas but with Maxwell's model, while Mr. Cook represents himself as thinking that Maxwell's formula gives the true distribution of velocities in gases, and under all circumstances. When therefore I found that Maxwell's formula did not explain the escape of carbon dioxide from the Moon, I searched for the cause of its insufficiency, with the result that I found those several respects in which it is insufficient which are enumerated in my paper in last June's Philosophical Magazine * (see also Astrophysical Journal for July 1904, p. 69).

I regret to see that Mr. Cook is under the impression that his data have been put in a false light in a passage which he quotes from my paper of last June : in this I think and hope he

* In the footnote at the end of that paper (see p. 700) readers are referred to an earlier paper "On the Physical Constitution of the Sun and Stars" in the Proceedings of the Royal Society for 1898. This should have been *in the Proceedings of the Royal Society for* 1868.

is mistaken. I was certainly most desirous of representing his
work correctly ; and in fact the numbers he gave and again
gives in the present letter, are, as I have already said, suffi-
ciently in agreement with those I had worked out.   It is not
here that we differ; but in the inference we draw from those
numbers.   Mr. Cook does not think them inconsistent with
the fact that the Moon's atmosphere has escaped, while I do.
He finds that a temperature of 270° C. above freezing-point,
or of 543° absolute, was requisite for the escape of carbon
dioxide.   But he does not refer to the further fact that no
such temperature can have prevailed in the lunar atmosphere
when it had one.   [In this discussion the temperature of a
gas is of course regarded from the usual limited point of
view, which represents it as proportional to the kinetic energy
of the translational motions of the molecules.]

The best determination which we seem to possess of the tem-
perature of a small fully absorbing body at the distance from the
Sun of the Earth and Moon seems to be Professor Poynting's
(see Phil. Trans. vol. 202. (1904) p. 535).   It is 300° Absolute,
which is the same temperature as 27° C. above freezing-point.
Again, for the upper limit which could be reached by the
temperature of the hottest part of the surface of an airless
Moon, he obtains 412° Absolute, which is the same as
137° C.

Accordingly the highest temperature at our command
seems to lie between 27° C. and 137° C., and probably nearer
the former temperature than the latter.   It thus appears that
the 270° C. which Mr. Cook requires is far beyond what we
are justified in regarding as possible.   Langley, who has
also investigated lunar temperatures, assigns still lower tem-
peratures than Poynting's.

I think, therefore, that the numerical results which
Mr. Cook obtains from misunderstanding Maxwell's formula
do not justify his statement that they " explain fully the
escape of the atmosphere from the Moon."   The correct
inference from Mr. Cook's results seems to be precisely the
opposite, and supports the inference which I drew that
Maxwell's formula is misapplied if employed upon this
problem.

Fortunately, the inductive method is also available for
investigating the rate at which gases actually do escape from
atmospheres.   And, still more fortunately, it is free from the
pitfalls which beset the steps of the mathematician who
attempts to employ the deductive method upon a physical
problem of this kind, unless he is very careful to keep in
mind that his argument consists of deductions from assumed

data ; and that, therefore, what it deals with is the succession of simpler events which would take place within a model, which the procedure he must of necessity adopt obliges him to substitute for the more complex machinery of nature—viz. : that model which is defined by the data which he has to make the foundation of his reasoning.

<div align="right">G. JOHNSTONE STONEY.</div>

30 Ledbury Road, W.
February 9, 1905.

XLIX. *The Slow Stretch in Indiarubber, Glass, and Metal Wires when subjected to a Constant Pull. By* PERCY PHILLIPS, *M.Sc.*[*]

The Slow Stretch in Indiarubber.

IN an investigation of the adiabatic and isothermal elasticities of indiarubber, it was observed that whenever indiarubber was subjected to a sustained pull it continued to yield slowly after the completion of the first large stretch. This slow yielding, being quite considerable in size, was very easily investigated, and the following general results were obtained.

Firstly, with a particular pull the stretch is a linear function of the logarithm of the time elapsed after the establishment of the pull. Thus, if x is the stretch and t the time since the pull was established,

$$x = a + b \log t;$$

where a and b are constants for the particular pull. This obviously does not hold for very small values of t. No doubt t is to be reckoned from an instant a little before the establishment of the pull, but if t has any value above a few seconds it is sufficiently exact to measure it from the instant of establishment of the pull.

Secondly, the constant b is proportional to the pull, and when t is measured in minutes the constant a is also roughly proportional to the pull.

Thirdly, if a pull is established for a short time and then removed, the indiarubber returns to its original length, according to the law $x = b \log \dfrac{t}{t_0}$, where x and b denote the same things as before, t represents the time which has elapsed since the pull was established, and t_0 the time since the pull was removed.

* Communicated by the Physical Society : read January 27, 1905.

Fourthly, if indiarubber is stretched to a certain length and retained at that length, then the pull required to keep it so stretched decreases for about the first day, according to the law

$$P = a - b \log t,$$

where P is the pull, t is the time which has elapsed since the initial stretch was established, and a and b are constants for a particular extension. b is proportional to the amount of the initial extension.

The apparatus used to investigate the slow stretching in indiarubber was very simple. One end of a strip of india-rubber bandage 30 cms. long was attached by a clip to a wall bracket, and a light scale-pan and a photographic scale with divisions of ·210 mm. were attached by another clip to the other end. A telescope with cross wires in the eyepiece was used to view the scale.

The mode of procedure was as follows :—

1st, the reading in the telescope was noted.

2nd, the required load was placed very gently in the scale-pan at a given time.

3rd, as each division of the scale passed the cross wire of the telescope the time was taken.

The following is a typical set of readings :—

Pull.	Reading on Scale.	Stretch in Scale-divisions.	Time.		$\log t.$
0	30·12				
500 grams	...	...	12 0	0 noon	
	76·8	46·68	2 m.	59 s.	·47
	76·9	46·78	3	50	·58
	77·0	46·88	4	42	·67
	77·1	46·98	6	0	·78
	77·2	47·08	7	40	·88
	77·3	47·18	9	24	·97
	77·4	47·28	11	40	1·07
	77·52	47·4	15	36	1·19
	77·61	47·49	20	18	1·31
	77·73	47·61	27	45	1·44
	78·43	48·31	2 30	0	2·18

Plotting the scale-readings against $\log t$, as in Curve No. I., we get a straight line, *i. e.* we have the relation

$$x = a + b \log t \quad \text{or} \quad \frac{x - a}{\log t} = b,$$

the symbols having the same meaning as before.

If x is measured in scale-divisions and t in minutes, then for this case

$$b = 1{\cdot}006 \qquad \text{and} \qquad a = 46{\cdot}2.$$

Curve No. I.

The constancy of $\dfrac{x-a}{\log t}$ is exhibited in the following table.

x.	$\log t$.	$\dfrac{x-46{\cdot}2}{\log t}$.
46·68	·47	1·02
46·78	·58	1·00
46·88	·67	1·02
46·98	·78	1·00
47·08	·88	1·00
47·18	·97	1·01
47·28	1·07	1·01
47·4	1·19	1·01
47·49	1·31	·98
47·61	1·44	·98
48·31	2·18	·97

If x is measured as stretch per unit length

$$b = {\cdot}0073 \qquad \text{and} \qquad a = {\cdot}3235.$$

On removing the load from the indiarubber it returned gradually back to its original length according to the law

$$x = a + b \log t - (a + b \log t_0)$$

i. e. $$x = b \log \left(\frac{t}{t_0}\right),$$

where t is the time which has elapsed since the load was put on, and t_0 that since the load was removed. The following is an ordinary set of readings :—

Load.	Reading on Scale.	$t.$		$t_0.$		$\log t.$	$\log t_0.$	$\log \dfrac{t}{t_0}.$
0	22·24							
200 grams	12·0		20 s.	....		$\bar{1}$·52		
at 3 P.M.	11·8		50 s.	...		$\bar{1}$·92		
	11·7	1 m.	25 s.	...		·15		
	11·6	2	20	...		·37		
	11·5	4	15	...		·63		
	11·4	7	5	...		·85		
	11·3	14	30	...		1·16		
	11·2	29	30	...		1·47		
	11·08	58	0	...		1·76		
	11·02	80	0	...		1·90		
0 grams	21·5	83	23		23 s.	1·92	$\bar{1}$·58	2·34
at 4.23 P.M.	21·6	83	38		38	1·92	$\bar{1}$·80	2·12
	21·7	84	23	1 m.	23	1·93	·14	1·79
	21·8	85	40	2	40	1·93	·43	1·50
	21·9	89	0	6	0	1·95	·78	1·17
	22·0	100	0	17	0	2·00	1·23	·77
	22·09	132	0	49	0	2·12	1·69	·43
	22·18	277	0	194	0	2·44	2·29	·15
	22·26	1153	0	1070	0	3·06	3·03	·03

Curve No. II.
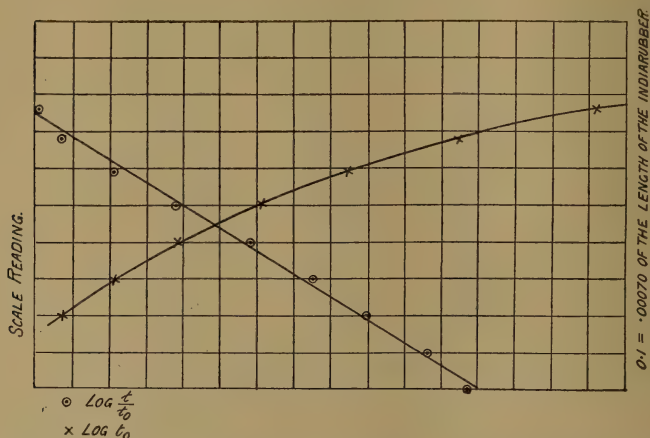
On Curve No. II. these results are exhibited, both $\log t_0$ and $\log \left(\dfrac{t}{t_0}\right)$ being plotted against the stretch so that they may be compared.

The relations of the constants a and b to the stretching force were then found by plotting the curves of stretching for the different loads in exactly the same way as is shown for the 500 gram load. The following results were obtained :—

Load.	a.	b.	$\dfrac{b}{\text{load}}$.
100 grams	6·8	·176	·00176
200 ,,	14·7	·385	·001925
300 ,,	23·5	·580	·001933
400 ,,	33·4	·800	·002000
500 ,,	46·2	1·006	·00201

These numbers show that b is nearly proportional to the load.

The effect of superposing two loads was then investigated, but this showed that the effect was by no means the sum of the effects which each would produce separately. *E. g.*, suppose the stretch due to the first load be given by

$$x = a_1 + b_1 \log t_1,$$

t_1 being the time which has elapsed since the first load was put on, and suppose the effect of the second load be given by

$$x = a_2 + b_2 \log t_2,$$

t_2 being the time since the second load was put on. Then the total stretch should be

$$x = a_1 + a_2 + b_1 \log t_1 + b_2 \log t_2.$$

No such relation was found to hold however, and the only conclusion which was drawn from these experiments is that when t_2 is small compared with $t_1 - t_2$ the indiarubber stretches almost as if the second load were the only one acting, and when t_2 is large compared with $t_1 - t_2$ the stretch is the same as if both loads had been put on at the same time. This shows that unless the duration of an experiment is small compared with the time which has elapsed since the indiarubber was last strained, an effect will be produced by that previous strain.

The alteration in pull required to keep a piece of indiarubber stretched to a certain length was next investigated.

This was done by means of a balance so arranged that the beam was lifted just off the supports. From one end of the beam the scale-pan was removed, and just under this end was placed a heavy weight to which the lower end of a piece of indiarubber bandage was attached by a clip. To the upper end of the bandage a hook was attached by means of another clip. At first the indiarubber was too short to reach to the beam of the balance, but at a certain time it was stretched just far enough to hook on to the beam, and weights were placed in the pan at the opposite end of the beam so as to just balance the pull of the indiarubber. The weight had to be altered from time to time as the stress altered, but as the beam of the balance was only just raised above the supports, the indiarubber was stretched by a practically constant amount all the time.

The readings in one case were :—

Pull in grams.	Time after the indiarubber was stretched.		$\log t$.
1472·5	1 m.	30 s.	·18
1467·5	2	15	·35
1462·5	3	15	·51
1457·5	4	35	·66
1452·5	7	30	·88
1448·5	10	0	1·00
1444·5	14	45	1·17
1440·5	19	24	1·29
1436·5	26	45	1·43
1430·5	48	0	1·68
1427·5	58	0	1·77
1412·5	210	0	2·32
1384·5	1436	0	3·16

If the pull be plotted against $\log t$ we get a straight line, so that the stress obeys the law

$$S = a - b \log t.$$

From the curve we immediately deduce a to be 1477 grams, and the constancy of $\dfrac{a-S}{\log t}$ is exhibited in the following table.

Little notice should be taken of the first few figures, for $a-S$ and $\log t$ are very small, and so a small error in measuring S is largely exaggerated.

Pull in grams.	log *t*.	$\dfrac{a-\mathrm{S}}{\log t}$.
1472·5	·18	25·0
1467·5	·35	27·2
1462·5	·51	28·5
1457·5	·66	29·5
1452·5	·88	36·0
1448·5	1·00	35·3
1444·5	1·17	36·1
1440·5	1·29	35·4
1436·5	1·43	35·3
1430·5	1·68	36·2
1427·5	1·77	35·8
1412·5	2·32	36·0
1384·5	3·16	34·3

Subsequent experiments have shown that after about a day the decay of stress becomes much more rapid until S becomes very small, and then S gradually approaches a zero limit. In the experiment of which the observations are given the pull after twelve months had fallen to 22 grams.

Stretching by different proportions of the unstretched length shows that *b* is proportional to the stretch.

All the ordinary assumptions which one can make, such as considering the stretch as consisting of elastic and viscous portions, seem to lead to the conclusion that the stretch should be a function of e^{-t} and not of log *t*.

Consequently it was thought that the amplitude of the up and down oscillations of a weight on the end of a length of indiarubber bandage might obey some other law than the usual one. Observations showed, however, that the usual law is obeyed, *i. e.* that the amplitude decreases in geometrical progression as the time increases in arithmetical progression.

In order to give some idea of the effect of alternations of temperature in the laboratory in producing errors, the temperature coefficients of the indiarubber were found while the indiarubber was in tension. The coefficient of expansion on cooling proved to be much larger than the coefficient of contraction on heating. Consequently the alternations of temperature in the laboratory would produce a slow creep on their own account: thus when the creep is very slow the effect of the alternations of temperature may be much greater than the true logarithmic creep. In all cases it was found that when the creep became very slow, *e. g.* a few days after the load was put on, it was a great deal faster than the logarithmic law would warrant.

To sum up the results for indiarubber :—

1. When indiarubber is stretched by a given load it stretches according to the law

$$\text{stretch} = a + b \log (\text{time}),$$

and b is directly proportional to the load.

2. When the load is removed the stretch still remaining is equal to

$$b (\log t - \log t_0),$$

where t is the time since putting on the load, and t_0 is the time since removing the load.

3. When two loads are put on successively their effects are not exactly superposable, but just after the second load is put on, the effect is as if the last load only were acting, and then it gradually alters till finally the effect is the same as if both loads had been put on simultaneously.

4. When indiarubber is stretched to a fixed length and retained there, the load required to maintain that stretch alters according to the law

$$S = a - b \log t,$$

where S is the stress and t is the time since the rubber was stretched ; b is proportional to the initial stretch.

5. The temperature coefficients of expansion and contraction of indiarubber in tension are quite different.

6. The decrement of the amplitude of vibrations in indiarubber obeys the ordinary law.

The Slow Stretch in Glass.

Having established these results for indiarubber, it was thought desirable to investigate whether the properties were possessed by other substances.

Glass being a substance which is known to exhibit a slow creep when under stress, it was next investigated. As the motion is so much smaller in glass than in indiarubber, it was necessary to use a much more exact measuring device.

A diagram of the apparatus is given in fig. 1. f is a glass fibre made by drawing out a glass rod to a fairly thin neck and then heating the neck and drawing it out to a length of about 1 foot. By this means a fairly uniform fibre was obtained, but it was still a little thinner in the centre than it was elsewhere. One of the thick ends of the fibre was attached as shown to a firm support S.

To the other end was attached a little table T, and a weight

carrier W which had a system of vanes V dipping into oil to damp vibrations.

To the upper end was also attached a circular piece of brass C from which depended three rods, *a*, of glass, and to the lower ends of *a* a ring of brass R was fixed. An optical lever O rested with two of its legs on the ring R and one on the table T. A telescope viewed the reflexion of a millimetre-scale in the mirror of the optical lever O.

This arrangement is that used by Mr. G. A. Shakespeare in this laboratory, the rods *a* serving to compensate for temperature alterations and for any give of the support. The length of the optical lever was ·820 cm. and the distance of the scale was 3 metres during some experiments, and 461 cms. in others.

In the same way as for the indiarubber experiments the weight was put on by hand as gently as possible, and the reading in the telescope taken at intervals. At first no consistent results could be obtained, but when the fibres were put on one side for a few weeks after their manufacture the subsequent results for stretching were quite concordant.

In the case of glass, as in that of indiarubber, the law of stretching was

Fig. 1.

$$x = a + b \log t,$$

the symbols having the same significance as before; and when the load was left on for a long time and then removed the creep back was proportional to the logarithm of the time which had elapsed since the removal of the load if the duration of the observations was short compared with the time since the load was put on.

The numbers given (p. 522) were obtained for a load of 779 kilograms per sq. cm. cross section.

Here again, for the first few numbers $x - a$ is rather small, and so a small error in x would make a large error in $x - a$.

a was found as before by plotting the stretch against $\log t$, and producing the straight line so obtained till it cut the axis, $\log t = 0$. The value of x at $\log t = 0$ is of course equal to *a*. If x is measured in scale-divisions and t in minutes, we see that b in this case is equal to 2·74: if x
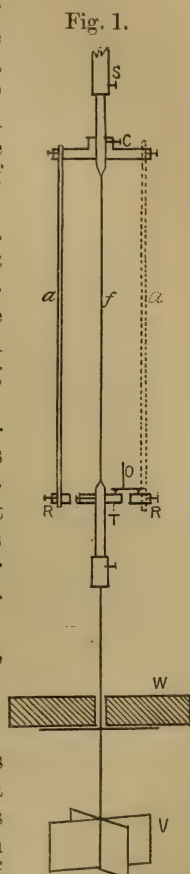
Stretch in Scale-divisions.	log t.	$\dfrac{x-a}{\log t}$.
289·7	·38	2·63
290·2	·50	3·0
290·6	·66	2·88
291·0	·86	2·68
291·3	·94	2·77
291·75	1·15	2·66
292·3	1·32	2·73
292·5	1·40	2·72
292·9	1·54	2·73
293·2	1·65	2·73
293·8	1·90	2·69
296·7	2·92	2·74

is the stretch per unit-length and t in minutes, $b = 1\cdot55 \times 10^{-5}$. For a load of 599 kilograms per sq. cm. cross section on the same fibre b was found to be

$$1\cdot16 \times 10^{-5},$$

and for a load of 422 kilograms $b = \cdot780 \times 10^{-5}$. Plotting these three values of b against the load we get a straight line, and when the load is 284 kilograms b becomes 0. This is shown by the approximate equality of the three values of $\dfrac{\text{Load} - 284}{b}$:—

$$313 \times 10^5$$
$$273 \times 10^5$$
$$300 \times 10^5$$

The Slow Stretch in Copper, Platinum, Silver, and Gold Wires.

It was next thought very desirable to find out whether metals behave in a similar way to glass and indiarubber with regard to slow stretch.

To investigate this an arrangement almost identical with that used for glass was employed, the rods a being of course replaced by wires of the same kind as that being stretched. Until the load used reached a certain value no slow stretch at all approaching in size to that occurring in glass could be detected, but above this value there was a creep obeying the same law as for the indiarubber and glass. This creep, however, caused a permanent extension, and when the load was removed there was little, if any, slow creep back.

The inquiry into the slow stretch of wires reveals astonishing differences in behaviour, even in specimens from the same

roll of wire, and no consistent results were obtained until wires were used which came from the same roll, and which were annealed at the same temperature for the same length of time, and were cooled at the same rate.

Having no convenient annealing furnace, I annealed the wires separately by an electric current. The whole annealing apparatus is represented diagrammatically in fig. 2.

Fig. 2.
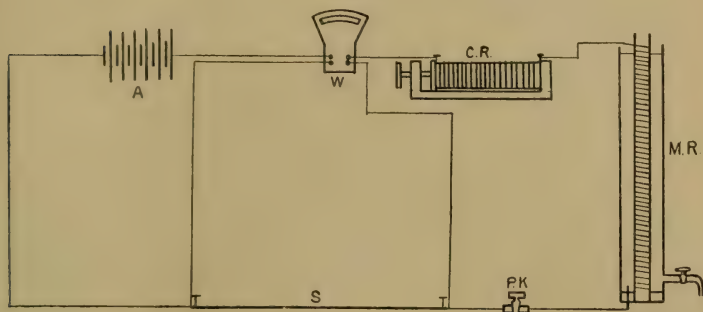
A represents the accumulator-cells which supply the current.

W is a Weston standard ammeter and voltmeter.

S is the wire which is to be stretched. It is enclosed in a glass tube to prevent air draughts, and when copper wires were used the glass tube was filled with coal-gas to prevent the oxidation of the wire.

CR is a carbon rheostat for adjusting the current.

MR is a rheostat for cutting off the current at the same rate at each annealing. It consists of two concentric tubes, the outer of glass and the inner of porcelain. The lower end is closed with a bung, and at the botton of the glass tube is a side tube with a tap. Round the inner tube is wound a long spiral of iron wire of large resistance compared with the rest of the circuit, and connected up as shown. The space between the two tubes is filled with mercury while the annealing continues. At the required time the tap is turned on and the mercury slowly flows out, thus putting in gradually more and more resistance.

PK is a plug-key.

It was found that the way in which the load was put on made some difference to the result.

Ultimately uniformity was obtained by using a flotation method which put the full load on in about 5 seconds.

Before the load was put on, the telescope was placed to view the scale : then the load was put on and the times were taken as the successive divisions on the scale passed the cross-wire of the telescope.

The following is a fairly typical set of readings for a copper wire :—

> Length of wire　　　= 28·2 cms.
>
> Diameter of wire　 ·= 　·0390 cm.
>
> ∴.　Cross-section　　　= 　·001195 cm².
>
> Weight of clips, &c. = 122 grams.
>
> Annealing current　 = 12 amps. for 5 minutes.

Load.	Reading on Scale.	Time.			t,	log t.
122 grams.	84					
722　　,,	...	11h 30m	0s			
	542	31	13		1·22	·09
	543	31	23		1·38	·14
	544	31	36		1 6	·20
	545	31	48		1·8	·26
	546	32	8		2·13	·33
	547	32	26		2·43	·39
	547·5	32	36		2·6	·41
	548	32	48		2·8	·45
	548·5	32	58		2·97	·47
	549	33	12		3·2	·50
	549·5	33	24		3·4	·53
	550	33	38		3·63	·56
	550·5	33	56		3·93	·59
	551	34	15		4·25	·63
	551·5	34	34		4·57	·66
	552	34	52		4·87	·69

t, as in the case of indiarubber, is the time in minutes which has elapsed since the load was put on. If log t is plotted against the scale-reading we again get a straight line, and therefore copper obeys the same law $x = a + b \log t$.

If x is measured in scale-divisions, b for this case is equal to 16·28, and $a = 456·7$.

In the following table (p. 525) the constancy of $\dfrac{x - 456·7}{\log t}$ is exhibited.

To bring b to absolute units we must know the value of the scale-division. This was found by removing the wire, and so placing a micrometer-screw gauge that the leg of the optical lever, which previously rested on the table t, was now resting on the top of the movable jaw of the screw-gauge.

x.	$\log t$.	$\dfrac{x-456\cdot7}{\log t}$.
458	·09	14·4
459	·14	16·4
460	·20	16·5
461	·26	16·5
462	·33	16·1
463	·39	16·2
463·5	·41	16·6
464	·45	16·2
464·5	·47	16·6
465	·50	16·6
465·5	·53	16·6
466	·56	16·6
466·5	·59	16·6
467	·63	16·4
467·5	·66	16·4
468	·69	16·4

As the screw was turned the readings in the telescope and in the gauge were taken :—

Scale.	Micrometer.	Scale.	Micrometer.	Scale Difference.	Micrometer Difference.
137·1	12·50	543·6	15·0	406·5	2·5
217·5	13·00	626·0	15·5	408·5	2·5
299·0	13·50	709·4	16·0	410·4	2·5
379·8	14·00	794·0	16·5	414·2	2·5
461·4	14·50	879·9	17·0	418·5	2·5
				2058·1	12·5

∴ 1 scale-division = ·00608 cm.

Expressing everything in absolute units we have

$$\text{Load per sq. cm. cross section} = 604 \text{ kilos,}$$

and when x is measured as stretch per unit length

$$b = ·000351.$$

Exactly the same process was gone through with different copper wires from the same coil. They were annealed with the same current (12 amps.) for the same time (5 mins.), and then their curves of stretching with various loads were obtained. It appears that the logarithmic law obtains right up to the load which causes the wire to pull out quite rapidly and break.

The following is the series of values of b which was obtained for this particular coil of copper wire treated in this manner.

	Load per sq. cm. in kilos.	$b.$
All up to	380 kilos.	0
	399·2	·000073
	458·0	·000174
	498·0	·000237
	531·0	·000297
	604·0	·000351
	700·0	·000428
	720·0	·000442
	795·0	·000542
	850·0	·000658

At   900   the wire drew out so far that the scale went right out of view, and finally the wire broke.

If we observed the form of the curve connecting b with the load (Curve No. III.a) we see that there appears to be a
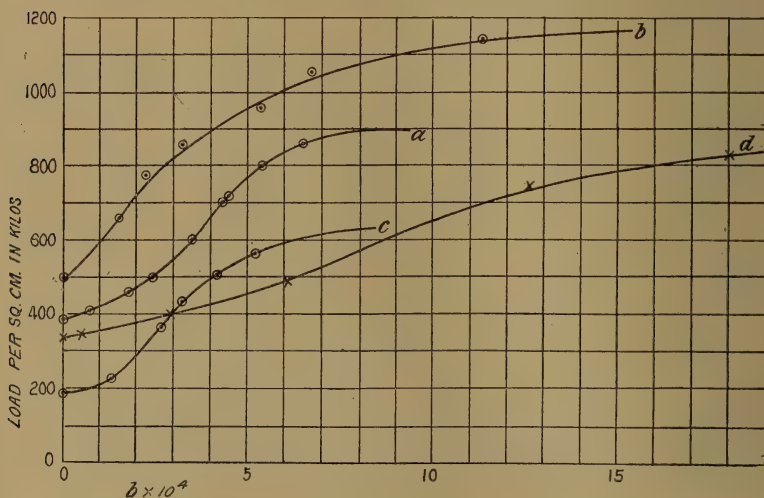
very definite point below which there is no very considerable slow creep, and there is a somewhat indefinite load above which the wire begins to stretch very rapidly. The range of load between these points is from 380 kilos. to 850 kilos. per sq. cm. cross section.

Exactly similar series of experiments have been conducted with platinum, silver, and gold wires. All these substances show the same general characteristics that between wide limits (to nearly three times the smallest load necessary to produce a creep) they stretch according to the law $x = a + b \log t$, and

that this creep starts at a very definite load and becomes very rapid at a less definite point.

For the Platinum wire :—

Diameter = ·0506 cm. ∴ cross-section = ·00202 sq. cm.

Annealing current = 8·5 amps. for 5 minutes.

	Load per sq. cm. cross section.	$b \times 10^4$.
All up to	500 kilos.	0
	654	1·525
	771	2·265
	854	3·22
	952	5·39
	1050	6·73
	1141	11·35
	1247	26·40
	1354	46·40
	1455	138·0
	1560	wire pulled out and broke.

The curve connecting b with the load is shown in III. b.

For Gold wire :—

Diameter of wire = ·043 cm.

Annealing current = 7·5 amps. for 5 minutes.

	Load per sq. cm.	$b \times 10^4$.
All up to	180 kilos	0
	222	1·30
	362	2·61
	432	3·19
	501	4·16
	570	5·03

At 640 kilos. the wire pulled out so much that the weight rested on the table underneath.

The curve connecting the load with b in this case is given in III. c.

For Silver wire :—

Diameter of wire = ·0486 cm.

Annealing current = 16 amps. for 5 minutes.

	Load per sq. cm cross section.	$b \times 10^4$.
All up to	330 kilos	0
	335	0·479
	396	2·81
	486	6·08
	731	12·68
	827	18·08

At 900 kilos the wire pulled out and broke.

The curve connecting b with the load for silver is given on Curve No. III. d. All these curves show the characteristics mentioned in the case of copper. No great importance must be attached to the absolute value of these numbers, for besides the great effect which is produced by the annealing temperature, by the length of time during which the annealing continues, and by the rate at which the wire is cooled down, there is an effect produced by the way in which the weight is put on. Added to this, different specimens of wire give very different results, though when pure wires were obtained from Messrs. Johnson & Matthey these differences were fairly small.

A fundamental difference between the slow stretching of indiarubber and glass and that of metal wires is that the former is reversible, *i. e.*, the creep back when the load is removed follows the same law, while in the case of the metals there is no appreciable slow recovery.

In the case of the stretching of these metals, it is just as difficult to make a reasonable assumption which shall give the law as it is with indiarubber. If we assume that the permanent stretch is produced by cleavage of crystals, we require that the rate of breakdown of the crystals shall be given by

$$\frac{dx}{dt} = \alpha e^{-\beta x}$$

where x is the stretch. Or, as it seems reasonable to suppose that the stretch is proportional to the number (n) of crystals broken down,

$$\frac{dn}{dt} = \alpha' e^{-\beta' n}$$

or, if N be the number of unbroken crystals,

$$\frac{dn}{dt} = \alpha'' e^{+\beta'' N}.$$

All previous assumptions have led to a limit, or else have attributed the characteristics of a liquid, and so have made the stretch a linear function of the time.

We see, however, that

$$\frac{d^2 x}{dt^2} = -\alpha\beta e^{-\beta x} \frac{dx}{dt}$$

$$= -\alpha^2 \beta e^{-2\beta x},$$

i. e. $\qquad \dfrac{d^2 x}{dt^2}$ is proportional to $\left(\dfrac{dx}{dt}\right)^2$.

We might therefore assume that after the very beginning of

the stretch no further crystals are broken, and that after this the resistance to the load is provided by ordinary viscous force between the faces of the cloven crystals according to the law which sometimes obtains in very viscous liquids, that the viscous force is proportional to the square of the velocity, This would suggest that even in pure metals there may be a non-crystalline matrix in which the crystals are embedded, and that this matrix acting as a very viscous liquid is responsible for this law of stretching. In the Phil. Mag. Oct. 1904, Prof. Trouton and Mr. Rankine have published a research in which they find this logarithmic law of decay of stress and strain in the case of lead wires. They did not require to anneal in any way, probably because any internal strain would decay very rapidly in such a soft metal. In the case of the metals which I used the logarithmic law did not obtain unless they had been annealed.

Prof. Trouton and Mr. Rankine also found that after about the first hour the stretching became a linear function of the time, and this also is probably due to their using such a soft metal.

In the case of gold, I found that after about a day the stretch approximated to a linear function of the time, but of course the linear stretch was a very much smaller proportion of the logarithmic stretch than was found with lead. The copper, platinum, and silver wires made no attempt towards approximation to a linear function, even after two days or more, but when the stretch became so slow that it was difficult to measure, there were considerable deviations from the logarithmic law.

There do not appear to be any absolute values of the stretch given in their paper, so that I cannot compare the size of the stretch in lead wires with that in copper, platinum, silver, and gold.

They also indicate that the assumptions which one usually makes with regard to the stretching of solid bodies do not give the stretch as a logarithmic but as an exponential function of the time, and it seems difficult to make any reasonable assumption which shall give the experimental law.

The Slow Stretch in Iron and Steel Wires.

In the case of iron and steel wires a bundle of each kind of wire was annealed for me in one of the annealing furnaces at Messrs. Perry & Co.'s pen works, but in all other respects the wires were experimented upon in exactly the same way as the other wires. The results, however, are entirely different. As before, there seems to be no slow stretching

until a load large enough to cause a permanent extension is used, but when such a load is used the stretching is no longer a linear function of the logarithm of the time. As there seems to be no simple connexion between the curves for stretching for different loads, a series of curves for iron wires is given on curve No. IV., and a series for steel on

Curve No. IV.
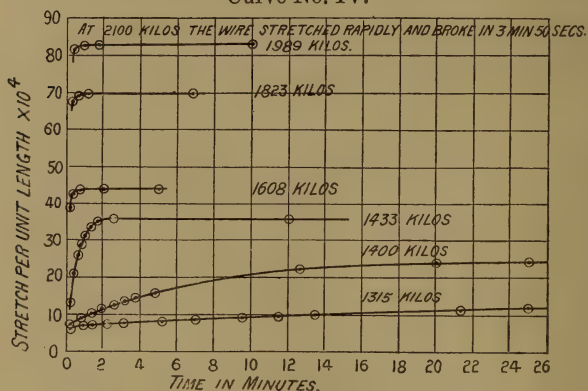
Curve No. V.
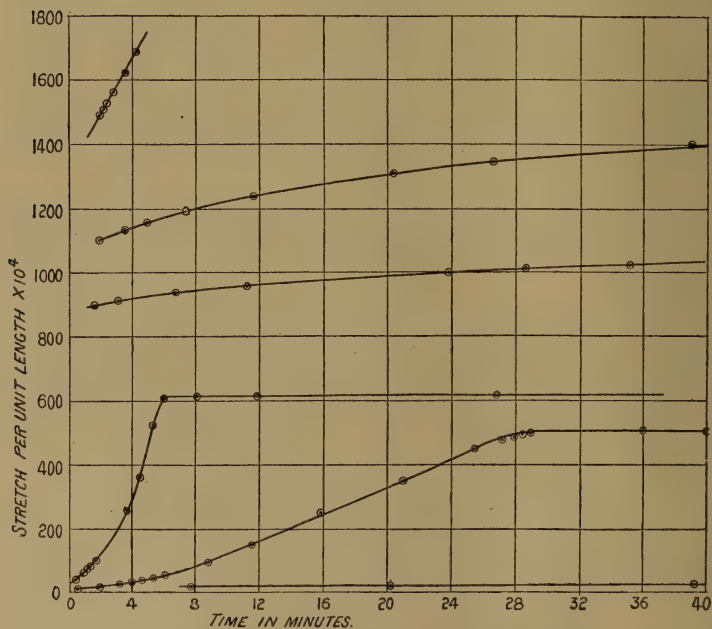
curve No. V.   Opposite each curve is given the load per sq. cm. used in the particular stretching.   It will be seen at

a glance how entirely different these curves are from those for the other metals investigated. I have no explanation for this difference, but it is well known that iron and steel are anomalous in many other properties. The steel wires after having been stretched are magnetized, and this suggests that a molecular change may take place in the steel, perhaps at a definite point in the stretching, causing the steel at that point to become harder.

Up to the present no other substances have been investigated, but it is my intention to investigate the behaviour of quartz fibres in the same way. They have been shown to be very perfectly elastic for small distortions, but it is quite possible that a slow creep does take place and could be detected if much larger distortions were used.

My best thanks are due to Prof. Poynting for his many valuable and kindly suggestions throughout the research.

Birmingham University,
   Oct. 1904.

L. *The Radiation from Ordinary Materials.* By NORMAN R. CAMPBELL, *B.A., Fellow of Trinity College, Cambridge*.*

[Plate VII.]

§ 1. THE experiments of Patterson, McLennan, Strutt, Righi, and others, have proved beyond doubt that part of the "spontaneous" ionization in a closed vessel containing gas is due to the influence of the walls. It is of the utmost importance to determine the nature of this influence —to ascertain whether it consists of Becquerel rays, and, if so, to discover the nature of those rays.

In this paper † two methods are described which have been used for the solution of the problem, the later being suggested by the earlier work.

§ 2. All the experiments consisted in the measurement of the spontaneous saturation current—or "leak," as it will be called hereafter—through vessels of various forms and materials. A preliminary description of the apparatus used for the measurement is desirable.

A Wilson gold-leaf electroscope ‡ was employed and found most satisfactory. The potential to which the insulated system had attained in a given time was ascertained by means

of the arrangement shown in fig. 1.   The point on the scale
of the observing microscope occupied by the leaf was noted,
the electrode earthed by inserting the wire C in the mercury

Fig. 1.
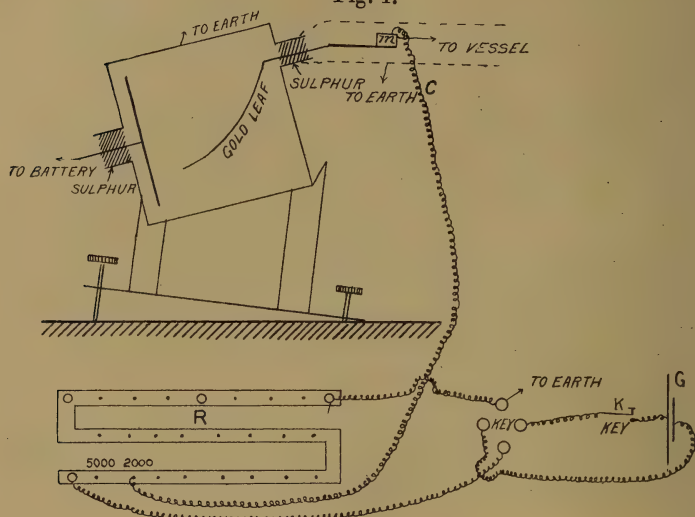
cup *m*, and resistances taken out of the box R until, on
depressing the key K, the leaf returned to the position noted :
from the value of the resistances and of the E.M.F. of the cell
G the required potential can be easily calculated.

The time for which the leak was measured was chosen so
that the potential acquired was about one volt, giving 55 to
60 scale-divisions ; the potential could be read with an
accuracy of one per cent.

The field necessary to send the saturation current through
the gas was derived from an excellent battery of small cells.
Care was taken that the current was saturated and that all
insulation (sulphur was used throughout) was acting
efficiently.   The direction of the field was occasionally
reversed ; in all cases in which the contrary is not explicitly
stated, the magnitude of the leak was independent of its
sign.

The capacity of the insulated electrode system, the product
of which with the potential acquired in unit time gives the
current through the gas, was measured by means of the device
described by McClelland *.   The current is given throughout
the paper on an arbitrary scale, in which the unit of capacity

* McClelland, Roy. Dublin Soc. Proc. 10. 18, p. 167 (Feb. 26, 1904).

is the capacity of the " Uranium Pot " used (a lead cylinder 17 cm. high and 9 cm. in diameter, with a central wire electrode), the unit of potential the volt, and the unit of time the minute.

Since the reading of one electroscope did not wholly occupy the attention of the observer, two exactly similar but entirely separate sets of apparatus were used : thus two experiments were proceeding at the same time. On several occasions the same experiment was tried successively with the two electroscopes, and excellent agreement between the results obtained.

§ 3. A few preliminary experiments were made with vessels of the same dimensions as the " Uranium Pot," but of the most diverse materials—metals, woods, paper, porous pots saturated with electrolytes, &c. The leak in all cases was of the same order of magnitude, the extremes differing in the ratio of 7 to 1.

§ 4. The first method by which information was sought as to the nature of the ionizing influence was the measurement of the relation between the ionization and the volume in vessels of special form.

Rectangular vessels were constructed of wood, of which one side could be moved parallel to itself through a considerable range (fig. 2). The walls were covered with the material

Fig. 2.
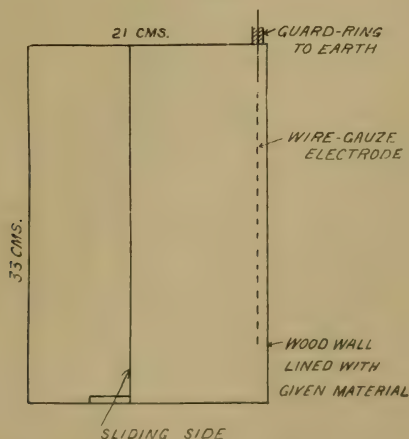
under investigation, carefully cleaned with sandpaper and charged to a high potential ; the current through the gas to an electrode of wire netting, suspended by a stiff wire passing through a sulphur plug and guard-ring in the top of the vessel,

was measured by the electroscope. The surface of the wire forming the electrode was so small that its influence was negligible compared with that of the walls of the vessels.

The front wall could be removed in order to adjust the sliding side ; it is supposed to be removed in the figure to show the interior arrangement. The dimension of the sliding side was about 33×23 cms., differing slightly in different boxes ; the range was some 22 cms.

The following materials were used as coverings for the walls :—

> Lead sheet .... ·1 cm.   thick.
> Tinfoil   .... ·0013 cm.   ,,
> Aluminium .... ·05 cm.   ,,
> Zinc   .... ·08 cm.   ,,
> Platinum   .... ·01 cm.*   ,,
> Carbon (wood coated with pure graphite).

The results of the experiments are expressed in the curves of fig. 1, Plate VII.† Each point marked on a curve represents the mean of at least fifteen observations—usually a far greater number. The plotting of each curve occupied from seven to ten days.

§ 5. Before proceeding to the discussion of the curves some attention should be given to a difficulty which seriously obstructed the progress of the work. This difficulty is the variation which was found to occur in the value of the leak through a given volume.

The variations were of two kinds :—

(1) A slight continuous and continual change over a range of 15 or 20 per cent. No serious trouble arose from this cause ; by taking the mean of some 25 readings a value could always be obtained which could be reproduced to within 4 or 5 per cent. at any subsequent time by replacing the sliding side in the same position—so long as changes of this kind *only* had occurred in the interval.

(2) Sudden discontinuous changes or "jumps," when the value of the leak might be altered by as much as 50 per cent. These were far more troublesome. They occurred on an average once in 2 or 3 days, and affected in the same sense the values all over the range. The leak could not be brought back to its original value by blowing in fresh air or cleaning the surface of the walls ; the occurrence of the variations

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* For the loan of the considerable quantity of platinum required, I am indebted to the generosity of Messrs. Johnson, Matthey & Co.

† The curve for platinum is so nearly coincident with that for aluminium that it has not been drawn.

could not be prevented nor their abolition secured by any of the numerous measures that were tried. They were indubitably changes in the ionization due to the walls, for they affected the leak at small volumes in a greater ratio than the leak at large volumes, but their cause has remained a mystery.

It might be thought that jumps of this kind would render impossible the plotting of such curves as have been drawn ; but fortunately they were well marked—there was never any doubt when one had occurred. By patiently continuing observations, a period between two jumps could always be found sufficiently long to enable a complete series of observations to be made throughout the whole range of the sliding side. But if the series had been made in a different interval, the constants of the curve—but not, so far as can be ascertained, its form—would have been markedly different.

Dr. Jaffé has recently published in this Magazine * a paper in which he has given some account of similar changes which he observed. Our experiments were in progress at the same time and in the same laboratory ; and it is worthy of note that while changes of type (1) occurred in general with both of us at the same time in the same sense, no correlation could be established between changes of type (2).

The vessels which I used were not perfectly air-tight, but it seems improbable that this fact is the cause of the changes, in view of the impossibility of restoring the leak to its former value by blowing in fresh air from outside the room.

§ 6. We will now consider the meaning of the curves of fig. 1, Plate VII. The difference between the curved portion near the origin and the straight line into which it develops suggests at once that two different agents are at work ; and we can easily see that the form of the curve can be explained on the assumption that the ionization is caused by an easily absorbable radiation proceeding from the walls acting in conjunction with a much more penetrating radiation. For this penetrating radiation, not being perceptibly decreased in intensity by passing through the air of the vessel, will cause ionization proportional to the volume ; the part of the curve due to this cause will be a straight line inclined to the axis. On the other hand, the ionization caused by easily absorbable radiation coming from the walls will increase with the distance apart of the sliding sides, only so long as a layer of air of a thickness equal to that distance is insufficient to absorb all the radiation. When all the radiation is absorbed in passing from one side of the box to the other, an increase in the

* Jaffé, Phil. Mag. Oct. 1904.

distance between the sides will cause no increase in the resulting ionization. The part of the curve due to this cause will increase rapidly to a constant value. Combining the two, we should expect a rapid rise of the curve near the origin followed by a straight line cutting the axis of ionization on the positive side. This is precisely the form of the curves in the figure.

Reasoning from the assumption of the existence of these two radiations, we can obtain some information as to their intensities.

Let each sq. cm. of the walls give out (a) an amount of absorbable radiation which, when totally absorbed by the air, causes ionization s, and (b) an amount of penetrating radiation which causes ionization v_1 per c.c.; (c) let the external penetrating radiation discovered by Cooke * cause ionization v_2 per c.c.

Then, if x is the distance apart of the movable sides, c and d the lengths of their edges, the surface of the box exposed to the air inside is

$$2cd + 2(c+d)x$$

and the volume of the air contained is cdx.

When x is so great that all the radiation (a) is absorbed, the ionization in the vessel will be

$$\{2cd + 2(c+d)x\}s \qquad \text{due to } (a)$$
$$\{2cd + 2(c+d)x\}v_1cdx \quad \text{due to } (b)$$
$$cdxv_2 \qquad\qquad \text{due to } (c).$$

If y is the total ionization the equation of the curve remote from the origin is

$$y = 2cds + \{2(c+d)s + 2c^2d^2v_1 + cdv_2\}x + 2cd(c+d)v_1x^2.$$

Now we see that the part of our curves remote from the origin is indistinguishable from a straight line; hence we may put $v_1 = 0$ and obtain

$$y = 2cds + \{2(c+d)s + cdv_2\}x.$$

Produce the straight portion of the curve to cut the axis of y; let the intercept on this axis be p. Then

$$p = 2cds \text{ or } s = \frac{2cd}{p}.$$

The tangent of the angle which the straight line makes with the axis of x is $\tan\theta = 2(c+d)s + cdv_2$. From the

* H. L. Cooke, Phil. Mag. Oct. 1903.

experimental curves we know c, d, p, θ, hence s and v_2 can
be calculated. The values found in this way are given in
Table I., columns 2 and 3.

TABLE I.

1	2	3	4	5	6	7	8
Material.	Without Screen.		With Screen.		Dimensions of Box.		
	$s \times 10^4$	$v_2 \times 10^5$	$s' \times 10^4$	$v_2' \times 10^5$	$cd.$	$c+d.$	Range.
Lead	3·86	3·22	3·84	1·31	777	56·4	21
Tinfoil	3·3	0·92	2·15	0·43	792	57·0	20
Aluminium..	1·0	3·19	1·0	0·88	710	53·8	22
Zinc	0·8	2·76	0·47	1·33	710	53·8	20
Platinum ...	0·92	3·36	0·0	2·2	780	56·6	21
Graphite ...	2·04	1·54	1·69	0·83	710	53·8	22

§ 7. The curved portion of the trace requires further con-
sideration. In this region the distance between the sides is
so small, that the absorbable radiation is not wholly absorbed
in traversing the layer of air, and consequently does not
exert its full ionizing effect. For this part of the curve the
ionization due to each sq. cm. of the walls will be an ex-
ponential function of x and λ, the absorption coefficient, say
$f(x, \lambda)$. Of course $s = f(\infty, \lambda)$.

If we could determine the form of the function we could
determine λ from the curve, but in order to do this we must
know the relation between the intensity of the radiation
emitted in each direction and the angle that direction makes
with the normal to the radiating surface. Thus, if the rays
were all projected normally,

$$f(x, \lambda) = \frac{I_0(I - e^{-\lambda x})}{\lambda}.$$

where I is the ionization caused actually at the surface. If
the rays are emitted equally in all directions,

$$f(x, \lambda) = 2\pi I_0 \left\{ \frac{1 - e^{-\lambda x}}{\lambda} + x \int_x^{\infty} \frac{e^{-\lambda r}}{r} \, dr \right\}.$$

Ideally it would be possible to determine from the earlier
part of the curve the function applicable to the case and the
value of λ for any metal. But actually it is not possible to
draw this portion with sufficient accuracy for the purpose;

the volumes are small and the probable error great. The matter is complicated by the influence of the corners of the box ; at the edges of any wall part of the rays, unless they proceed normally from the surface, will be cut off by the adjacent wall and fail to exert their full effect. When account is taken of this, $f(x, \lambda)$ becomes too complicated for further progress to be made in this direction. It might appear at first sight possible to assume that the rays were emitted equally in all directions ; for the properties along any line in an infinite slab of material—and for this purpose the thicknesses used may be considered infinite—should be independent of the direction of that line. But the values of λ obtained on this assumption are so utterly discordant with those found in the experiments described below that the assumption must be rejected. Probably the inequalities of the surface have an influence which cannot be neglected. Much better agreement is also found between the calculated and plotted curves on the assumption that all rays are projected normally.

For these reasons the attempt to deduce values of λ from the curves has been abandoned. But it is readily seen that the values of λ cannot be the same or nearly the same, for different materials. The distance from the origin at which the curve becomes appreciably straight will clearly be some measure of the penetration of the rays : it is evident then that λ for carbon and tin is less than for lead and aluminium.

Experiments are in preparation in which the complicated effects of the corners will be eliminated, and, it is hoped, greater accuracy obtained. It may then be possible to determine I_0 and λ, and to ascertain whether v_1 is or is not accurately zero.

§ 8. Observations of the same nature were subsequently made in which the boxes were surrounded by some 150 kilos. of lead forming a layer about 3 cms. thick ; this screen should reduce by 2/3 the intensity of Cooke's penetrating radiation.

The resulting curves are given in fig. 2, Pl. VII., and the values of s and v_2 deduced from them in Table I., columns 4 and 5. It is to be noted that the value of s is reduced in the cases of tin, carbon, zinc, and platinum, but remains unaltered in lead and aluminium. That is to say, in the former materials, the intensity of the radiation proceeding from the walls themselves is reduced by a screen outside the walls. We have thus strong confirmation of the conclusion reached by Wood [*], that part of the spontaneous radiation from ordinary

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[*] A. Wood, Proc. Camb. Phil. Soc. vol. xii. p. 477 (1904).

materials is secondary radiation excited by external pene-
trating rays. In the case of platinum all the absorbable
radiation appears to be secondary ; the conclusion is remark-
able, but I have no reason to think that the experiments with
platinum are less accurate than any of the others. It
will be noted that the value of v_2 is not the same for all
metals either in column 3 or 5, as might have been expected
if the radiation corresponding to v_2 was external to the
apparatus. But the variations mentioned in § 5 must be
borne in mind. In addition, it should be pointed out that
the value of v_2, being calculated as the difference of two
comparatively large quantities, is liable to considerable error,
and that the introduction of a correction for the effect of the
corners of the box would cause a notable alteration.

§ 9. It was now considered that sufficient evidence had
been obtained of the existence of easily absorbable rays from
ordinary materials ; experiments were therefore devised by

Fig. 3.
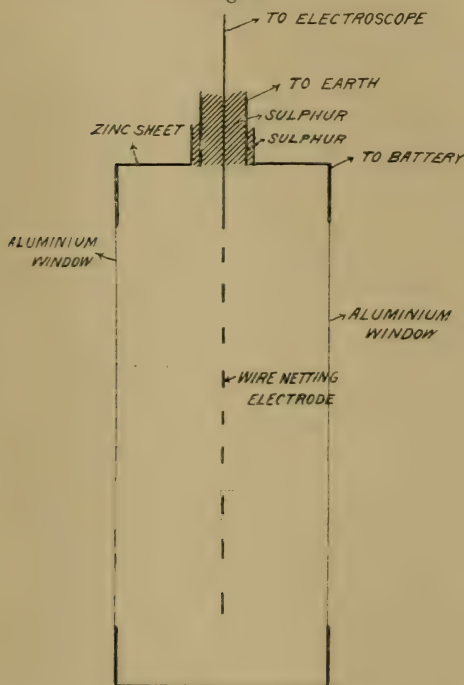
which it was hoped to establish more directly the existence
of the rays and to arrive at some knowledge of the value of λ.

A rectangular vessel of zinc sheet (fig. 3) was constructed,

$15 \times 15 \times 6$ cms., in one side of which was a window 12×12 cms. covered with a sheet of thin aluminium-foil ; the thickness of the foil used in most of the experiments was ·00033 cm.; similar results were obtained with a thickness of ·00053 cm., and also with a sheet of very thin paper rendered conducting with graphite.

The observations consisted in ascertaining the leak through the vessel when no solid body was within a distance of one metre from the window, and then noting the effect of bringing up to the window a plate of the material the radiation from which was to be investigated.   At first the vessel was charged to a high negative potential, but troubles occurred which were traced to the deposition on the surface of the " excited activity " which is known to collect on all negatively charged bodies exposed to the atmosphere.   The amount of the excited activity, the rays from which could penetrate the window, varied with the freedom of access of the outside air to the vessel, and hence the leak was constantly changing.   In the later experiments the vessel was always charged positively.

§ 10. If our conclusion is correct that the influence of the walls on the ionization consists of rays projected from them absorbable by some 6 cms. of air, we ought to be able to detect the influence of the rays through the aluminium foil, which is equivalent in surface density to ·69 cm. of air. Accordingly when plates of lead, tin (foil or plate), aluminium, zinc, iron, brass, copper, graphite, wood, or glass were brought up to the window, an unmistakable increase in the leak was detected—in the case of tin the increase was as much as 30 per cent.   It is clear that the increase caused by any material will rise with the penetrating power as well as with the intensity of the rays given off from it ; it is, then, in no way inconsistent with Table I. that tinfoil gave a larger effect than lead, and carbon an effect nearly as great.   With some materials, such as paper and ebonite, the increase was so small that it cannot be positively asserted to exist.

Strutt [*] has found large differences between the effects obtained with different specimens of the same material, but though I have tried several samples of each of the materials mentioned, appreciably the same value was obtained for each.   Strutt must have been extraordinarily unfortunate in securing samples contaminated with radioactive impurities, for McLennan, Righi, and Wood have found remarkable constancy in the value given by different samples of the same metal.

[*] Strutt, ' Nature,' vol. lxvii. p. 369.

Tinfoil and tin-plate gave notably different values, but tinfoil is, I believe, an alloy ; specimens of foil and plate agreed amongst themselves. Only in the case of wood were large differences found, but so porous a material, after lying some time in a laboratory where considerable quantities of pure radium salts have been used, might be expected to acquire a spurious radioactivity.

§ 11. An attempt was made to measure the coefficient of absorption of the rays. Three methods were employed :—

(1) By moving the plate away from the window so as to interpose a layer of air of known thickness.

(2) By interposing a layer of aluminium-foil.

(3) By cutting a second window in the other side of the zinc vessel, covering it with a layer of aluminium-foil of different thickness, and comparing the increase caused by the plate first before one window and then before the other.

In these ways we can show that the coefficients of absorption of the rays from different materials are not the same, and obtain some estimate of their value. But the attempt to ascertain the value accurately is frustrated by the difficulty we have met before : we do not know the relation between the intensity of the rays from the surface and their inclination to the surface normal. Thus, if we place a layer of thickness d and absorption λ' on the face of the material, the effect of the rays will be cut down in the ratio $e^{-\lambda'd}$ if the rays are projected normally, and in the ratio $\int_0^1 e^{-\frac{\lambda'd}{y}} dy$ if they are equally projected in all directions. In Table II.

TABLE II.

$\lambda'd$.	Ratio.	
	A.	B.
0·25	0·78	0·52
0·5	0·61	0·34
0·75	0·47	0·26
1·00	0·37	0·15
2·00	0·135	0·038

the ratios of reduction are given for different values of $\lambda'd$, (A) when the rays are projected normally, (B) when they are emitted equally in all directions.

In what follows we shall calculate all quantities on the assumption that the rays are ejected normally ; we shall thus get a qualitative description of the phenomena which will not be quantitatively correct. We will give one set of readings to exemplify each method.

Method 1.—Tin-plate: window ·00033 cm. of aluminium.

Distance of plate from window=x.	Increase of leak=I.	$\log_e 1000$ I.	$\lambda = \dfrac{\log_e \mathrm{I}_1 - \log_e \mathrm{I}_2}{x_2 - x_1}$.
0·5 cm.	·021	3·04	
			> .... 0·6
0·75 ,,	·018	2·89	
			> .... 0·72
1·00 ,,	·015	2·71	
			> .... 0·66
1·50 ,,	·008	2·08	

Mean   0·66

The value of λ obtained by moving the plate from 0 to ·5 cm. was always abnormally small and therefore rejected.

Method 2.—Lead.

I=increase of leak through one leaf over window.	$\log_e 1000$ I.	I′=increase with two leaves over window.	$\log_e 1000$ I′.	$\lambda'(\mathrm{Al})$*	$\lambda(\mathrm{air})$*
·023	3·14	·0095	2·23	2700	1·3

The increase given by zinc and aluminium was too small to be measured by this method.

Method 3.—Lead.

Window covered by foil ·00033 cm. thick.	Window covered by foil ·00053 cm. thick.	$\lambda'(\mathrm{Al})$.	$\lambda(\mathrm{air})$.
I = ·023	I′ = ·014	2500	1·18

In the first column of Table III., column 1, are given the values of λ deduced from the curves by a simple geometrical method on the assumption that all rays are projected normally ; in columns 2, 3, 4 are given the values found by methods 1, 2, and 3 above. In Table IV. the increases which should be caused by the various materials placed outside the window are calculated on the same assumption and compared with those actually found.

§ 12. The agreement in Table III. and in Table IV. is only qualitative ; but it must be remembered that the assumption on which the figures are calculated is almost certainly incorrect.

* The absorption coefficients for air and aluminium are compared throughout on the assumption that they are proportional to the densities of those substances.

TABLE III.

Material.	Absorption Coefficient.				
	1	2	3	4	Mean.
Lead	0·91	1·28	1·3	1·18	1·17
Tin	0·45	0·66	0·71	0·8	0·65
Carbon	0·27	0·52	0·47	0·52	0·44
Aluminium	0·61	...	...	1·00	0·8
Zinc	0·36	...	...	0·7	0·58
Platinum	0·56	...	...	...	0·56

TABLE IV.

Material.	Increase of Leak.	
	Observed.	Calculated.
Lead	0·023	0·0245
Tinfoil	0·025	0·0305
Aluminium	0·0075	0·0084
Zinc	0·0055	0·0077
Carbon	0·020	0·022

But I think the experiments have proved sufficiently that the coefficient of absorption is not the same for the rays from different materials—that each material emits rays of a quality peculiar to itself.

The importance of this conclusion, if accepted, can hardly be overestimated. If the rays from different materials are of different quality, it is impossible that they should be given out by traces of a radioactive impurity common to all. Again, the presence of a radioactive impurity—a suggestion that has received considerable support—seems hardly able to explain the remarkable constancy of the intensity of the radiations emitted by samples of metals derived from different sources.

§ 13. Experiments in which the plates of metal outside the window were backed with slabs of iron 2·5 cms. thick confirmed the conclusion previously reached that part of the radiation from zinc, tin, and carbon is secondary, excited by a penetrating radiation from outside.

§ 14. No evidence whatever could be obtained of the existence of rays from ordinary materials more penetrating than those which have hitherto been considered. If the window of the box was covered with a layer of aluminium ·003 cm. thick, no increase amounting to 2 per cent. of the normal leak was caused by placing before the window any of the ordinary materials used in these experiments. Such a thickness of aluminium would scarcely cause an appreciable diminution in the intensity of rays as penetrating as the β rays from radium or uranium. Hence, again we are led to the conclusion that v_1 of § 6 is zero.

§ 15. The conclusions which are considered to be established by the experiment described in this paper may be put briefly as follows :—

(1) That the influence, which the walls of a containing vessel are known to exert upon the spontaneous ionization of the enclosed air, may be attributed to a radiation proceeding from the walls.

(2) That part of this radiation from certain materials, such as tin, zinc, graphite, and platinum, is analogous to the secondary radiation excited by Röntgen and other rays, being caused by the penetrating ionizing rays which Cooke has cut off with thick lead screens.

(3) That the absorption coefficient of air for this radiation is comparable with that of air for the α rays from radium.

(4) That the absorption coefficient is different for different materials, and hence it is unlikely that the radiation is due to radioactive impurities ; it is more probably an inherent property of the material *.

(5) That there is no evidence of the existence of rays from ordinary materials of a penetrating power considerably greater than that of the α rays from radioactive elements.

It is obviously a matter of great importance to determine the nature of the rays from ordinary materials—their charge, if they carry a charge, their velocity, and so on. But the investigation of these properties will tax the ingenuity of the ablest experimenter.

These experiments were carried out in the Cavendish Laboratory, Cambridge. When this is stated it is hardly necessary to add that a large portion of whatever success has attended them is due to the inspiring advice of Prof. Thomson.

Cambridge, Jan. 1905.

* Most of the radioactive elements give off an emanation. It is here suggested that all elements are radioactive to some degree. It might be expected that some would give off an emanation. All efforts to directly detect such an emanation from lead, zinc, and aluminium have hitherto failed.

LI. *Radioactivity and Chemical Change.* By NORMAN R.
CAMPBELL, *B.A., Fellow of Trinity College, Cambridge**.

§ 1. IN the course of the experiments described in the
preceding paper (see p. 531) the effect of cleaning
the surface of a metal on the intensity of the rays emitted
from it was investigated. It was found that no mechanical
cleaning with sandpaper produced a change, but that an
increase in the ionization caused by a metal was brought about
by cleaning the surface with an acid that dissolved it. This
observation suggested that chemical action might be attended
by the emission of ionizing rays, and further experiments
seemed to confirm the suggestion.

A shallow dish with a bottom of thin aluminium leaf was
placed above the window of the vessel described in § 9 of the
preceding paper; when certain interacting reagents were
placed in the dish a notable increase in the leak through the
vessel ensued.

§ 2. When these experiments had proceeded for a short
time, it was found that a similar effect due to chemical action
had been described by M. Gustave Le Bon†. But it was
soon remarked that the chemical actions which were attended
by the largest effect were those which gave out considerable
quantities of heat. Was it heat, and not chemical action, that
was the cause of the effects noticed?

The question seemed to be answered in the affirmative
when it was found that a metal plate heated to 80° placed
just outside the window caused an increase in the leak greater
than that given by the most violent chemical action; while
melting ice or an action attended by absorption of heat
caused a remarkable decrease of the leak.

§ 3. A similar increase in the leak through a vessel when
heated had been observed by Wood‡, and traced to a change
of the surface electrification of the insulating material between
the high potential vessel and the earthed guard-ring. A new
apparatus on a far larger scale was constructed in which the
device adopted by Wood for the cure of his trouble was
introduced.

The new vessel (fig. 1) was a cylinder of galvanized iron
1 metre in diameter and 19 cms. high. A sheet of wire
netting, which served as electrode, was suspended by a wire
passing through the insulation which was arranged as shown

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* Communicated by the Author.
† Gustave Le Bon, *Revue Scientifique*, Nov. 1902.
‡ A. Wood, *infra*, p. 550.

in the figure ; by this means all induction effects due to surface electrification on the insulation were avoided.

Fig. 1.
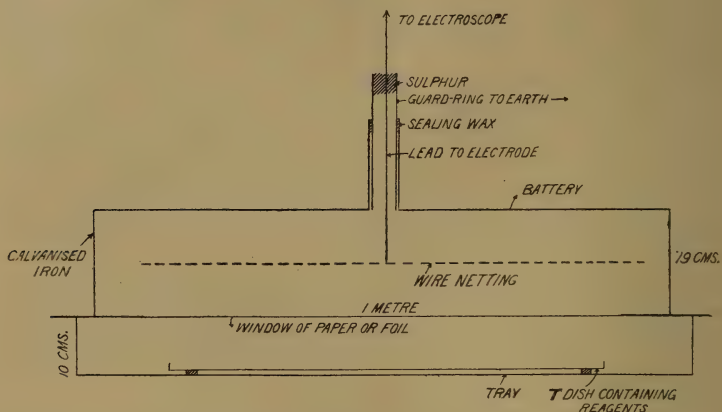
The bottom of the cylinder was covered by a window which in various experiments consisted of paper coated with graphite, aluminium-leaf, or tinfoil. Over the window was placed the shallow dish of galvanized iron T, which prevented the deposition on the window of the excited activity from the atmosphere, when the vessel was charged negatively. The electroscope was arranged as in the previous paper. The time required for the electrode to reach a potential of 1·3 volts was about one minute ; individual readings could be taken at the rate of four in five minutes.

It should be noted incidentally that the variations in the leak of this large vessel (see *ante*, § 5) were extremely small, never more than 5 per cent. The effects which were now observed when heat was applied by placing a bunsen-burner under the tray were somewhat complicated ; their nature and cause has not been fully elucidated, but only investigated so far as to enable the question to be decided whether chemical change is accompanied by radioactivity. It will be seen that the heat effect is clearly " instrumental,"—unconnected with the essential processes of ray-emission from ordinary material ; hence further search into its origin has been postponed until more urgent problems are decided.

§ 4. The tray was placed directly over the opening in the bottom of the cylinder which was not covered by a window.

Immediately after the bunsen had been lighted a change in the leak was observed whose nature and sign varied with

the conditions. In general a considerable increase (4 to 5 times) was observed : but on other occasions no change occurred or a slight decrease was noted. The determining condition appeared to be the amount of dust in the vessel : it was impossible to get the air of a vessel of the size and construction used quite dust-free, but when all precautions against dust had been taken the change on heating was very small.

If the bunsen was left in action the leak gradually returned to its normal value ; when that was reached no further change occurred whether the heating were continued or the flame removed and the vessel allowed to cool down.

The effect of chemical action was investigated by placing one of the reagents in a dish on the tray and pouring on it, through a tube, a solution of the other reagent. Gases which are evolved in chemical action are known to be strongly ionized, and hence in these experiments no actions in which gases are evolved were tried.

The following actions which are attended by no considerable heat-change were found to have no perceptible influence on the leak. (Class A) :—

Zinc on copper sulphate, ferrous sulphate oxidizing in air, ferrous sulphate oxidized by permanganate, barium chloride and sodium sulphate, ammonia and copper sulphate, sodium chloride dissolving in water.

The following actions which are attended by considerable heat-changes gave effects exactly similar to, but much less intense than, those observed when the tray was heated by a bunsen-burner : when the flame increased the leak, these actions increased it ; when the flame decreased the leak, so did the actions. (Class B) :—

The solution in water of calcium chloride, phosphorus pentoxide, sulphuric acid, ammonium sulphocyanide, sodium thiosulphate ; strong potash and sulphuric acid, strong soda and hydrochloric acid. Action of water on lime, and plaster of Paris.

§ 5. A window was used consisting of a large sheet of thin paper rendered conducting by graphite, or a dilute solution of calcium chloride, the tray being placed over the window.

The most extraordinary and complicated heat effects were observed in this case. In general an increase occurred when heat was applied, and a decrease—passing through zero and amounting to a leak against the field—during cooling. But these changes are not of importance for our present purpose, since no experiments were made on chemical actions with

this window in use. It is probable that the effects were caused by the indifferent conductivity of the paper.

§ 6. The window was formed of sheets of tinfoil ·0013 cm. thick, equivalent in surface-density to ·0035 cm. aluminium or 7·35 air. The effect of heat was of the same nature as in case 1, sometimes an increase, sometimes a decrease; but the absence of dust did not seem to diminish the effect. No change occurred while cooling down after heating.

Actions of Class A gave no effect.

Actions of Class B gave a small effect of the same sign as the heat effect.

Actions of Class C, *i.e.* actions evolving gases ($Zn + HCl$, $Cu + HNO_3$, $Fe + H_2SO_4$, $CaCO_3 + HCl$) gave a slight increase occurring about a minute after the action began. It is only reasonable to suppose that this was due to the diffusion of the ionized gases evolved through the small holes, which could not be avoided, in the tinfoil window.

§ 7. Window of aluminium-foil ·00053 cm. thick supported on wire netting.

The effect of heat was of the same nature as in case 3, sometimes an increase, sometimes a decrease ; but fortunately the cases in which the heat decreased the leak were far the most numerous. Only in these cases were the effects of chemical action tried.

Actions of Class A gave no effect.

Actions of Class B gave a small decrease.

Actions of Class C gave an increase beginning some time after the action, and greatly diminished by blowing a rapid current of air over the surface of the reagents. It was also found in this *and in every case* that the increase caused by a chemical action was in no way diminished by placing over the dish in which the action was proceeding a sheet of zinc ·08 cm. thick. If the effect had been due to rays emitted by the reacting substances, it is impossible that these rays should not have been diminished in intensity by absorption in the zinc. If the increase were due to heat or to the evolution of ionized gases, the zinc sheet might be expected to have little influence.

§ 8. We thus find :—

(1) That chemical actions which evolve no heat cause no increase in the leak.

(2) That chemical actions which evolve heat sometimes cause an increase and sometimes a decrease; the sign of the change being the same as that of the change due to heat.

(3) That chemical actions which evolve gases cause an increase, but that increase is more readily explained by the ionization of the gases than by a process of ray-emission.

§ 9. It is not difficult to explain how M. Le Bon arrived at the conclusion that chemical change is accompanied by radioactivity. All the actions with which he obtained large effects are actions evolving a considerable amount of heat. Not only would he be liable to the heat effect which I have been unable to avoid, but also the action, due to change of surface electrification on the insulation, to which reference is made in § 3 ; for the whole floor of his vessel was formed of a block of sulphur. He mentions no experiments made with the object of proving that the increase of leak was really due to chemical action, and not to some secondary cause.

§ 10. The nature of the connexion between the heating of the vessel and the increase of the leak has not been explained. It does not lie in any change of the insulation. The only part of the insulation which could become electrified is that exposed to the large difference of potential between the vessel and the guard-rings ; any induction effect due to a change of the electrification of this portion is precluded by the design of the apparatus. That the resistance of the insulation was not affected by the heat was proved by direct experiment.

There is considerable evidence of the existence of some kind of polarization at the walls of the vessel, as if ions moved up to the charged surface and clung there without giving up their charge. For a leak considerably larger than the normal is always found for a short time after the direction of the electric field is reversed, or put on afresh after being left off for some hours. It may be that ions from this polarized layer are driven off from the part of the surface directly heated, and give rise to a larger current. It is noticeable that pouring boiling water into the tray has no effect upon the leak ; the water may absorb the ions which are detached by the action of the heat.

On the other hand, it is noticeable that the effect of the heat is larger and more persistent when the heated surface is separated by a thin layer of foil or paper from the volume of air whose ionization is measured, than when no such layer is present.

§ 11. It seems then permissible to conclude :—

(1) That there is no evidence that chemical change is accompanied by radioactivity.

(2) That the increase in the spontaneous leak, the observation of which has led some investigators to suggest such a connexion, is due to the heating of the walls of the vessel ; the nature of the relation between the heating and the increased leak requires further elucidation.

Cavendish Laboratory, Cambridge,
        January 1905.

LII. *Spontaneous Ionization of Air in Closed Vessels and its Causes.* By ALEXANDER WOOD, *B.Sc. Glasgow, B.A. Cantab., late* 1851 *Exhibition Scholar of the University of Glasgow; Emmanuel College, Cambridge* [*].

Preliminary Discussion.

THE conductivity of air and other gases is now generally attributed to the presence of free ions, and as the free ions existing at any time are continually recombining, it follows that some agency must be at work splitting up the molecules and so maintaining the supply constant. Ionization of this kind, in which no artificial ionizing agent is employed, was at first thought to be an intrinsic property of the gas, and was known as " spontaneous " ionization. Later, however, the hypothesis that it was due to the action of a penetrating radiation constantly passing through the atmosphere in all directions was advanced, and has since received so much experimental support that it has come to be universally adopted. McLennan [†] surrounded the vessel in which he was measuring the ionization with water to a depth of about a foot, and found that this reduced the leak by about 35 per cent. At the same time Rutherford and Cooke [‡] surrounded their vessel with a screen of lead and observed a similar reduction of the leak. This pointed strongly to the existence of the hypothetical penetrating radiation. On the other hand, C. T. R. Wilson's " tunnel experiment " [§] rather tended to throw doubt on this conclusion. It was at this point, and before the publication of the more recent research by Cooke [||], that the present series of experiments was undertaken at the suggestion of Professor J. J. Thomson, whose kindly interest and helpful advice it is a real pleasure to be in a position to acknowledge. The earlier experiments were made with a view to confirming the results of McLennan and Rutherford & Cooke. The conclusion reached in the course of the investigation was that the natural ionization of the gas in a vessel had at least a threefold origin—part of it being due to a penetrating radiation from without, part to a secondary radiation from the walls of the vessel excited by the former, and part to an intrinsic radiation from the walls completely independent of the external radiation. The

existence of this last radiation is very important from its bearing on the question of the radioactivity of ordinary materials. Strutt * had previously found that the ionization of the gas in a vessel varied with the material of which the walls of the vessel were made. He found considerable differences, however, between different samples of the same material, and was inclined to conclude that the intrinsic radiation from the walls of the vessel was due to traces of radioactive impurities. Such very small traces would be sufficient to account for the effects observed, and the distribution of radium is so universal, that the explanation seemed highly probable. An obvious way to attack the problem was to repeat the experiments of Strutt, using different samples of the same metals, and to compare the results. This was done in the present investigation, and it was found that the order obtained for the various metals tried was exactly the same as that given by Strutt with one exception. Not only so, but the numbers obtained showed a considerable agreement, especially when the difficult character of the research is taken into account. Similar experiments were also performed in the Cavendish Laboratory by Campbell †, and the order obtained by him was again the same. McLennan's ‡ results for lead, tin, and zinc further confirm this order ; and so it seems reasonable to conclude that although radioactive impurities may cause differences between the values obtained for different samples of the same metal, those differences only partially conceal the fact that each metal gives out a specific intrinsic radiation, the amount of which varies very much from one metal to another. This is strong evidence in favour of the view that all matter is radioactive, and suggested the desirability of collecting more evidence bearing on the point. A search was accordingly commenced for any traces of emanation from the metals used in the experiments. This search was unsuccessful in obtaining any direct evidence of the existence of such emanations. It was found that if a current of air was passed over heated tin, something was carried off with it which was capable of increasing the ionization in a closed vessel. It was at first supposed that this might be an emanation, but when the gas from the heated tin was drawn through a tube to which a strong electric field was applied, it failed to affect the ionization in the vessel. The immediate cause of the increased ionization was therefore, presumably, free ions. Those may, of course, have been formed by the action of an emanation which itself

was too short-lived to reach the measuring vessel, and, indeed, this seems a very probable explanation. Other indirect evidence of the existence of an emanation will be cited towards the end of the paper; and it will be seen that, although the research cannot claim to have proved that ordinary matter is radioactive, it has at least established a strong probability in favour of that view.

The vessel in which the ionization was originally measured was of tin, about 13 cm. high and 6·25 cm. in diameter. The top of the tin was perforated to admit an ebonite plug about 2 cm. in diameter. Through this plug there passed a brass tube about 1 cm. in diameter containing a sulphur plug which carried the electrode. The vessel was kept charged to a potential of about 250 volts by means of a battery of small storage-cells, one terminal of which was connected to earth, while the other was connected to earth. The brass tube was also connected to earth, and as it projected right through the ebonite plug it prevented all possibility of leakage from the vessel to the electrode across the insulation. The electrode was a stout brass wire one end of which passed down almost to the bottom of the vessel, while the other was connected to the electroscope. As the electroscope has been very fully described by Wilson *, it is unnecessary to give the details of its construction here. It was found possible to make it extremely sensitive, but great sensitiveness was found to introduce compensating disadvantages which more than outweighed the gain. Thus minute changes in the potential of the cells made the gold-leaf very unsteady and rendered it difficult to take an accurate reading. The apparatus also became very sensitive to small differences of temperature, the convection currents caused inside the electroscope moving the gold-leaf through several divisions of the microscope-scale. For those reasons it was found most convenient to work with a sensitiveness of from 30 to 50 divisions per volt. This sensitiveness was sufficient to enable one to measure accurately the amount by which the electrode system became charged up in an interval of twenty minutes. To one end of a stout brass wire, the other end of which carried the gold-leaf of the electroscope, was attached a mercury-cup into which dipped the electrode-wire. Both wires were screened throughout their entire length with earthed metal screens so as to prevent induction effects from neighbouring charged

* Proc. Camb. Phil. Soc. vol. xii. part ii. (1903).

bodies. Just above the mercury-cup the screen was perforated to admit an insulated wire, the extreme end of which was bare. This wire was connected to a potentiometer arrangement, and so by inserting it into the mercury-cup it was possible to bring the electrode system to zero, or any other convenient potential. At first a paraffin-key was used

Fig. 1.
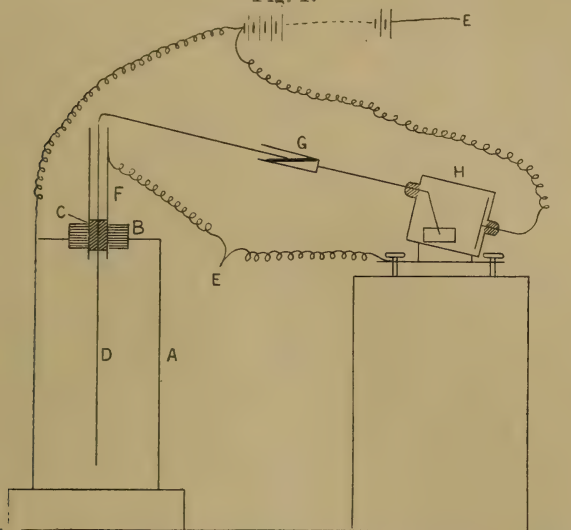
A. Vessel. B. Ebonite Insulation. C. Sulphur Insulation. D. Electrode.
E. Earth. F. Guard-tube. G. Mercury-cup into which potentiometer-wire may be inserted. H. Wilson Electroscope.

to connect the electrode and electroscope, but the paraffin caused endless difficulties by acquiring charges, and was ultimately discarded in favour of the arrangement described. This reduced the insulation to a minimum, and was found to work very satisfactorily. The method of experimenting was as follows:—The potentiometer-wire was inserted in the mercury-cup and the system was brought to zero potential, the position of the gold-leaf being read off on the microscope-scale. The potentiometer was then adjusted till the wire was brought to a potential of one volt, the position of the gold-leaf being read again. The difference between the two readings gave the sensitiveness of the electroscope in divisions per volt. The potentiometer-wire was now once more brought to zero, the reading of the gold-leaf taken and the wire withdrawn from the mercury-cup. This left the electrode connected up to the electroscope, but otherwise insulated. After twenty minutes the reading of the gold-leaf was taken again, the

potentiometer-wire reinserted, and the zero reading repeated.
If there was little change in the zero during the experiment,
the change was allowed for, but if the change was at all
large the reading was rejected. The sensitiveness was now
redetermined, and the voltage to which the electrode had
been raised was found from the deflexion and the mean of
the values of the sensitiveness before and after the twenty-
minute period. The capacity of the electrode system remained
the same, and so this voltage was a measure of the ionization
of the air in the vessel. The extremely small capacity of the
electroscope made it invaluable for measuring the small
charges obtained. A rough determination of the capacity of
the electrode, connecting wire, and gold-leaf showed it to be
about one electrostatic unit. It was found that when the
field was first applied, the leak in the vessel was considerably
above the steady value which it afterwards attained ; and to
avoid possible errors from this source the field was usually
left on all night, and the connexions so arranged that the
screens used to cut down the ionization could be put in
position and removed without taking off the field. The
probable cause of this phenomenon will be discussed later.
The lead screen used in the experiments was cylindrical in
form, and consisted of six layers of sheet-lead, making in all
a thickness of about 1 cm. The ionization in the vessel was
first measured without any screen—the mean of six twenty-
minute periods being taken as the correct value ; and the
vessel was then surrounded with the screen and the ionization
again measured. The two measurements were repeated
alternately several times, and from the values obtained the
mean ionization with and without the screen was found. The
results of five such sets of observations are given below. The
first set is given in full, and the initial falling off in the leak
will be noticed. In this particular case it lasted for nearly
two hours, but usually the time was much shorter.

Vessel Unscreened.

Zero.	Reading.	Deflexion.	Sensitiveness.	Leak.
14·0	65·5	31·0	30·00	1·930
32·0	53·0	21·0	28·25	0·730
35·0	52·5	17·5	28·50	0·610
33·5	51·0	17·5	29·75	0·590
32·5	49·0	16·5	29·25	0·564
32·0	46·0	14·0	28·37	0·493
28·0	40·5	12·5	27·37	0·464
28·5	41·5	13·0	26·00	0·490
28·5	40·5	12·0	25·50	0·471
29·0	41·0	12·0	26·00	0·461
28·0	41·0	13·0	25·50	0·510

In this experiment, and usually throughout the research, the vessel was charged positively, but the field was occasionally varied in sign. In no case did this change affect the results.

Vessel Screened.

Zero.	Reading.	Deflexion.	Sensitiveness.	Leak.
30·5	42·0	11·5	27·5	0·418
29·0	41·0	12·0	27·5	0·436
28·0	41·0	13·0	27·75	0·468
29·0	39·0	10·0	27·0	0·370
29·0	40·0	11·0	26·0	0·423
24·5	33·5	9·0	25·75	0·350
			Mean Leak ...	0·416

Vessel Unscreened.

Zero.	Reading.	Deflexion.	Sensitiveness.	Leak.
70·0	84·0	14·0	27·5	0·509
70·0	81·5	11·5	26·75	0·430
66·0	79·0	13·0	25·75	0·504
65·0	79·5	14·5	25·5	0·490
68·0	80·5	12·5	25·5	0·490
68·0	80·5	12·5	26·0	0·481
			Mean Leak ...	0·497

Vessel Screened.

Zero.	Reading.	Deflexion.	Sensitiveness.	Leak.
55·5	69·5	14·0	32·0	0·438
56·0	68·0	12·0	31·5	0·381
54·5	68·5	14·0	31·0	0·452
54·5	68·5	14·0	30·5	0·459
58·0	71·0	13·0	32·0	0·419
59·0	71·0	12·0	32·5	0·369
			Mean Leak ...	0·419

Vessel Unscreened.

Zero.	Reading.	Deflexion.	Sensitiveness.	Leak.
52·0	66·0	14·0	30·50	0·459
51·5	65·5	14·0	30·25	0·462
51·5	67·0	15·5	30·00	0·516
50·0	64·0	13·0	30·00	0·433
50·5	63·0	12·5	29·75	0·420
51·5	65·5	14·0	29·50	0·474
			Mean Leak ...	0·461

Summary of Results.

Vessel Unscreened.	Vessel Screened.
0·481	0·416
0·497	0·419
0·461	—

Mean ...0·48   Mean ... 0·42

Reduction of Ionization due to screen, 12 per cent.

The following is a summary of another similar series of five sets of readings :—

Screened.	Unscreened.	Screened.	Unscreened.	Screened.
0·529	0·692	0·557	0·727	0·586
0·588	0·666	0·571	0·666	0·586
0·589	0·680	0·586	0·702	0·612
0·599	0·640	0·586	0·656	0·642
0·603	0·677	0·628	0·661	0·628
0·573	0·651	0·571	0·661	0·612
Mean .. 0·576	0·669	0·588	0·679	0·611

Summary.

Vessel Screened.	Vessel Unscreened.
0·59	0·67

Reduction of Ionization due to screen, 12 per cent.

It was thought that it might be interesting to extend these experiments to screens of different materials. Wood and paper were used, but the effect produced by them on the ionization in the vessel, if any, was so small as to be within the limits of experimental error. So far as one could judge, however, the wood gave a slight increase and the paper a slight decrease.

The apparatus used in the above experiments had one or two drawbacks. For instance, the paper screen referred to above consisted of old volumes of the Phil. Trans., and was about 45 cm. thick. If the ionization vessel was placed inside and the electroscope outside, the long connecting-wire which was necessary added considerably to the capacity of the electrode system, and so diminished the sensitiveness of the apparatus. On the other hand, if ionizing vessel, electroscope, and microscope were all enclosed within the screen, then a large quantity of material was necessary in order to get sufficient thickness all round. An attempt was therefore made to improve the apparatus previously used, so as to combine the great sensitiveness of the Wilson electroscope with the compactness of the self-contained apparatus commonly used in experiments of the kind. It was impossible to use the electroscope in its original form both as electroscope and ionizing vessel. The volume of air enclosed was so small and the capacity of the plate so large, that the leak was not measurable. The new apparatus was of brass, cylindrical in form, about 14 cm. high and 10 cm. in diameter. In the centre of this cylinder, supported on a thin quartz pillar, was a small rectangular brass plate set at an angle of 30° to

the vertical. Opposite this plate, and occupying as nearly as possible the same relative position as in the Wilson electroscope, was suspended a gold-leaf. Through an ebonite plug in the bottom of the cylinder a vertical wire passed, to the end of which a horizontal cross-wire was soldered, and this could be turned round so as to make contact with the plate. This charging rod was connected to one terminal of a battery of small storage-cells, the other terminal of which was connected to the case of the instrument and to the gold-leaf. This arrangement was the inverse of that of the Wilson electroscope. In this case the gold-leaf was kept at constant potential (zero) while the potential of the plate was allowed to vary. By altering the tilt of the instrument by means of levelling-screws attached to the base, and by varying the potential of the plate, the combination of tilt and potential giving maximum sensitiveness was found. In this position, a variation of one volt in the potential of the plate gave a deflexion of the gold-leaf through about 18 divisions. The potential of the plate which gave this sensitiveness was about 200 volts. It was charged up by means of the charging rod and then insulated. As the potential of the plate fell the gold-leaf was deflected from its initial position, and as this position was in a region of uniform sensitiveness, the deflexion was a measure of the change in potential of the plate, and consequently of the ionization in the vessel. As the reliability of the results depended entirely on the insulating power of the quartz pillar, this was very carefully tested, and great care was taken to have it as perfect as possible. It was obtained fresh from the blowpipe and put in position with forceps, great care being taken not to finger it. A small capsule containing calcium chloride was enclosed in the apparatus, and it was found that with those precautions the quartz insulated splendidly. It was tested as follows : The leak was first of all determined in the ordinary way. Some fine wire was now twisted several times round the quartz pillar, about halfway down, and connected to the charging-rod. In this case there could be no leak to earth by means of the quartz. The leak was now redetermined, and the two values were practically identical. This new form of apparatus (fig. 2, p. 558) was not without new defects of its own. In the first place, it was extraordinarily sensitive to small changes of temperature. The large volume of the vessel in which the gold-leaf was now enclosed meant a great increase of convection-currents whenever any non-uniform changes of temperature took place. This defect was remedied by encasing the instrument in asbestos, and surrounding it with a cardboard screen fitted

with mica windows to shield it from draughts and air-currents. Then, again, the field inside the vessel was far from being uniform, as the plate was not symmetrical with respect to the cylinder. This, probably, accounted for a small

Fig. 2.
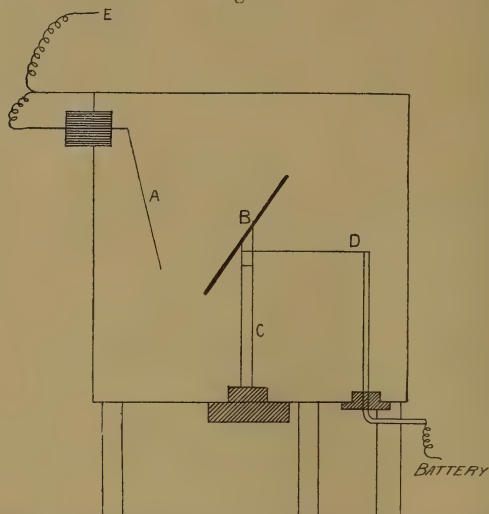
A. Gold-leaf.   B. Brass Plate.   C. Quartz Pillar.   D. Charging Rod.
E. Earth.

oscillation of the leaf which was noticed. On the other hand, the apparatus was complete in itself, very easy to surround with screens of different materials, and sufficiently sensitive to give a leak easily measurable in an hour. It was found that the following method of experiment gave the most satisfactory results. The plate was charged up, the charging-rod left in contact with it, and after a short time the position of the gold-leaf was read every half-minute for five minutes. The mean of these readings gave the zero position of the gold-leaf, and eliminated small errors due to its vibratory motion. The plate was then insulated, and the position of the gold-leaf again read in the same way. The mean of these readings gave its initial position, and a similar set at the end gave its final position, and so the deflexion for the period. At the end of the experiment the zero was redetermined by bringing the rod again into contact with the plate, and any small change was allowed for. With this arrangement the experiment was repeated. Blocks of wood were piled round the instrument to a depth of 29 cm., and a small but distinct increase of the ionization was observed.

Summary of Observations.

	Initial Reading.	Final Reading.	Change of Zero.	Deflexion.
Without Screen.	32·0	72·7	1·5	39·2
	22·3	72·3	6·4	43·6
	23·2	63·7	0·4	40·1
			Mean ...	41·0
With Wooden Screen.	27·1	75·1	3·0	45·0
	25·0	72·6	2·6	45·0
	22·4	67·3	0·4	44·6
			Mean ...	44·9

Increase due to Screen, 8·5 per cent.

It seemed probable that this increase was due to some radiation from the wood, and so the effect of interposing a thin lead screen between the wood and the instrument was tried. The apparatus was surrounded with a screen of sheet-lead about 2 mm. thick, and the ionization carefully measured. The wooden blocks were now piled up outside the screen, and the ionization was once more measured. The measurement without the wooden blocks was now repeated, and it was found that the wood no longer produced any appreciable effect. Screens of iron and paper were also tried, two thicknesses of iron being used. The first screen consisted of a cast-iron box about 1 cm. thick, which completely surrounded the apparatus. The results obtained with it are summarized in the following table :—

	Initial Reading.	Final Reading.	Zero Change.	Deflexion.
Without Screen.	9·8	54·4	1·3	45·9
	14·4	65·5	6·0	45·1
	21·1	67·5	0·5	45·9
	22·7	71·6	2·2	51·5
	22·9	69·2	6·7	53·0
			Mean ...	48·3
With Iron Screen.	11·2	55·5	3·1	41·2
	20·2	70·2	3·6	46·4
	19·6	68·1	3·3	45·2
			Mean ...	44·3

Diminution due to Screen, 8·0 per cent.

A second iron screen, 5 cm. thick, gave the following readings :—

	Initial Reading.	Final Reading.	Zero Change.	Deflexion.
Without Screen.	17·4	79·3	2·9	50·0
	.21·0	63·8	8·6	50·6
	22·1	72·1	0·3	50·3
	18·8	65·3	1·9	44·6
			Mean ...	48·9
With Iron Screen.	22·3	64·7	1·4	43·8
	11·9	48·8	2·4	34·5
			Mean ...	39·1

Diminution due to Screen, 20·0 per cent.

Thus the second screen, although about five times the thickness of the first, produced a diminution only a little more than twice as great. The paper screen was now retried—the apparatus being now surrounded with books to an average thickness of about 35 cm. The effect on the ionization was extremely small, but a diminution amounting to about 3 per cent. could be distinctly observed in the mean of a number of experiments. Thus the experiments show that, with the exception of wood, all the materials tried as screens reduce the ionization, and there seems therefore to be every reason for concluding that the ionization is due, in part at least, to a radiation of a very penetrating kind coming from some external source. The result obtained in the case of wood may be very simply explained on the hypothesis that the increase of ionization is due to a radiation of very feeble penetrating power emitted by it, which more than compensates for the absorption produced by a screen of the thickness used. This hypothesis is further strengthened by the fact that the effect is entirely cut off by a thin screen of lead, and also by the fact that Cooke * found an exactly similar result with bricks. That every material does absorb the penetrating radiation to some extent, is rendered probable by the fact that even paper when in sufficient thickness was found to effect a small reduction of the ionization. It has been suggested by J. J. Thomson that every material both emits and absorbs a penetrating radiation capable of ionizing a gas, and that the effect produced on the ionization in a vessel by a screen of any material will depend on the ratio between the

* Phil. Mag. October 1903.

emission and the absorption for the material, and on the thickness of the screen. On this view, as the thickness of the screen is increased, the amount by which the radiation passing through it is altered approaches a constant value, and after a certain thickness has been reached, any further increase in thickness will not affect the quantity of radiation which passes through the screen. The result obtained with iron seems to confirm this view, as one screen five times the thickness of another produced a diminution of the ionization only a little more than twice as great. It is further borne out by the result obtained by Cooke, who found that, no matter how thick his lead screen, it was impossible to reduce the ionization by more than about 30 per cent. The increase found in the case of the wooden screen also seems at first sight to confirm this view, but the fact that the radiation producing this increase is entirely cut off by 2 mm. of lead shows it to be of quite a different character from that which we have been considering. This much may be regarded as certain, however, that the ionization in a closed vessel is in part at least due to the action of a very penetrating radiation from an external source. This leads naturally to the question of the source. Cooke tried, unsuccessfully, to obtain some evidence as to its direction by measuring the ionization in his vessel with a thick lead screen in different relative positions. The conclusion he came to was that the radiation came equally from all quarters. It seemed possible, however, that the more sensitive apparatus used in these experiments might reveal differences which his apparatus had failed to detect, and in any case the result was worth confirming. The apparatus was placed on the lower side of a strong vertical rectangular wooden framework, the two extremities of the lower side being mounted on two wooden blocks. The framework was so arranged that a rectangular screen of lead, about 30 cm. square and about 3·5 or 4·0 cm. thick, could be slipped under the lower side, or rested on the upper side, and in those two positions was equidistant from the centre of the apparatus, and so in each case subtended the same angle there. Thus radiation coming equally from above and below would be equally absorbed by the screen in the two positions. The method of measuring the ionization in the vessel was slightly altered, with some gain in accuracy. Between the earth and the earthed terminal of the battery was inserted a potentiometer arrangement by means of which the voltage of the battery could be opposed by any fraction of 4 volts. At the end of the period of time over which the leak was being measured, the position of the gold-leaf was carefully noted. The

charging-rod was then brought into contact with the plate, and the potentiometer adjusted until the leaf was brought back to the position noted. The potentiometer-reading then gave the difference between the initial and final potentials of the plate. This method assumed of course that the potential of the cells remained constant in the interval, an assumption justified by the fact that it was found to alter only by a very small fraction of a volt between Monday morning, when they were first used, and Saturday morning, when they were recharged. The first experiments were performed in a room on the top floor of the building. They were inconclusive, but seemed to indicate a greater screening action with the lead above than with it below. The effect was so small that although a large number of experiments were performed— fifteen—it was thought that it might be due to errors of experiment, or even, if real, might be traceable to local conditions. The experiments were therefore repeated in the basement. In this case the effect observed was of the same order, but opposite in direction, the mean of a large number of experiments indicating a greater screening action with the screen below. The failure of those experiments to give definite results is not very surprising, when we reflect that the total reduction of ionization due to a plane screen—assuming the radiation to come equally from all directions—is only about 6 or 7 per cent. To attain an accuracy sufficient to detect a variation in so small a quantity is extremely difficult in work of the kind. The great obstacle in the way of any attempt at accuracy in those experiments, and in all experiments of the kind, is that the ionization in a vessel is subject to well-marked variations, for which it has hitherto been found impossible to assign a cause. The leak in a closed vessel varies in an irregular manner not only from day to day, but even from hour to hour. During the course of the present investigation, attempts were made to connect those variations with variations in meteorological conditions, such as pressure [*], temperature, clearness of sky, and the like, but without success. The changes sometimes amounted to as much as 10 per cent.; and were frequently found to accompany sudden changes of weather. For instance, in several cases when a dull day suddenly became fine, an increase of the leak was noticed, but this rule was not universal. That this particular effect may have been largely, if not wholly, due to a temperature effect on the insulation, will be seen from experiments to be subsequently described.

[*] Elster and Geitel's experiments have led them to connect the variations with variations of pressure.

Having arrived at the conclusion that the ionization in a closed vessel is partly due to a penetrating radiation from without, there remains the further question of the part, if any, played by the walls. The question is one of the very greatest interest and importance from its bearing on the question of the radioactivity of ordinary materials.

C. T. R. Wilson * measured the ionization of various gases in small sealed vessels, and found that, except in the case of hydrogen, the ionization is approximately proportional to the density. This is what would happen if the radiation were a penetrating one from outside, or one from the walls, of sufficient penetrating power to pass from side to side of the vessel without complete absorption. Patterson found that in a large iron vessel the ionization varied directly as the pressure for pressures below 80 mms., while it was practically independent of the pressure for pressures greater than 300 mms. If the whole of the ionization in a closed vessel is due to a very penetrating radiation either from outside or from the walls, this result would be difficult to explain : but if we suppose that such radiation is responsible only for a part of the ionization, and that the remainder is due to a radiation from the walls of comparatively feeble penetration, the results can be very simply explained. For pressures greater than 300 mms. the gas is sufficiently dense to absorb all this feeble radiation, and so no increase of pressure, with consequent increase of density, will affect the amount of ionization which this radiation will produce. As the pressure is reduced, however, a point will be reached when the radiation will be able to traverse the vessel without complete absorption, and below this pressure the absorption and, presumably, the ionization produced will be proportional to the pressure. The conclusion that the ionization is partly due to a radiation from the walls was further supported by some experiments of Strutt †. He measured the ionization in a vessel lined with different materials, and found that it varied considerably, being large for lead and tin, and small for zinc and aluminium. This pointed to the existence in each case of a radiation peculiar to the material.

The experiments next undertaken had a twofold object :—
(1) To ascertain whether samples of the same metals as those used by Strutt would give values of the ionization similar to those obtained by him ; and (2) to ascertain if possible whether the radiation from the walls of the vessel was primary or secondary in character, whether it was an intrinsic property of

* Proc. Roy. Soc. lxix. p. 277 (1901).
† Phil. Mag. June 1903.

the material or a secondary radiation excited by the more pene-trating kind. This latter point was attacked by determining the action of a lead screen on the ionization in vessels of different materials. If the radiation from the walls be primary and an intrinsic property of the material, it will be unaffected by an external screen, and so, in the case of a metal possessing this property to any extent, we should expect a lead screen to produce comparatively little effect on the ionization, while in the case of a metal possessing the property to a less extent, or even lacking it altogether, we should expect the same screen to produce a greater effect. Thus the pro-portionate reduction of the ionization due to a screen would, on this view, vary considerably with the material of which the vessel was constructed. On the other hand, if the radiation from the walls be secondary in character, excited by the more penetrating kind, then we should expect both to be cut down in the same proportion, and hence, even if the absolute amount of secondary radiation excited by the penetrating radiation varied with the material, the proportionate reduction due to a given screen would still be the same for all. Thus, if we measure the proportionate reduction of ionization due to any particular screen in vessels of the same dimensions but of different materials, our results ought to enable us to say whether the radiation coming from the walls is primary or secondary in character.

The apparatus used was similar to that employed in the first experiments. The vessels were all of the same dimen-sions—6·25 cms. in diameter and 13 cms. high. The electrode was of the form previously described, and each vessel was fitted to receive it, so that the same electrode and insulation was used throughout the whole series of experiments. As the electrostatic screens for the electroscope and electrode wire were kept in the same positions, the capacity remained the same all through, and the ionization in any one vessel, as measured by the Wilson electroscope, could be directly com-pared with a similar measurement for any other. Instead of determining the sensitiveness of the electroscope at frequent intervals, and finding the potential to which the electrode had been raised, from the deflexion of the gold-leaf, on the assumption that the sensitiveness was uniform over a range of one volt, a modification of the method described on page 561 was used. The potentiometer-wire was inserted in the mercury-cup and the electrode system brought to zero. The wire was then withdrawn and, after twenty minutes had elapsed, the position of the gold-leaf was carefully read. The potentiometer-wire was then once more inserted, and the resistance adjusted until the gold-leaf was brought back to

the same position. The potentiometer reading then gave the
final potential of the electrode, and as it had initially been at
zero this was a measure of the ionization in the vessel. This
method of reading the electroscope gave results independent
of small changes in the zero and of variations in sensitive-
ness in different parts of the scale. With the volume of air
used—about 400 c.c.—it was found that the electrode was
charged to about 0·4 volt in twenty minutes, an amount
which could be measured with a considerable degree of
accuracy. As a rule five sets of four readings each were
taken with each vessel. During the first, third, and fifth,
the vessel was freely exposed to the penetrating radiation;
while during the second and fourth it was completely sur-
rounded by a lead, or other, screen. Means of each set of
readings were taken, and from them the mean leak with and
without the screen was calculated. This of course gave the
proportionate reduction due to the screen. The vessels were
made of lead, tin, iron, aluminium, and zinc. In confirma-
tion of Strutt's results, it was found that the ionization varied
very greatly with the material. The following table gives
the average value obtained for the various metals, on a quite
arbitrary scale. For purposes of comparison the correspond-
ing values obtained by Strutt, reduced to the same arbitrary
scale by equating the values for zinc, are placed alongside :—

Material.	Ionization.	Strutt's Value.
Lead	5·2	4·2
Tin	4·7	4·4 to 6·3
Iron	2·9	—
Aluminium........	2·7	2·7
Zinc	2·3	2·3

It will be at once admitted that the two sets of values
show a surprising agreement. This practically disposes of
the suggestion that the differences in the ionization in vessels of
different materials are due to the presence of radioactive
impurities. Probably enough impurities of the kind do play
some part. For instance, in the case of tin Strutt found a
considerable difference between different samples, while two
samples of platinum gave 3·8 and 7·4 respectively. A sample
of the same metal, tested in the course of this research, gave
2·4—a number much smaller than the smaller of the two
given by Strutt. That such discrepancies are due to the
radioactive impurities seems more than likely, but it seems
impossible on this hypothesis to account for the agreement
shown above. Campbell, working in the Cavendish Labora-
tory, has obtained some results, a summary of which he has
kindly placed at my disposal. He obtains an order for the
different metals identical with that given above. McLennan

too gives as the result of similar experiments, that lead produces an ionization twice that for zinc, while tin is intermediate between the two. This, again, is in agreement with the table given above. If the ionization produced by the various metals were really due solely to the presence of radioactive impurities, it is inconceivable that vessels constructed from different samples of the metals, and tested by entirely different experimental methods, should give results in such close agreement. It follows that the ionization due to a metal has a certain definite value which is characteristic of the metal—a conclusion which affords the strongest support to the hypothesis that all matter is radioactive.

The first screen used in these experiments was of lead, about 1·3 cms. thick. The diminution of the ionization in a lead vessel was measured in the way indicated above. With a view to getting some idea of the degree of accuracy of the results, three separate determinations were made. These gave 9 per cent., 13 per cent., and $11\frac{1}{2}$ per cent. respectively. As this was the smallest diminution observed for any metal, and as the probable error is greater the less the diminution, it is improbable that the figures given in the following tables as proportionate diminution are wrong by more than one, or, at the very most, two units. Two determinations of the proportionate reduction for the zinc vessel gave 26 per cent. and $26\frac{1}{2}$ per cent. respectively. A summary of the results for each metal is appended.

Lead.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·540	·490	·571	·518	·530
·507	·501	·545	·485	·540
·556	·485	·523	·461	·571
·534	·479	·540	·485	·530
Mean ...·534	·489	·545	·487	·543

Mean Ionization $\left\{ \begin{array}{l} \text{Unscreened... ·541} \\ \text{Screened ...... ·488} \end{array} \right.$ Proportionate Diminution *10* per cent.

Tin.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·496	·368	·461	·400	·479
·518	·387	·496	·354	·467
·496	·374	·502	·393	·507
·524	·406	·513	·393	·496
Mean ...·508	·384	·493	·385	·487

Mean Ionization $\left\{ \begin{array}{l} \text{Unscreened... ·496} \\ \text{Screened ...... ·384} \end{array} \right.$ Proportionate Diminution *22½* per cent.

Iron.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·298	·222	·280	·238	·291
·284	·238	·333	·182	·269
·305	·238	·276	·214	·261
Mean ...·300	·228	·287	·212	·274

Mean Ionization $\begin{cases} \text{Unscreened...} & ·287 \\ \text{Screened ......} & ·220 \end{cases}$ Proportionate Diminution *23½* per cent.

Aluminium.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·276	·222	·255	·207	·246
·261	·222	·246	·214	·249
·276	·206	·246	·230	·243
·261	·214	·255	·246	·250
Mean ...·268	·216	·250	·224	·247

Mean Ionization $\begin{cases} \text{Unscreened...} & ·255 \\ \text{Screened ......} & ·220 \end{cases}$ Proportionate Diminution *13½* per cent.

Zinc.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·238	·168	·261	·171	·222
·230	·173	·206	·165	·238
·238	·176	·217	·167	·254
·206	·157	·209	·169	·214
Mean ...·228	·168	·223	·168	·232

Mean Ionization $\begin{cases} \text{Unscreened...} & ·228 \\ \text{Screened ......} & ·168 \end{cases}$ Proportionate Diminution *26½* per cent.

Similar experiments were now made with an iron screen. This screen consisted of large rectangular blocks of iron about 5·5 cms. in thickness. The results are tabulated in the same way as before.

Lead.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·529	·449	·566	·479	·518
·551	·445	·518	·461	·496
·540	·485	·502	·485	·507
·524	·491	·524	·485	·512
Mean ...·536	·467	·527	·477	·508

·Mean Ionization $\begin{cases} \text{Unscreened...} & ·524 \\ \text{Screened ......} & ·472 \end{cases}$ Proportionate Diminution *10* per cent.

Tin.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·561	·443	·437	·449	·518
·546	·443	·485	·437	·491
·529	·425	·461	·406	·491
·544	·449	·485	·425	·502
Mean ...·545	·440	·467	·429	·500

Mean Ionization $\begin{cases} \text{Unscreened... ·504} \\ \text{Screened ...... ·435} \end{cases}$ Proportionate Diminution $12\frac{1}{2}$ per cent.

As the mean of the third set of observations was so far from agreement with the means of the first and fifth, and for another reason to which attention will be called later, another series of readings was taken with the same vessel.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·513	·496	·561	·496	
·546	·467	·546	·502	
·551	·461	·529	·496	
·548	·473	·561	·456	
Mean ...·539	·474	·549	·486	

Mean Ionization $\begin{cases} \text{Unscreened... ·544} \\ \text{Screened ...... ·480} \end{cases}$ Proportionate Diminution $11\frac{1}{2}$ per cent.

Iron.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·294	·222	·276	·238	·284
·298	·238	·261	·238	·319
·333	·222	·270	·214	·319
·312	·222	·291	·214	·325
Mean ...·309	·226	·274	·226	·312

Mean Ionization $\begin{cases} \text{Unscreened... ·292} \\ \text{Screened ...... ·226} \end{cases}$ Proportionate Diminution 24 per cent.

Aluminium.

Unscreened.	Screened.	Unscreened.	Screened.	Unscreened.
·276	·261	·291	·230	
·291	·276	·304	·206	
·304	·271	·291	·217	
·291	·264	·246	·222	
Mean ...·290	·268	·283	·219	

Mean Ionization $\begin{cases} \text{Unscreened... ·287} \\ \text{Screened ...... ·243} \end{cases}$ Proportionate Diminution 15 per cent.

The results for the lead screen may be more briefly tabulated thus :—

Material.	Ionization (unscreened).	Ionization (screened).	Diminution (absolute).	Diminution (proportionate).
Lead	·541	·488	·053	10 p. cent.
Tin	·496	·384	·112	$22\frac{1}{2}$,,
Iron	·287	·220	·067	$23\frac{1}{2}$,,
Aluminium...	·255	·220	·035	$13\frac{1}{2}$,,
Zinc	·228	·168	·060	$26\frac{1}{2}$,,

From those figures some facts about the cause of the ionization in the vessels may be at once deduced. If part of the ionization were due to the external radiation, and the remainder to an intrinsic radiation from the walls of the vessel, then the first would be cut down by the same amount in each case by the screen, while the second would remain unaltered. Hence the absolute diminution of ionization would be the same for all the materials. We see, however, from the above table that it varies from ·035 in the case of aluminium to ·112 in the case of tin. Further, as the ionization produced by the external radiation would be the same in all the vessels, the greater the ionization in a vessel the greater would be the proportion of it due to intrinsic radiation, and consequently, the smaller the ionization the greater would be the proportionate diminution due to a screen. We find, however, that in the case of aluminium, a material for which the ionization is small, the proportionate diminution is small also. If, on the other hand, part of the ionization were due to the external radiation and the remainder to a secondary radiation from the walls excited by it, then we should expect both primary and secondary radiation to be diminished in the same ratio by a screen. Therefore, although the amount of secondary radiation excited by the primary might, and probably would, vary with the material, still, whatever its amount, we should expect it to be cut down in the same proportion as the primary, and so we should expect the ionization in all the vessels to be diminished in the same ratio by a screen. One glance at our table shows us how far this is from being the case, the proportionate diminution for lead being little more than one third of that for zinc. Our results then point to a threefold origin for the ionization in closed vessels. Part of it is due to the external radiation, part to a secondary radiation from the walls of the vessel excited by it, while the remainder is due to an intrinsic radiation from the walls, altogether independent of the external radiation. Considering only the radiation from the walls, where that

is mainly secondary the proportionate diminution due to a screen will be greatest, while in those cases where an intrinsic radiation from the walls predominates the proportionate diminution due to a screen will be least. Applying this criterion to the metals examined, we find that in tin, iron, and zinc the radiation must be largely secondary, while in lead and aluminium it must be largely intrinsic. The results for the iron screen are briefly summarized in the following table :—

Material.	Ionization (unscreened).	Ionization (screened).	Diminution (absolute).	Diminution (proportionate).
Lead	·524	·472	·052	10 p. cent.
Tin	·504	·435	·069	12½ ,,
Iron	·298	·226	·072	24 ,,
Aluminium...	·287	·243	·044	15 ,,

It will be seen that the proportionate diminution due to this screen is in three out of the four cases identical with that produced by the lead screen. The exception is tin, where the difference is very marked indeed. Nor is this discrepancy due to experimental error, as the determination was carefully repeated and confirmed.

Having thus established a probability in favour of the view that all matter is radioactive, a systematic search for emanations from the various metals used was commenced. The method adopted was to heat the vessel and endeavour by this means to drive out any emanation which was stored up in the metal. Preliminary experiments gave every indication of the existence of such emanations. When the vessel was heated the leak increased very greatly at first, but soon fell off again, and when the vessel was allowed to cool, the leak fell to a very low value, which seemed on the face of it to correspond to the leak given by a de-emanated metal. The apparatus employed was exactly the same as that previously described, except that care was taken to screen the electroscope from the source of heat, which in this case was a Bunsen burner. Lead, brass, and tin were used, and all three gave a result similar in kind but differing in degree and in the time constant. The apparatus was now duplicated, and two vessels were used, each with an electroscope. The vessels were fitted with two delivery-tubes ; in the first vessel one of those tubes was connected to a long indiarubber tube which passed out into the open air, while the other communicated with one of the delivery-tubes of the second vessel ; the second tube from this vessel was connected to a pump. A current of air could

thus be drawn from the first vessel to the second at any desired speed. The air as it entered the vessels was freed from dust by passing through a plug of cotton-wool, and dried by passing through another tube containing calcium chloride. A steady current of this dry, dust-free air was now drawn through the two vessels, and the ionization in each measured. The first was then heated, and measurements taken in the second while the heating was in progress, and in both when the heating had been stopped. As tin had given the most marked effect in the preliminary experiments, this metal was used for the first vessel, and as zinc had given the smallest normal leak it was chosen for the second as being likely to make any small increase in the leak in the second vessel easy of detection. It was hoped by this method to ascertain whether the increase of the leak noticed on heating a vessel was due to something of the nature of an emanation which could be drawn from one vessel to another. The experiment proved conclusively that such was indeed the case. When heat was applied to the tin vessel, the ionization increased not only in this vessel but also in the zinc one, showing that whatever was causing the increased leak in the first vessel was capable of being carried over in a current of air from one vessel to another. So far the source of heat used was a Bunsen burner, but this was found to be rather inconvenient, as it was impossible to be sure of reproducing the same temperature. This defect was obviated by wrapping a thin sheet of micanite round the vessel and winding on this a resistance of german-silver. By passing a current of about 2 amperes through this, the vessel could be heated to about 120° C. The insulation was thoroughly tested up to this temperature, and found to act perfectly. The foregoing experiments were now repeated with the apparatus thus improved, and similar results were obtained. They were accompanied, however, by a suspicious circumstance which at once threw doubt on their genuineness and led to the detection of a source of error previously unsuspected. The leak at first increased when heat was applied. If now the temperature was kept constant, the leak fell off again to its initial value, and remained constant at that value until the source of heat was removed. While the vessel was cooling, on the other hand, the leak fell to an extremely low value and in some cases it was even reversed in sign. It was this manifest absurdity of a leak against the field which suggested that the results were vitiated by some source of error. This source was by no means easy to find, but the effect was ultimately traced to the insulation. An enlarged diagram

of the arrangement of the insulation is given in fig. 3. Apparently when the vessel is charged up the ebonite takes some charge also. This charge acts inductively on the electrode, sending electricity of the same sign to the gold-leaf, and so apparently increasing the leak. This would explain why the leak always seemed to be greatest when the field was first applied. If now the ebonite is able to hold more charge when hot than when cold, the observed phenomena admit of a very simple explanation. While the temperature of the ebonite is rising the charge on it will be increasing, and so the apparent leak will be large ; while the temperature is constant the leak will be normal, and while the ebonite is cooling the reverse effect

Fig. 3.
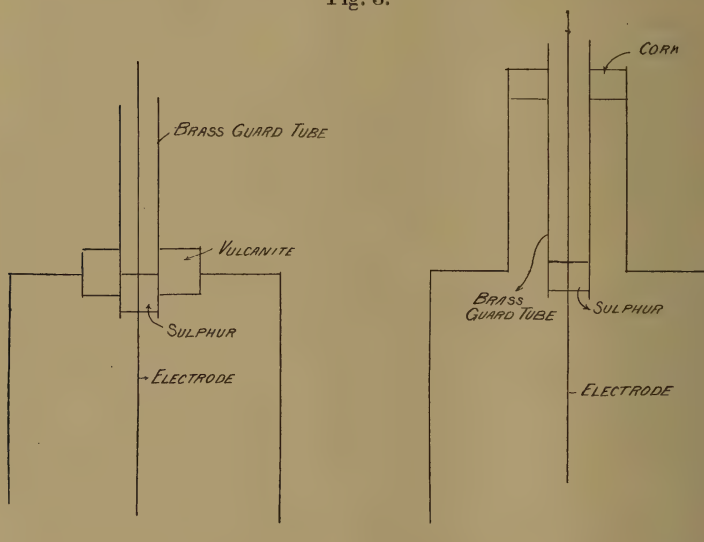
to that observed during heating will take place, and the apparent leak will be abnormally small, or, if the temperature has been high enough, the leak will even appear to be negative. The following experiments may be adduced in favour of this view :—If the field was applied to the vessel for some time, and after the ebonite had had time to charge up the vessel was earthed, heating the vessel produced a large apparent leak. On the other hand, if the vessel had not been charged, then, unless the field were on, no apparent leak could be obtained. The changes in the leak were thus dependent on the charge on the ebonite, and occurred whether the field was on or not. In order to get rid of this source of error, the distribution of the insulation was changed, as shown in the diagram.

The ebonite was replaced by cork, as this latter was a sufficiently good insulator for that part of the apparatus, and was less apt to charge than the ebonite. The wall of the vessel was carried up in the form of a tube and the insulation put at the mouth of the tube, the earthed brass tube projecting through the cork down into the vessel, and so screening the electrode. If the vessel after being charged was now earthed and heated, there was no apparent leak in any direction. Nevertheless, when the experiments were repeated, it was found that if the vessel was heated a distinct increase of the leak could be observed. The next point was to ascertain whether this increase was due to an emanation. A current of air from outside was drawn through (*a*) a glass-wool plug to free it from dust ; (*b*) a drying-tube filled with calcium chloride ; (*c*) a tube containing granulated tin ; and, lastly, through a vessel in which the leak could be measured. A rapid, steady current of air was passed through the apparatus and the leak measured in the tin vessel. The tube containing the granulated tin was now heated with a bunsen, and the ionization in the tin vessel showed a distinct increase. The following are the figures for one experiment. The numbers given in the column headed " Ionization " are the voltages to which the electrode system was charged up in ten minutes from the time indicated in the first column. The numbers in the second column are the potentiomer readings.

Time.	Potentiometer Readings.	Ionization.
$9.20\frac{1}{2}$	710	0·830
$9.32\frac{1}{2}$	750	0·857
9.44	750	0·857
9.55	630	0·773
$10.6\frac{1}{2}$	685	0·813

Glass tube containing the granulated tin gently heated $10.18\frac{1}{2}$ to 10.21.

10.18	730	0·844

Glass tube heated to the melting-point of tin 10.29 to 10.36.

10.29	880	0·936
10.40	650	0·788
$10.51\frac{1}{2}$	750	0·857
$11.2\frac{1}{2}$	770	0·870

Glass tube heated again to the melting-point of tin 11.14 to 11.17.

11.14	900	0·947
11.27	780	0·876
11.48	760	0·864

It will be observed that in each case after heat was applied, the leak in the measuring-vessel showed a distinct increase. This increase only lasted for a very short time, but it must be remembered that the current of air was passing all the

time, and so would draw fresh air into the vessel. The decrease in the leak which sometimes followed—*e. g.* after the second heating in the above experiment—is no doubt to be attributed to the diminished density of the air at the higher temperature. It now became necessary to ascertain whether this increase was due to an emanation or merely to free ions. For this purpose a brass tube was inserted between the glass tube containing the granulated tin and the measuring-vessel down the centre of which passed a wire which was kept charged to a potential of 280 volts. The outside of the tube was kept to earth, and as the tube was about 65 cms. long it must have been practically impossible for free ions to make their way along it to the vessel. With this change in the apparatus, heating the tin produced no effect on the ionization in the vessel. The following are the details of two experiments carried out with samples of the same tin, the experimental arrangements being identical in the two cases, except that in the first experiment there was no field applied to the brass tube, while in the second a field of 280 volts was applied.

Time.	Potentiometer Readings.	Ionization.
10.40	760	0·863
10.51	790	0·884
11.2$\frac{1}{2}$	760	0·863
11.13$\frac{1}{2}$	750	0·857

Glass tube heated 11.24$\frac{1}{2}$ to 11.28.

11.24$\frac{1}{2}$	1200	1·092
11.35$\frac{1}{2}$	750	0·857
11.47$\frac{1}{2}$	750	0·857
11.58$\frac{1}{2}$	750	0·857
12.10	720	0·837
12.21$\frac{1}{2}$	740	0·850

Glass tube heated 12.32$\frac{1}{2}$ to 12.35$\frac{1}{2}$.

12.32$\frac{1}{2}$	800	0·889
12.44	750	0·857

The glass tube was now emptied and refilled with some fresh tin and the experiment repeated, this time with the field on.

Time.	Potentiometer Readings.	Ionization.
9.27	700	0·823
9.38	750	0·857
9.50$\frac{1}{2}$	790	0·884
10.1$\frac{1}{2}$	750	0·857

Glass tube heated 10.13$\frac{1}{2}$ to 10.16$\frac{1}{4}$.

10·12$\frac{1}{2}$	620	0·765
10·24	760	0·863
10·35	725	0·840

Glass tube re-heated 10.46 to 10.49.

10.46	770	0·870

Thus in the second experiment there is no appreciable increase of the leak as a result of heating the granulated tin. This makes it probable that the increase previously observed was due to free ions, and as a proof of the existence of an emanation the experiment breaks down. All that one is justified in concluding, however, is that the existence of the emanation has not been proved ; as it is still perfectly open to us to conclude that those free ions, which are the immediate cause of the increase of the leak, have themselves been produced by the action of an emanation, the life of which is so short that it is disintegrated before reaching the vessel. The same experiments were performed on lead, and in this case no increase could be found, either with or without the field. Now lead ·is the very metal of all those tried in which one would look with the greatest confidence for an emanation ; because in the first place it is a heavy metal, and therefore likely to be radioactive, and in the second place the intrinsic radiation for lead was found to be greater than for any other metal.

Although no direct evidence of the emanation has been found, the following experiments admit of a simple explanation on the emanation hypothesis. On one occasion when a lead vessel which had been in use was cleaned with nitric acid and set up afresh, the natural leak was found to be distinctly increased. If the surface of the lead was now scraped and the vessel replaced, the leak was found to have resumed its normal value. This increase could be produced again and again ; after each treatment with acid, the vessel was carefully washed in running water for half-an-hour or more, and then thoroughly dried in a current of hot air, and each time there was a marked increase in the natural leak. This increase of course varied very much—in some cases being as much as 300 per cent , and in no case being less than 30 per cent., although occasionally as low as that. In order to study this effect more carefully a piece of thin sheet-lead was made into a lining for the zinc vessel. It was first polished with fine emery and then placed inside the zinc vessel. A mean of six observations of the natural leak in the zinc vessel with this lining gave 0·278. The lead lining was now removed and treated for a few minutes with strong nitric acid. It was then carefully washed, dried, and replaced. The mean leak was found to have increased to 0·328, showing an increase of about 18 per cent. The polishing process was now repeated, and the natural leak went back to 0·228, while after another treatment with nitric acid it rose again to 0·296. The effect of hydrochloric acid on tin was next tried. The

same zinc vessel was used, but this time it was lined with a thin sheet of carefully polished tin. The mean leak was now 0·281 ; the lining was removed, treated with strong hydrochloric acid, washed, dried, and replaced. The leak was found to have increased to 0·366—an increase of more than 25 per cent. Repolishing reduced the leak again to 0·315. The acid was in each case tested for radioactivity, but without success. Now by no known process can radio-activity be produced, and it seems extremely improbable that the treatment of metals with acids is any exception to this general rule. Another possible explanation suggests itself, however. In a paper communicated to the Phil. Mag. recently, Mr. Beilby* has gone very carefully into the changes produced in the surface of a metal by polishing. He concludes that a film is drawn over the surface, filling up the interstices between the crystals of the metal. This film is removed by treatment with acids. It seems not improbable, therefore, that this surface-film hinders the escape of emanation from the metal, and so reduces the leak in the vessel. The action of acids, on the other hand, dissolves off this film, and so allows the emanation to have free access to the air in the vessel. An extension of those experiments is at present being carried out. Further indirect evidence of the existence of an emanation is supplied by some experiments of McLennan†, confirmed in the course of the present research. A metal vessel was filled with fresh, dry, dust-free air and the leak measured immediately. Measurements of the leak were also taken subsequently at short intervals, and it was found that the leak at first fell and then rose to a value higher than its initial value, at which it remained constant. The initial decrease was attributed to the decay of emanation drawn from the air, and it is difficult to account for the subsequent increase of the leak on any other hypothesis than the escape of emanation from the metal, especially as it differed both in time-constant and amount for different metals. All this evidence is of course indirect, but taken in conjunction with the argument founded on the agreement of the results obtained by different observers for the ionization produced by the various metals tried, it points strongly to the conclusion that all matter is radioactive and that the disintegration processes going on in radium and the other radioactive elements are going on also, though to a very much smaller extent, in all matter.

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* Phil. Mag. August 1904.   † Phys. Rev. iv. 1903.

LIII. *Flat-Wavelet Resolution.*—Part III.
By G. Johnstone Stoney, *M.A., Sc.D., F.R.S.*[*]

Part III.—*How to exhibit in Microscopes and Spectroscopes the resolution of the light into its component Undulations of Flat Wavelets, and How to employ this resolution as our guide in making and in interpreting experiments.*

23. THE letters *u f w* were employed in Part II. of this series and continue to be used throughout the present paper as an abbreviation for *Undulation of Flat Wavelets*. By an undulation is meant a train of similar waves ; and the reader is requested to bear in mind that throughout each of the u f w's with which we shall have to deal, the light will be of one wave-length, and the wavelets of which it consists will be alike and each wavelet uniform throughout its whole extent—so that the wave-length, the intensity, and the state of polarization of the light is the same throughout the unlimited extent of each such undulation. The extent of each u f w is theoretically unlimited in time and in lateral directions, and it is possible to carry the analysis so far that the extent of each u f w shall be unlimited in every direction ; but nevertheless it is legitimate and often convenient to deal with the part of a u f w which lies during a specified time within a portion of space bounded by a closed surface, and to provide for the outlying parts of the undulation by imagining the medium which happens to lie within the closed space to be indefinitely extended. The state of things described in this paragraph follows from what has been already established in previous papers of this series : see also the Appendix to this paper.

24. The first part of the inquiry, which will be found at p. 570 of the Report of the British Association for 1901, contains a proof by the Method of Reversal of the theorem that light traversing a space S occupied by any uniform transparent medium, can always be completely resolved into u f w's of the kind described above, travelling across that space.

A less direct proof of this theorem was published by the present writer in 1895 (see ' English Mechanic ' for December 1895, § 38290), and he was then and remained until lately under the impression that he was the first to point out that every disturbance within a uniform wave-propagating medium is susceptible of being resolved in this way. But he has

* Communicated by the Author.

lately had his attention directed to one of Sir George Stokes's early papers in which the resolution is distinctly enunciated. Stokes's words are ". . . . , for we may represent an arbitrary disturbance in the medium, as the aggregate of series of plane waves propagated in all directions" (see Stokes's Collected Papers, vol. i. p. 192). Sir George Stokes here refers to the theorem incidentally and without proving it, or giving a reference to where a proof may be found; from which the natural inference seems to be that it could be cited as a well-known theorem so long ago as 1845, when Stokes's paper was published. The present writer desires to call especial attention to this, lest it should be supposed that he claims the discovery of the theorem. What he has done has been only to rediscover the theorem when it seemed to be forgotten, and to show the conspicuous use that can be made of it in investigating optical problems.

25. In the chamber study of optical problems it is found to be most convenient to employ this resolution into absolutely flat wavelets; but in making experiments the components usually consist of wavelets which are either somewhat convex or somewhat concave. This is permissible, inasmuch as the superiority of the resolution of light into undulations of absolutely flat wavelets over other kinds of resolution, depends on the circumstance that each such component undergoes no change as it advances through space; and, accordingly, a resolution into components consisting of *nearly* flat wavelets is allowable in all cases where the approximation to being flat is sufficient to carry the nearly flat components across the space to which the experiment extends without more change than may legitimately be left out of consideration. Moreover, cases sometimes arise in practical work, where a resolution into conspicuously curved components is found useful. To be prepared for such eventualities it is proposed to show, in an Appendix to the present paper, that a resolution into concave or convex wavelets is legitimate, and to study how the three resolutions into concave, flat, and convex wavelets stand related to one another.

26. There does not seem to be any such thing in nature as light of a single wave-frequency. The nearest approach to it that has been found is such light as is met with in the rays of a spectrum that consists of narrow lines: these rays include wave-frequencies ranging from ϕ to $\phi + \delta\phi$, where $\delta\phi$ is finite though small. Notwithstanding this, when we have occasion to deal with monochromatic light, we shall provisionally treat it as though it were light of the single frequency ϕ, reserving the necessary correction till we

have occasion to consider the conditions which must be fulfilled in nature; amongst which we shall find it necessary to include a compliance with the equation of energy. The proposed assumption is convenient, as by adopting it we simplify the elementary mathematics we have to use.

27. Light, so far as it may be regarded as light propagated exclusively by a medium, is light everywhere except within one or two wave-lengths of some situation where there is a transfer of energy between the medium and an external agent, or an agent that we may regard as external : *i. e.*, an agent which generates light or absorbs it or otherwise affects it. With this understanding, light that is propagated by a uniform isotropic medium, which is perfectly transparent, consists of waves of alternating electromagnetic stresses propagated in obedience to the differential equation of electromagnetic waves (see Maxwell's ' Electricity and Magnetism,' Part iv. Chapter xx.), viz. :

$$\nabla^2 V = \frac{1}{k^2} \frac{\partial^2 V}{\partial t^2}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9)$$

where k is the speed of propagation of light of wave-frequency ϕ in the medium. This differential equation is also the diff. equ. of dynamical waves propagated in a uniform isotropic medium, wherever the displacements are sufficiently minute to admit of their being geometrically superposed. Hence events will occur in suitably chosen dynamical waves which correspond to the actual events in the electromagnetic waves ; and we are therefore justified in using the terminology of dynamics—disturbance, energy, motion, acceleration, force, transversal, and so on—understanding these terms in a sense generalized in such wise as will extend them to the corresponding events in electromagnetics. This proves to be a great convenience. It is even legitimate for us to fix our attention upon the progress of events in a dynamical model, in studying which these terms can be employed in their usual dynamical acceptation, since what we are occupied upon is *a physical resolution** of the disturbance in the medium, and every such resolution is common to all the successions of events whose connexion with their physical causes is represented by equation (9)—the fundamental differential equation of the connexion which here prevails.

28. Any complete set of events which that differential equation can indicate as occurring in space may be called one of the *fields* of the differential equation : and if the field

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* A resolution may be physical (that is such as may actually occur in nature) or merely kinematical.

is a complex one it may be resolved in various ways, into simpler fields. In order that the resolution may be entitled to be called a *physical* resolution, *all* the dynamical conditions of the problem must be complied with. These include the *general* dynamical conditions of motion, along with the numerous *special* conditions which arise out of the definite kind of physical causes that operates within the medium with which we are dealing. The most useful of the general conditions are the three following :—

    1. Each component field must be a solution of the fundamental equation, which in the present case is equation (9).

    2. The sum of the energies of the components must be equal to the energy of the *complete* event that is to be resolved.

    3. The equation of Least Action must be complied with.

Each of these is available as a criterion of physical resolutions, being in so far a criterion that its not being true of any given resolution demonstrates that this resolution is not a physical resolution. By a physical resolution of one of the complex fields that may result from a differential equation is to be understood one in which, if some of the components were absorbed, shifted aside, or in some other way withdrawn, the others would continue unchanged. A good example of a physical resolution is the resolution of white light into its various colours. This is a resolution which can be effected by natural processes none of which need to import energy into the system : indeed some (those which act by absorbing some of the colours) withdraw energy from it.

29. In the statement of the second condition it is essential to insert the word " complete," inasmuch as what equation (9) represents is the operation of physical causes *inherent in the medium,* which are competent without external aid to propagate undulations forward without loss of energy or change of velocity. Now, a medium which fulfils the necessary conditions *completely* will be a uniform medium subjected to the following conditions :—

    A. It must not be acted upon by external sources of light or absorbers of light.

    B. It must not be subjected to boundary conditions, and will therefore be of unlimited extent.

    C. It must not emit or absorb light, but merely transmit it.

It is only in a medium which completely fulfils these conditions that the progress of events will be in *exact* accordance

with the mathematical results obtained from the differential
equation.

30. Accordingly, in dealing with the media we meet with
in making experiments we must contrive such a succession
of imagined events as will justify us in conceiving the actual
medium (which may be air, or glass, or oil, or any of the
media in which microscopical objects are mounted) replaced
by a medium fulfilling the foregoing conditions. This can
be accomplished in perfection by employing the methods of
investigation placed within our reach by the Principle of
Reversal. This has been done in the first paper of the present
series, that published in the Report of the British Association
for 1901, so far as concerns resolution into flat wavelets; and
the slight modification of the proof to make it available for
resolutions into convex or concave wavelets is given in the
Appendix to the present paper.

In these proofs the space within which the resolution is
effected is any space lying within the sphere *r* mentioned in
those papers, where we are at liberty to assign any length
we please to *r,* the radius of this sphere (see the diagram on
p. 596).

31. To give definiteness to our research we shall fix upon
a definite instance, selecting one which we shall find useful.
We shall make it our definite aim to resolve into its plane
wavelet components the light which crosses the air space that
intervenes between the cover-glass protecting a microscopical
object and the front lens of the objective (here assumed to be a
" dry " objective) when the microscope is focussed upon the
object. The light that crosses this air space has come either
wholly or partly from the microscopical object. It may also
partly have come from surrounding unoccupied portions of the
field of view. We shall use the phrase microscopical objects, in
the plural, to imply the *whole* contents of this field. It will
facilitate our descriptions to suppose the microscope pointed
directly downwards, in which case the air space spoken of
above will be a horizontal stratum of air which we may call
stratum S. We shall also picture this stratum as not of
limited extent, but as extending indefinitely between two
bounding planes—the horizontal plane in which the front
face of the front lens of the objective lies, and the parallel
plane in which the upper surface of the cover-glass lies. It
is the resolution into u f w's of the light within this air space
as it travels upwards, that can be exhibited experimentally in
the microscope, as will be explained further on. We shall
find it convenient to make the same supposition of unlimited
extension sideways, in the case both of the cover-glass and

of the medium in which the microscopical object is mounted; so that the reader is requested to picture all three—stratum S, the cover-glass, and the mounting medium—as transparent and of unlimited extent horizontally.

32. In order to be able to employ the Method of Reversal in our investigation, we here make the legitimate hypothesis that the air within stratum S is replaced by a medium which is *perfectly* transparent, and which we may call *transparent* air. We may further suppose the apparatus that is over this air space to be removed—the microscope tube and all it carries—and that transparent air fills the entire space over stratum S, which we shall call space K. Under these circumstances the light from the microscopical objects, *i. e.* from the microscopical object and its surroundings, will travel uninterruptedly forward, crossing stratum S and advancing into K, the added air space above it. Let us suppose that the emission of light from the microscopical objects has been allowed to go on for some definite time, say for one minute. At the end of the minute let Reversal of the light take place, the effect of which will be that the light which had been travelling upwards through space K, will now retrace its steps, will recross stratum S, will re-enter the cover-glass, will pass from it into the medium in which the microscopical object has been mounted, and will finally converge upon the microscopical objects from which it had emanated. This is what would happen if Reversal alone took place; but we shall find it useful to suppose that at the instant of Reversal, the microscopical object is removed and mounting material put in its place. Under these circumstances the light upon its return journey after Reversal, would recross stratum S and the cover-glass, and advancing into the mounting medium would in it converge upon and form an image of the microscopical object, which image is entitled to the name of its *standard image*, inasmuch as, although inconveniently small, it is the most perfect image of that object that the light in question is competent to form, *i. e.* the most perfect as regards fulness of detail and accuracy of delineation. We shall give to it the name of Image B′.

33. A slight modification of the procedure we have followed will furnish another standard image, which shall include precisely the same detail as Image B′, and will be formed under conditions that are more convenient for the use we intend to make of it. To form this image, proceed as before until the First Reversal has been effected in space K. This, as before, causes the light to retrace backwards the course which before the reversal it had pursued upwards. At the instant of

reversal let the cover-glass, object, mounting medium, and everything else below stratum S be suppressed, and let the whole of the space below stratum S, which we may call space Q, be filled with the same medium—air or oil—as occupies stratum S and the space K. The light returning downwards after reversal, will then form within space Q an image, which we may call Image B, that will exhibit exactly the same amount of detail as Image B′; the chief difference being that if the medium occupying S, K, and Q be air, B will be located a little further down than B′, with a similar but smaller displacement if the medium be water or oil. The slight displacement is of no significance, since it is unaccompanied by any change in the amount of detail which the images furnish. And there is now the great convenience that only one medium—air, water, or oil—intervenes between image B and the front lens of the objective ; whereas three media—air or oil in stratum S, glass in the cover-glass, and mounting material—had intervened between image B′ and the front lens of the objective.

34. A further step in our investigation can be made by supposing the light to undergo a second Reversal. For this purpose, let the reversed light after forming image B, be allowed to continue its downward course in space Q for another minute and let it be then reversed. Let us imagine that simultaneously with this Second Reversal, the microscope-tube and the rest of the apparatus above stratum S are allowed to resume their positions, while the cover-glass and all subjacent apparatus are still absent. The following events would then happen :—The light directing its course upward after the Second Reversal, would converge upon and form the same standard image B as before, after passing which it would continue to ascend as though it had been emitted from the several puncta of that image. It would, accordingly, proceed to fill air space S with precisely the same light as that which had before the first Reversal been emitted by the real microscopical objects. From this we learn that the light which in the ordinary use of the microscope reaches the front of the objective from objects under examination, after having passed in succession through the medium in which the microscopical objects are mounted, the cover-glass that protects them, and the air space between it and the objective—that this light is *identical* with what would come from Image B, pursuing the latter part of its course to the objective through the air space S ; and that therefore it will be legitimate to regard the light that crosses the air space S as having emanated from that image, and being incapable of presenting to us, in

any image that can be formed from it by any optical instrument, more of the detail of the microscopical object than that which is contained in image B. In fact no optical instrument that can be made can form an image that will contain so much of the detail as does image B, inasmuch as to do so it should be able to receive and deal with *all* the light emitted from the object which would enter the air space S if both the cover-glass and the air space S had been extended laterally without limit. Now in practice the more oblique rays of this light fail to get through the objective, since the NA of a dry objective cannot be quite so much as 1·00, nor can that of an oil-immersion objective be quite so much as the index of refraction in the oil—conditions which would need to be fulfilled in order that the objective might be able to grasp the whole of the light.

35. This circumstance makes another standard image more useful in studying the microscope than image B. This, which we may call image C, is the image which would be produced by the same supposed succession of events as image B, with the sole difference that the light subjected to Reversal is not to consist of all the u f w's into which the light emitted by the microscopical objects may be resolved, but only of such of them as advance in directions not so much inclined to the optic axis as to make it impossible for the objective of the microscope to transmit portions of them. The image formed in this way would occupy the same position as image B in space Q, which by one of the hypotheses made is to be regarded as occupied by an extension downwards of the air or oil which occupies stratum S.

36. No image formed by light whose waves are not infinitesimal in length can reproduce the whole of the detail present in any real object, all beyond a certain limit of fineness being necessarily absent from the image. The image which could furnish all detail, would need to be formed by light of which the wave-length is cipher. This impossible image is called the Geometrical Image, and may be spoken of as image A. Image B contains as much of the detail as light of the wave-lengths that form that image can by any possibility reproduce; while image C, which is formed from u f w's whose inclination to the optic axis is limited by the NA of the objective, becomes a different image when an objective with a different NA is in use; but is for each objective the standard image for that objective, reproducing the most minute detail of the object, which could by any possibility exist in any image of the object formed by an objective of that aperture. Accordingly, it is legitimate for us to treat

the light which reaches the objective as being light which was emitted by the several puncta of image C, and as having come from them direct through air (or oil) to the front lens of the objective.

37. Hitherto we have for the most part dealt only with the case where the medium between the cover-glass and the objective is air. If the objective we are using is an oil-immersion objective, the procedure will be essentially the same; only we have now to substitute theoretically transparent oil for real oil, to imagine space K filled with an extension of this oil, and to suppose the standard images B and C to be formed in a continuation downwards of the same medium.

38. We shall freely make use of the conception presented above, that the light coming to the objective has emanated from the standard image—either B or C. This is a much simpler apparatus than the real apparatus which consists of the microscopical object, with molecular motions and other detail immensely too fine to be reproduced in any real image; and yet the simpler machinery we have substituted would supply to the objective precisely the same light as that which really reaches it.

39. Moreover, we may imagine the light emitted by each punctum of the standard image to be resolved into its plane wavelet components; so that, if N is the number of optical puncta in the standard image there will be N u f w's traversing space S in *each* upward direction. The number of these however may be greatly reduced. Those of them travelling in any one direction may be combined into a single resultant u f w advancing in that direction; so that finally the whole of the light in space S is resolved into u f w's *of which one only needs to be provided for each upward direction.* It is the resolution of the light when it attains this final form, that we shall find it possible to exhibit experimentally in the microscope.

40. To do so, we have to trace the further progress of these resultant u f w's. Those of them whose guide lines—lines perpendicular to the planes of the waves—are not too much inclined to the optic axis of the microscope will, parts of them, not only enter but will be transmitted through, and will each of them be brought to a focus by the objective. What thereupon happens may be strikingly illustrated by a very simple experiment. Look towards some distant landscape, and hold one of the objectives of the microscope at a distance of about ten inches from the eye, with its front lens towards the landscape and its back lens towards the observer. An inverted

image of the landscape will then be seen near the back of the objective. The light which forms this image consists of conical beams each with its vertex at some punctum of the distant landscape and its base on the tiny front lens of the objective. If the experiment is made with a high-power objective, it is well to observe how exceedingly small that front lens is. From this it is obvious that the light that enters the objective consists of beams which are very nearly parallel, or in other words of beams each of which is part of an undulation of waves that differ very little from being flat; and each punctum of the image that is formed may be regarded as the concentrated light of one of these beams.

Fig. 11.

The same succession of events takes place when the microscope is in use. In the air or oil space which lies in front of the objective, the light that is advancing upward may be resolved into its component u f w's ; and the portions of these that are transmitted by the objective undergo the same treatment as the beams of light that enter a telescope from a celestial object. Each of them is by the objective brought to a focus in a plane, or quasi-plane, perpendicular to the optic axis and near to the back lens of the objective. There it forms the kind of image which Airy investigated as the image of a star, consisting of a central " spurious disk " of light surrounded by appendage rings. But little of the light is expended in forming the appendage rings, so that we may, as a first and usually as a sufficient approximation, regard the whole of the light of each u f w as concentrated into the central speck. The number of u f w's is unlimited,

and therefore the amount of light in each of them is
infinitesimal. It requires the cooperation of a sheaf * of them
to produce an optical punctum p, which the human eye can
see. But as is proved in theorem X (see Phil. Mag. Feb.
1903, p. 274) a single u f w may be substituted for all the
u f w's of such a compact little sheaf, as will contain sufficient
light to affect the eye. The image which results from the
presence of all the little groups of spurious disks that constitute
puncta bright enough to be visible, is the image marked with
an x in fig. 11, which is the image seen near the back lens
of the objective on looking down the tube. To see it properly
it is best to put a blank eyepiece into the top of the tube—
that is the mounting of an eyepiece, without lenses—whose
eyehole (and the eyehole should be small) will serve to keep
the pupil of the eye on the axis of the instrument. When
accurate experiments are being made it is advisable that the
eyehole should not be on the outside of the blank eyepiece,
but so far sunk down in it as to bring it to the level of
image D, the image of the microscopical object formed by
the objective. The image at x as seen through the blank
eyepiece may be called the *Concentration Image*, as every one
of its visible puncta is the concentrated light of one of the
small sheafs of u f w's, into which the light between image C
and the objective may be resolved.

41. This concentration image renders the greatest service
to the student of microscopical vision. To understand the
extent of this service, imagine a "reference hemisphere"
with its "indicator diagram" (see the annexed figure (fig. 12),
and the Phil. Mag. for Feb. 1903, p. 268) constructed with its
base upon the plane in which image C lies, and with its optic
axis coincident with the axis of the microscope†. Then the
concentration image presents to the observer in visible form,
upon a plane at the height x, drawn parallel to the base-
plane of the reference hemisphere, that part of the indicator
diagram which he has occasion to employ when investigating
microscopical problems. This appears upon comparing the

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* By a *sheaf* of u f w's is meant a group of them such that if the
guide lines of the undulations—lines perpendicular to their wavelets—
are drawn through a given point, these guide lines will form an acute
solid cone with that point as vertex.

† The reader is requested to conceive the reference hemisphere con-
structed of such a size that its radius—which is also the radius of the
indicator diagram—shall be that length on the NA scale of figures 13
and 14, which expresses the index of refraction of the medium between
the cover-glass and the front lens of the objective. This medium is in
practice either air, water, or oil; of which the indices of refraction are
1, 1·33, and 1·515.

geometrical properties of that image *, as given in sections 33 and 34 of the paper on " Microscopical Vision " in the Phil. Mag. for December 1896, with those of the reference

Fig. 12.
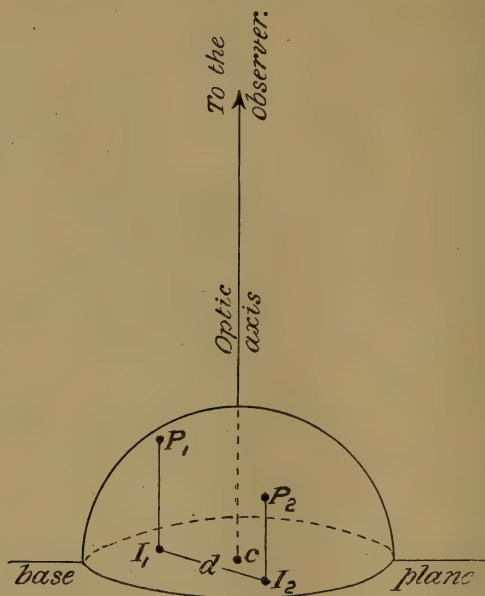
Reference-hemisphere, of which the circular base is the indicator diagram.

hemisphere and indicator diagram, as given in sections 7–10 of the paper in the Phil. Mag. of February 1903. From these geometrical properties we can deduce the rules to be observed in making experiments.

In all microscopes the medium above the objective into which the light emerges, is air ; but the medium under the objective from which the light is supplied to it differs in different cases. It is air when we employ what are called " dry " objectives, and it is water or oil with " immersion " objectives. Lagrange's formula—viz. :

$$n \sin \alpha = \text{M} \sin \beta,$$

where M is the number of times that the linear dimensions

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* In the paper here referred to the *concentration image* is called *image x*.

of image D (see fig. 11) exceed those of image C—is funda-
mental in inquiries such as that upon which we are engaged.
Now the symbol *n* in that formula (which is the ratio of the
indices of refraction of the media below and above the ob-
jective) is unity in the case of dry objectives, 1·33 for water-
immersion objectives, and 1·515 for oil-immersion objectives.
The first rule to be observed is as to the use of a scale in
making figures to represent the indicator diagram and that
part of it which is presented in image *x*. NA, the numerical
aperture of the objective, is $n \sin u$, where *n* is the *n* of
Lagrange's formula and *u* is the semi-angular-aperture of the
objective, as seen from the point where image C intersects
the optic axis. This is a fraction in the case of dry objectives
and a number greater than unity in the case of immersion
objectives ; and the rule is that we are to adopt such a unit
of length for our scale (which we may call the NA scale) as
will cause the radius *of the visible concentration image* to be
NA times that unit. It follows from the geometrical pro-
perties referred to above, that if we lay off *n* times the
unit of length on the same scale, and employ the length so
found as the radius of our reference hemisphere and indicator
diagram, then will the concentration image be the central
and only visible part of that indicator diagram.

42. This is illustrated by the accompanying figures which
represent the indicator diagram and the concentration image
(the image seen at *x*) for four of the objectives and two of
the condensers with which the author is accustomed to work,
viz. :

> An achromatic of NA=0·17.
> An apochromatic of NA=0·65.
> An apochromatic of NA=0·95.
> An oil-immersion objective of NA=1·35.
> A dry condenser of *na*=0·9.
> An oil-immersion condenser of *na*=1·3.

Two cases have to be distinguished :

(*a*) With dry objectives $n=1$; and as in our figures we are
using a two-centimetre scale as our NA scale, the indicator
diagram is to have a radius of 2 cm., and the annexed
figures show what portion of it is made visible when, after
focussing the microscope upon an object, we remove the
eyepiece, insert a blank eyepiece and then look down the
tube. An excellent object to employ in making this experi-
ment is the diatom *Coscinodiscus lineatus*, as the details in
the concentration image, the visible part of the indicator

diagram, become singularly distinct when the iris diaphragm under the condenser is sufficiently closed. If we arm the

Fig. 13.
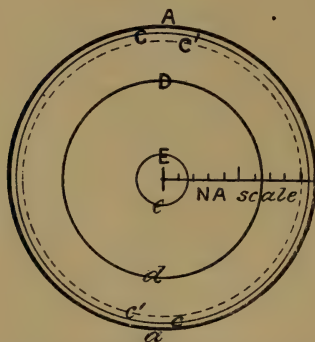
The parts of the indicator diagram made visible by dry objectives *.

microscope in succession with different dry objectives, and look at the contents of their concentration images produced by this diagram, we shall see how much more of those details become visible with an objective of large than with an objective of small NA, in other words how much more of the complete indicator diagram the objective of large aperture makes visible.

(*b*) When we employ oil-immersion objectives *n*, which is the ratio of the index of refraction of the oil under the objective to that of the air above it, is 1·515. This therefore is the *n* which we are to employ in our figures as the radius of the indicator diagram, in order that on the same scale NA may represent the radius of the concentration image. Hence if we are employing an oil-immersion objective of NA = 1·35, the outer circle of the annexed figure will be the boundary of the complete indicator diagram and on the same scale the next circle will be the part of it which is visible on looking down the tube towards the back lens of the objective. Each optical punctum of this image *x* is the concentrated light of one of the small conical sheafs of u f w's, into which the light which has been transmitted by the objective might

* In this diagram the outer circle represents the entire indicator diagram. The circle C*c* is the part of it made visible by an objective of NA = 0·95; the circle D*d* is the visible part when the NA of the objective is 0·65; and the circle E*e* is the part made visible by an objective of NA = 0·17.

The dotted circle is the disk of light which each punctum of the flame can furnish when a condenser of *na* = 0·9 is accurately adjusted.

have been resolved before it entered the objective, *i. e.* as it existed in the oil space between the cover-glass and the objective. It is the resolution of this light which is exhibited to us by the concentration image.

Fig. 14.
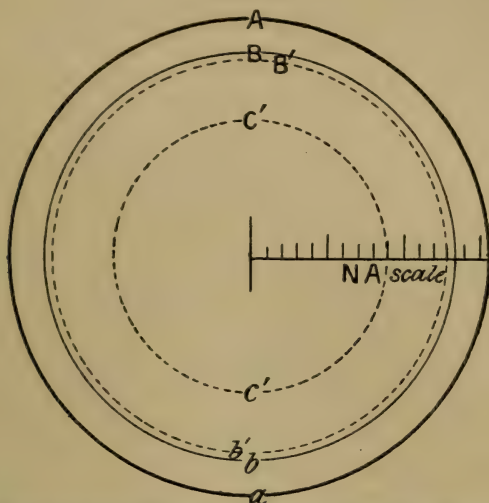
The part of the indicator diagram made visible by an immersion objective *.

43. It is well known to microscopists that in working with objectives of large aperture, it is necessary to adjust with care the correction-collar when the objective has one, or else the tube length; and that with dry objectives this adjustment will vary with the thickness of the cover-glass and of the mounting material through which the light has to make its way before reaching the objective, and with immersion objectives it will depend upon the index of refraction of the mounting material, and upon its thickness, in other words, upon the interval between the object and the under surface of the cover-glass. The reason of this becomes apparent by studying the concentration image. We may conceive that image divided into annuli, and the microscopical image (D, in fig. 11) ought to be formed by the satisfactory cooperation of the cones of rays coming from these several annuli. Now,

* Aa represents the entire indicator diagram; Bb is the part of it made visible by an objective of $NA = 1\cdot35$; $B'b'$ is the disk of light which each punctum of the flame can furnish if an immersion condenser of $na = 1\cdot3$ is in satisfactory adjustment. $C'c'$ is the similar disk furnished by a dry condenser of $na = 0\cdot9$, when in adjustment.

on the hypothesis that the objective has been accurately corrected for aplanatism there is only one distance of object, with its corresponding one distance of the image on the other side of the objective, for which the aplanatic correction can be satisfactorily made, and it is only when object and image are at these unique distances from the objective that the light from *all* the annuli of the concentration image can concur to reach the same focus. This adjustment is known to be one which has to be made with extreme care to bring out the full efficiency of a good objective of large NA.

44. All this is well known to microscopists : but it is not certain that they always equally appreciate that the same care must be bestowed on adjusting condensers of large aperture in order to secure the best, or even good results.

What adds very much to the usefulness of image *x*, the concentration image, is 1°, that it can be made to exhibit *na*—*i. e.* the numerical aperture of the condenser; 2°, that it supplies useful information as to the light transmitted from the condenser to the microscopical object ; and 3°, that by its help the important adjustment of the distance of the source of light from the condenser which is above referred to, can be made. When this adjustment has been effected the light which comes through a minute hole in a screen placed in front of the flame, or other source of light, will produce light in the concentration image which will extend in that image out to the dotted circle with radius *na*, as estimated on the NA scale, if the microscope has been armed with an objective of as large or larger aperture. If the NA of the objective is less than the *na* of the condenser, then the size of the concentration image, the visible part of the indicator diagram, is too small to show the whole of the disk of light which the condenser is competent to produce from each punctum of the flame. Accordingly, in order that we may be able to avail ourselves of the full aperture of a wide angled condenser the adjustment referred to above must be made with care, and in fact quite a moderate departure from it diminishes in a considerable degree the efficiency of the condenser.

45. Before proceeding to make use of the concentration image in carrying out the experiments which will occupy our attention in the next part of this series of papers, it appears desirable to call attention to the circumstance that although the concentration image as seen on looking down the tube of the microscope is a good, and even a very good, representation of the central part of the indicator diagram, it is not an exact reproduction of it. This is partly on

account of that substitution of tan β for sin β which is referred to on p. 503 of the Phil. Mag. for December 1896, where β is the small angle to which the letter β is assigned in the figure on p. 586 above ; and partly because the iris diaphragm and stops under the condenser of a microscope are always situated farther down than is desirable. The construction of the instrument makes it impossible to bring them up into coincidence with the principal focus of the condenser. Now it is an image of these and of the brightness bounded by them which fixes the apparent position of the concentration image as we see it : on which account each punctum of that image as seen by the observer is the concentration, not of light that has passed up through the air or oil space under the objective as a strictly parallel beam, but of light that was slightly convergent when it traversed that space. These two circumstances cause the concentration image to be a reproduction of the central part of the indicator diagram which is slightly distorted, but the distortion is so slight that it falls short of what could be detected by the unassisted eye and it causes no real embarrassment in making even delicate experiments.

46. The transition from the microscope to the spectroscope is an easy one. In fact, part of the apparatus which has to be put together to make a microscope, is essentially a spectroscope—viz. : the stop under the condenser, the condenser itself, the microscopic object on the stage, and the objective. To see this, employ as the microscopic object a ruling of equidistant parallel lines, such as one of those bands of lines (ranging from 5000 up to 120,000 in the inch) which have been exquisitely ruled on the under side of cover-glasses by Mr. H. J. Grayson of Melbourne : place under the condenser a stop in which there is a single narrow slit parallel to the band of lines ; and then look through the " blank " eyepiece at the concentration image which is formed near the back (*i. e.* uppermost) lens of the objective. In this concentration image will be seen a simple image of the slit—the zero spectrum, as Lord Rayleigh well designates it—attended on either side by spectra of the first and succeeding orders which are produced by the ruling under examination. In fact, the condenser has become the collimator of a spectroscope and the objective is its telescope. And, as in this case, so with the ordinary types of spectroscope, the spectrum, as seen in the telescope, is essentially a concentration image, each visible punctum of which is the concentrated light of one of the little sheafs of u f w's into

which the light between the prism and the telescope may be resolved—or the light between the diffraction-grating and the telescope, if the spectroscope be one furnished with a diffraction grating instead of a prism.

So, again, the images seen in astronomical telescopes are of the same nature as concentration images, since each visible punctum of the image is the concentrated light of a beam of parallel-light.

Aided by the information given to the microscopist by the concentration image, it becomes possible for him to make many interesting experiments *and to understand them*. These experiments when interpreted bring to light many instructive facts. For example, they will show that the image of an extended object produced by a lens is not—as has been too often supposed—the aggregate of the brightnesses of the spurious disks with attendant rings, of its several puncta ; they will show that the resolvability of two puncta upon a microscopical object depends not only upon the interval between those puncta but also in large part upon the detail present elsewhere upon the object ; they will further show that there are different kinds of resolution, and will indicate the conditions which will result in resolution of the different orders. Other experiments explain the obscure phenomenon of optical contact, or exhibit instances in which colour can be introduced into an image by excluding light of that colour from the objective, and so on. To describe and interpret some of these experiments must be our next task.

Appendix (see p. 578).

Any disturbance which can be propagated by a uniform transparent* medium can be resolved into components each of which is an undulation either of flat wavelets, or of spherical wavelets which may be either concave or convex. And the relation in which these three resolutions stand to one another can be traced.

47. To prove this, it is only necessary to recall the succesive steps of the proof of Theorem I., which are given on pp. 571 and 572 of the B.A. Report for 1901, which will need but little modification for our present purpose. To make the process clear we shall exhibit the successive steps in an individual instance. We shall select as the disturbance to be resolved that which takes place within a given glass lens while light emanating from external objects is traversing it.

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* By a transparent medium is meant one through which waves are propagated without loss of energy.

No material of which a lens can be made is *perfectly* transparent, and accordingly the first thing to be done is to conceive absolutely transparent glass substituted for the actual glass of the lens. It is the luminous disturbance which would occur within this transparent medium that we propose to resolve. We may, moreover, confine our attention to light of one wave-length λ, since any other wave-lengths that are present can be dealt with in the same way.

48. Let us next picture a superficial stratum of the lens which need not be more than one or two wave-lengths in thickness. Within this layer there will be "turmoil," *i. e.* motion differing from ordinary wave motion, due to the light passing from one medium into another, of the kind which Stokes investigated in certain cases (see Stokes's Collected Papers, vol. ii. p. 56). If this surface layer be divided into minute elements P_1, P_2, &c., then it is legitimate (as in Huygens' construction), and it will be convenient, to regard the light within the lens as consisting of undulations of spherical waves emanating from these P's. By this means we include with the light coming from abroad, and without making the investigation more complex, the special surface action which takes place when light passes from one medium into another. The disturbance we shall resolve will then be the actual luminous disturbance which would occur in the inside of the lens, excluding its surface layer, if the glass were perfectly transparent.

49. If, as will usually happen, the light from each punctum of the objects that transmit light to the lens, is not uniform, but suffers from one minute portion of time to another, abrupt or continuous changes, then the same will be true of the "motion" in each of the P's. But this need give no trouble. The light from a given punctum of the object which reaches one of the P's—suppose P_1—may in the first place be resolved into light polarized in two planes at right angles to one another ; and irregularities in the periodicity of the vibrations in each of these directions are susceptible of being analysed by that extension of Fourier's theorem which admits terms with periods that are incommensurable. This analysis will furnish a series of terms each of which represents regular periodic vibration. Accordingly in our further resolution it will be sufficient to deal with that one of these which produces waves in glass of wave-length λ. To simplify matters, we may suppose all the P's to be annihilated except that one whose emissions we are going to analyse. Furthermore, we shall make the hypothesis that the transparent glass of which

the lens is supposed to consist, is made a medium which extends through all space.

50. Let us suppose P_1 to be placed at the point in space which is represented by the centre of the accompanying figure.

Fig. 15.
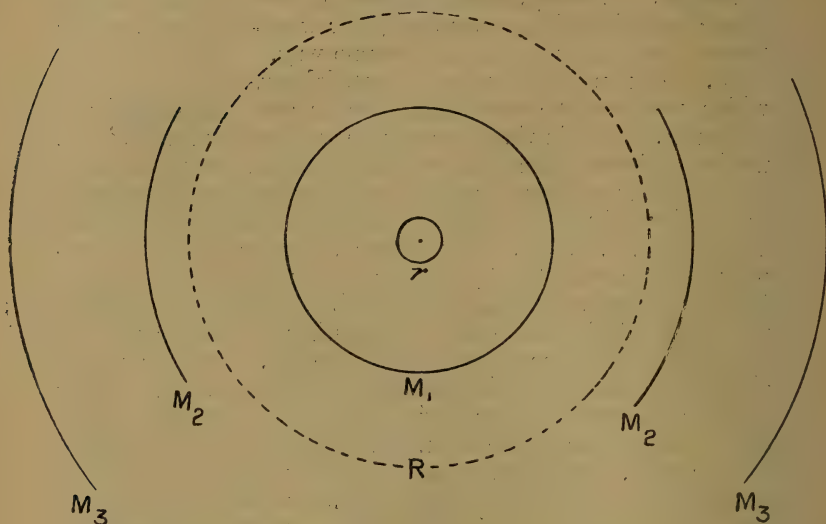
Diagram of distances in space which may be immense *.

Round P_1, which is on the surface of the glass lens, draw the sphere r† of such a size that the lens lies completely inside it. Let us next imagine the transparent glass to be extended beyond the lens so as to become a uniform medium filling all space. It will be convenient to suppose that the lens lies to the right-hand side of the centre, in which case the light we have to deal with will be the light emitted by P_1 towards the east or right-hand side of the diagram. The part of this light which is of wave-length λ, is an undulation of hemispherical waves travelling from the centre of the figure towards the east or right-hand side. We must recognize P_1 as a source of energy external to the medium; and accordingly the motion of the medium in its immediate neighbourhood will be of the kind which we have designated turmoil. But if we describe round P_1 a tiny sphere, not shown in the

* In this diagram P_1, the punctum emitting light, is supposed to be situated at the centre of the spheres. It is surrounded by a tiny sphere ρ (i. e. with radius ρ) which is too small to be represented in the figure. The spheres M_1, M_2, and M_3 have radii which equal the distances to which light travels in the medium in the periods τ, 2τ, and 3τ respectively.

† Sphere r means the sphere which has r as radius.

diagram. with a radius ρ of one or two wave-lengths, the turmoil will not extend beyond it : all beyond will be a regular undulation of luminous waves. The tiny sphere we may call sphere ρ.

51. We must contrive means of securing one more condition, namely, that the medium shall be freed from outside interference. To effect this, we shall employ the Principle of Reversal, and imagine the following series of events. Let P_1 begin to emit its light at the instant $t=0$. Of that light we need only attend to that which enters the lens. This is an undulation of hemispherical waves advancing towards the east, that is towards the right-hand side of our diagram. Let this emission of light continue from the epoch $t=0$ until the instant $t=\tau$, and let the right-hand half of sphere M_1 represent the front of the undulation at that instant. At that instant the luminous undulation extends from sphere ρ (which is too minute to be represented in the diagram) on to sphere M_1 ; and during the whole of the period from $t=0$ to $t=\tau$ there have been within the tiny sphere ρ both P_i, the source of the energy thrown into the medium, and turmoil surrounding P_1. *These we get rid of by supposing the whole of the contents of sphere ρ to be annihilated at the instant $t=\tau$, and quiescent medium substituted for them.* This in no way interferes with the further progress of the luminous undulation, which when the instant $t=3\tau$ arrives will find itself within the eastern half of the spherical shell between spheres M_2 and M_3. At the instant $t=3\tau$ let Reversal take place. All the motions in the medium are reversed, and the outflowing waves suddenly become inflowing waves. They will retrace their steps, converge upon the central punctum * of the diagram, as upon a focus, and after passing it will diverge towards the west, that is towards the left-hand side of the diagram. At the instant $t=8\tau$, the undulation finds itself within the left-hand half of the spherical shell between M_2 and M_3. At this instant let a Second Reversal take place. The waves which since the First Reversal have been moving westward, resume an eastward course, converge upon the focus at the centre and after passing it diverge towards the east. In this way they reproduce between $t=10\tau$ and $t=11\tau$ *precisely the same transmission of light to the east of sphere ρ, as had prevailed before the reversals in the period from $t=0$ to $t=\tau$*; and with the advantage that this transmission of light

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* The mathematical point has no place in physics. Its place is taken, so far as it can be taken, by the physical punctum, *i. e.* by the smallest speck of which we have to take notice in the problem upon which we are engaged. Its size depends upon the conditions of the problem, and differs from one problem to another. The *luminous* punctum may be taken to be a sphere with a diameter of from one-half to one wave-length.

is now no longer encumbered with interference from outside the medium and the resulting turmoil within sphere ρ.

52. The rest of what we have to do has now been made easy. Inside M_2 draw a concentric sphere, R. When travelling westward after the First Reversal the waves of the undulation come in succession to coincide with the right-hand half of sphere R; and when travelling eastward after the Second Reversal they successively coincide with the left-hand half of sphere R. On both these occasions the undulation is travelling inwards towards the centre of the diagram. On either of these occasions we may imagine a spherical shell one wave-length in thickness which is bounded by sphere R, to be divided up into elements δS, and each of these made the centre of spherical wavelets directed inwards. This is in fact Huygens' construction with the improvement suggested on p. 539 of the B.A. Report for 1902. If the resolution takes place on the western half of sphere R (after both reversals) then between $t = 10\tau$ and $t = 11\tau$, it causes slightly convex wavelets to sweep across the glass lens which lies within sphere r. If the resolution takes place on the eastern half of sphere R (after the First Reversal), we must allow the wavelets to advance westward until $t = 8\tau$, submit them then to Reversal, after which they travel eastwards and between $t = 10\tau$ and $t = 11\tau$ will cross the lens which lies within sphere r, in the form of slightly concave wavelets. Finally, we may increase τ *ad libitum*, and assign any size we please to sphere R; and the limit when both are increased indefinitely produces the resolution into flat wavelets. Furthermore, the construction makes obvious the respects in which the resolutions are related to one another. To every concave or convex wavelet there corresponds a flat wavelet consisting of precisely the same kind of light.

Although we have only spoken of the resolution of the light emitted by P_1, this is sufficient; for an exactly similar resolution is available for the light emitted by any of the other P's, and therefore for all of them. Thus the whole of the light which crosses the glass lens may be resolved either into flat or into convex or into concave wavelets sweeping across that portion of space.

53. Again, when dealing with microscopes and spectroscopes we have had no occasion to deal with other than uniform isotropic media; but the resolution into flat wavelets equally applies to any luminous disturbance traversing a uniform crystalline medium. The only change that has to be made is to substitute everywhere throughout the proof the wave-surface in the crystal for the spheres of which we have hitherto made use.

LIV. *The Origin of Radium*. *By* Bertram B. Boltwood [*].

IN previous papers [†] it has been shown that the quantities of uranium and radium which occur together in certain minerals are directly proportional to one another, and the important bearing of this relationship on the origin of radium has been pointed out. In the present paper the same general method of investigation has been applied to twenty-two separate samples comprising twelve distinct mineral species, and the constant proportionality between the uranium and the radium has been further confirmed [‡].

In order to determine the relative amounts of radium contained in any two samples of material, it is only necessary to determine the activity of the maximum or equilibrium quantity of radium emanation produced by equal weights of the two samples. The ratio between the activities of the two quantities of emanation will be the same as that between the quantities of radium which produced them. Since the activity of a given quantity of emanation can be measured with great ease and with extraordinary accuracy considering the infinitesimal quantities of matter dealt with, the quantitative measurement of the radium in this manner, through the medium of the activity of its emanation, affords a most convenient and accurate means of determining the relative quantity in which this element is present.

When a solid substance containing a radium salt is allowed to stand, the emanation which is formed is, for the most part, occluded by the solid. If the solid is dissolved the emanation escapes from the solution, slowly at ordinary temperatures and rapidly if the solution is heated to boiling. It is therefore possible in this way to completely separate the emanation from the radium salts, and under proper conditions

† Engineering and Mining Journal, lxxvii. p. 756 (1904); Nature, lxx. p. 80 (1904); Amer. Journ. of Science, xviii. p. 97 (1904).

‡ McCoy has published (*Berichte d. D. Chem. Gesell.* xxxvii. p. 2641, 1904) some numbers obtained from the direct measurement of the radioactivity of a number of uranium minerals, which seem to show that for the samples examined the activity is roughly proportional to the content of uranium. He assumes on the basis of this proportionality that the radium contained in each mineral is directly proportional to the uranium present, a conclusion which, under the circumstances, is only justified on the assumption of the further hypothesis that all of the other radioactive constituents (polonium, actinium, radio-lead) are also directly proportional to the uranium. This latter relationship still lacks experimental confirmation. McCoy's method is moreover quite unsuitable for minerals containing any notable quantities of thorium, as he himself acknowledges.

to collect it and measure its activity. Most solids have, however, a certain emanating power, *i. e.,* they lose at ordinary temperatures a certain proportion of their emanation. In the case of thorium and radium compounds this phenomenon has been examined by Rutherford [*], who has observed that the emanating power of the compounds is subject to great variations under different physical conditions. Rutherford found that the emanating power of ordinary thoria is permanently reduced about 90 per cent. when this substance is heated to a white heat, while the effect of high temperature on compounds of radium is likewise to decrease their power of giving off emanation at ordinary temperatures.

It will be shown in the course of this paper that marked differences in emanating power are exhibited by different mineral species and by different varieties of the same mineral substance. In determining the equilibrium quantity of emanation from these minerals it is therefore necessary to consider this factor, and either to determine it separately or else to compensate for it. In the methods described in previous papers, the compensation was accomplished by dissolving the mineral in a closed vessel and allowing the solution to stand until equilibrium had been reached. This plan, however, involved practical difficulties in the cases of many mineral substances, and has been abandoned in favour of the more direct method of separately determining the emanating power of each sample of material used in the investigation.

The measurement of the emanating power of the various substances was conducted as follows :—A sample of any given mineral having been selected, it was first crushed and then very finely pulverized in an agate mortar. A portion of this powder, weighing from 0·5 to 1·5 gram, was carefully weighed out and introduced, between short plugs of cottonwool, into a glass tube about 15 cm. in length and 7 mm. in diameter, the ends of which were drawn out into short capillary tubes. A slow current of air was then drawn through the tube in order to remove any emanation clinging to the solid, and the tips of the capillary tubes were sealed in the flame of the blowpipe. The tube was allowed to stand for forty days, and the emanation which it contained was removed by drawing through it about 100 c.c. of air, which was collected in a burette, allowed to stand for about fifteen minutes, and then introduced into an electroscope. The observed activity of the emanation divided by the weight of the material taken gave the activity of the emanation lost by one

* *Phys. Zeit.* ii. p. 429 (1901).

gram of the substance. A series of experiments showed that this factor could be more quickly and equally accurately determined by sealing up the sample for a definite period of about four days, measuring the activity of the emanation which then accumulated, and calculating the maximum value by the equation

$$I = \frac{I_t}{1 - e^{-\lambda t}},$$

where I_t was the value of the activity measured.

The arrangement of the apparatus used for the separation and collection of the radium emanation contained in the various samples is shown in the accompanying diagram. The flask A was about half full of boiled water and was closed with a rubber stopper, through which passed a glass
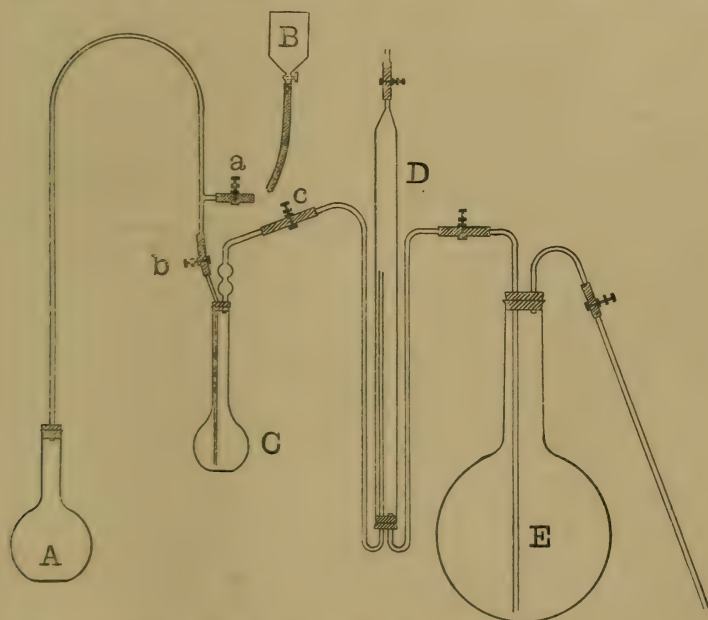
tube extending to b. This tube had a branch at a, to which was attached a short section of rubber tubing and a screw pinchcock. The flask E contained a dilute, boiling solution of sodium hydroxide (1 per cent.), a portion of which at the beginning of the operation was displaced to fill the burette D [*]. Some of the powdered sample was weighed out into

[*] For further details concerning the manipulation of this apparatus see Amer. Journ. of Science, xviii. p. 380 (1904).

the flask C. The quantity taken depended on the percentage of uranium contained in the substance and varied from 0·1 gram to 5 grams, the smallest quantities being taken of those minerals in which the content of uranium was high. The flask C had an approximate capacity of 200 c.c. and was closed with a tight-fitting rubber stopper which carried two tubes, one extending almost to the bottom of the flask, and the other only a short distance below the stopper. The air in this flask was largely displaced by a slow current of carbon-dioxide gas introduced through the longer of the tubes *. The tubes were connected by short lengths of rubber tubing to the rest of the apparatus as shown in the diagram, the stopper of the flask was removed and about 50 c.c. of the reagent used for the decomposition of the mineral was quickly introduced. The pinchcock b was closed and the flask was shaken with a rotary motion in order to thoroughly mix the reagent and the powdered mineral. The flask was warmed gently, the agitation of the liquid being continued without interruption, until the decomposition of the mineral was complete. During the heating operation the pinchcock c was opened, and the gas displaced by expansion was allowed to accumulate in the burette D, from which it was transferred to a second storage burette connected with the latter. The heating was discontinued and the pinchcock c was closed. The next step depended on the nature of the reagent used to decompose the mineral. If the reagent was concentrated sulphuric acid, a substance employed in a number of cases, the rubber tube at a was connected with a carbon-dioxide generator, the pinchcocks a and b were opened, and the gas was allowed to flow into C as the gaseous contents of the latter contracted on cooling. When the acid mixture in C had become quite cold, a process which could be greatly hastened toward the latter end by the cautious application of cold water, the pinchcocks a and b were closed, the carbon-dioxide generator was disconnected, and the tube at a was connected with the reservoir B containing water. The cock a was opened, permitting the water to fill the shorter limb of the tube Ab, and about 100 c.c. of water were admitted to the flask C through the cock b. The contents of the flask were agitated by shaking during the process of dilution, and the cock c was opened to allow the gas to escape into the burette D. When the reagent used in the flask C was a dilute acid, the dilution of the solution could be carried out

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* The object in using the carbon dioxide was that by absorbing it later the final volume of gas obtained was of convenient volume for introduction into the electroscope.

as soon as the decomposition of the mineral was completed. The tube attached to the reservoir B was disconnected, and a lighted burner was placed under the flask A. The water in the flask A was boiled for about five minutes, the steam being permitted to escape through the side tube *a*. The pinch-cock *a* was closed and the pinchcock *b* was opened, the steam from the flask A passing into the flask C. The pinchcock *c* was opened, and the gases were driven from the flask C into the burette D. The boiling of the water in A was continued until no further accumulation of permanent gases could be detected in the burette D. This portion of the operation required about 20 minutes, and the gases as they accumulated in the burette D were transferred to the storage burette. When the last of the gas had been transferred to the latter burette, it was treated with a small quantity of strong sodium-hydroxide solution which removed the remaining carbon dioxide. The residue of gas was usually from 50 to 100 c.c. in volume, and consisted chiefly of air which had been intentionally left in the flask C. It was allowed to remain in the storage burette for about 15 minutes, and was then transferred through a tube containing dry calcium chloride to an air-tight electroscope. At the end of three hours the activity of the emanation in the electroscope was measured. The observed activity of the emanation, expressed in terms of the number of divisions of the scale passed by the gold-leaf in one minute, was divided by the weight of the sample taken, and gave the activity of the emanation from one gram of the substance under investigation. The sum of the activity of the emanation thus obtained from one gram of the sample plus the activity of the emanation spontaneously evolved from one gram of the sample at ordinary temperatures, gave the activity of the total or equilibrium quantity of emanation corresponding to the radium contained in one gram of the material tested.

The analytical determination of the uranium in the minerals was conducted as follows * :—

The minerals soluble in dilute acids were treated with either pure, dilute (1 : 1) hydrochloric acid, or with dilute hydrochloric acid containing a little nitric acid. The solution was evaporated to dryness and heated until the excess of acid was driven off. The residue was moistened with concentrated hydrochloric acid, water was added and the liquid was heated to boiling. The residue of silica was filtered off, and the filtrate was treated with an excess of

* The method used is a modification of that described by Brearley, ‘Analytical Chemistry of Uranium,’ 1903.

hydrogen sulphide. The precipitated sulphides were filtered
off, the filtrate was boiled to remove the excess of hydrogen
sulphide, and the salts in solution were oxidized with a small
quantity of nitric acid. About 0·1 gram of microcosmic salt
($NH_4NaHPO_4 . 4H_2O$) was added to the solution, which was
allowed to cool somewhat, and was then poured into a solution
containing 10 grams of sodium carbonate (crystals) dissolved
in about 150 c.c. of warm water. The mixture was heated to
boiling and boiled gently for five minutes, cooled to about
50° C., and 10 grams of ammonium chloride added. The
mixture was again heated, gently boiled for five minutes
longer, and allowed to stand until cold. The precipitated
phosphates were filtered off, the filtrate boiled in a large
beaker until the greater part of the ammonium carbonate
was driven off and the uranium phosphate was precipitated
from the solution. Just enough nitric acid was added to
completely dissolve the precipitate, and the solution was
boiled to expel the last trace of carbonic acid. About
0·1 gram of microcosmic salt was added to the solution and
then 10 grams of pure crystallized sodium thiosulphate.
The solution was boiled for 15 minutes and the precipitate of
sulphur and uranium phosphate was filtered into a (platinum)
Gooch crucible, washed and ignited. The residue of phos-
phate was dissolved in the smallest possible quantity of hot,
concentrated nitric acid, the solution was separated from the
asbestos-felt, the excess of acid almost neutralized with
ammonia, and the clear solution poured into a solution of
3 grams of ammonium carbonate in 30 cc. of water. A slow
current of hydrogen sulphide was now passed into the
mixture for about two minutes and the liquid was heated
cautiously to boiling. The liquid was allowed to cool and
the precipitate, which in some cases was very slight, was
filtered off. The filtrate was boiled until the excess of
ammonium carbonate was expelled, acidified with nitric acid,
0·1 gram of microcosmic salt added, and the uranium re-
precipitated as phosphate by adding 10 grams of sodium
thiosulphate and boiling for 15 minutes. The precipitate
was filtered into a Gooch crucible, washed thoroughly with
water, ignited gently until the sulphur was removed, and
then heated to a low red heat and allowed to cool in a
desiccator. The uranium contained in the green pyro-
phosphate of uranium thus obtained was calculated on the
assumption that the green salt contained 68·55 per cent. of
uranium [*], an assumption which was frequently verified by
converting the green salt into the normal yellow modification

[*] Brearley, *loc. cit.*

(containing 66·81 per cent.), by solution in nitric acid, evaporation of the solution, and general ignition of the residue. It is important that the precipitate, obtained after treating with hydrogen sulphide the ammonium-carbonate solution of the first uranium precipitate, be tested for uranium, by dissolving it in hydrochloric acid, nearly neutralizing the free acid with ammonia, and adding potassium ferrocyanide. If a reddish precipitate, indicating the presence of uranium, is obtained, it is best to repeat the entire analytical operation, although fairly satisfactory results can usually be obtained by the following procedure :—The precipitate of ferrocyanides is filtered off and treated on the filter-paper with a solution of ammonium carbonate. The filtrate is boiled with a small quantity of pure sodium hydroxide until it no longer has the odour of ammonia, and the precipitated sodium uraninate is filtered off, dissolved in nitric acid and added to the filtrate from the original precipitate. The precipitate never contains appreciable quantities of uranium unless (1) the solution contains too much free nitric acid, (2) the hydrogen sulphide is passed too long into the solution, or (3) the solution in ammonium carbonate is heated too strongly.

Many of the minerals used in this investigation were not soluble in dilute acids and required the action of hot, concentrated sulphuric acid to effect their complete decomposition. These minerals were treated with a large excess of concentrated sulphuric acid and a little hydrofluoric acid in a platinum dish, and heated until sulphuric-acid fumes were copiously evolved and the mineral was completely decomposed. The resulting mixture was allowed to cool and was then treated with about 100 c.c. of cold water. The whole was heated to boiling, oxidized with bromine water, and filtered. The filtrate was diluted to a volume of about 250 c.c., heated to boiling, and an excess of ammonia added. The precipitate was filtered off, treated with dilute hydrochloric acid, the excess of acid partially neutralized with sodium hydroxide, and the solution was poured into a solution containing 15 grams of sodium carbonate. The mixture was boiled for five minutes, allowed to cool somewhat, 10 grams of ammonium chloride added, and the rest of the operation carried out as previously described. The quantities of the minerals taken for the analysis varied from 0·1 gram to 1 gram, depending on the percentage of uranium present, the larger quantities being of course taken of those minerals containing the least uranium.

Like many other analytical operations, this one requires a

certain amount of familiarity on the part of the operator in order to insure complete success, but it is comparatively rapid, and there is every reason for believing in the reliability of the results.

The chemical character of the specimens of minerals examined was most diverse, and the content of uranium varied from 75 to 0·3 per cent. The more important characteristics of the samples used are given below :—

1. *Uraninite.*—This substance, which is commonly known as pitchblende, is a compound of uranium dioxide (UO_2) and uranium trioxide (UO_3), approximating to the general formula U_3O_8. It contains lead and the rare earths (usually including thorium). This particular sample was a very pure variety and was completely soluble in dilute nitric acid.

2. *Uraninite.*—This specimen contained very noticeable quantities of sulphide of iron (pyrite).

3. *Gummite.*—This substance is a decomposition product of uraninite to which the doubtful formula (Pb,Ca) $U_3SiO_{12}.6H_2O$ has been given. The sample used was a portion of a large specimen, was orange-yellow in colour, and was completely soluble in dilute hydrochloric acid. It contained small quantities of the rare earths in addition to the elements indicated in the formula.

4. *Uraninite.*—The Joachimsthal variety, which has been the source of most of the radium of commerce. This specimen contained small quantities of the sulphides of iron and lead.

5. *Uranophane.*—A decomposition product of gummite having the general composition represented by the formula $CaO.2UO_3.2SiO_2.6H_2O$. This sample had a chalk-like structure, was pale yellow in colour, and contained only traces of lead and the rare earths.

6. *Uraninite.*—A very impure variety, containing considerable sulphur, iron, and tin.

7. *Uranophane.*—This specimen was taken from a compact piece, canary-yellow in colour, and containing appreciable quantities of lead and the rare earths. $SiO_2 = 13\cdot4$ per cent.

8. *Thorogummite* (?).—A mineral of doubtful composition, containing a considerable proportion of thorium, lead, and iron. Is chocolate-brown in colour, and occurs as a decomposition-product of uraninite. $SiO_2 = 13\cdot4$ per cent.

9. *Carnotite.*—A mineral having a composition approaching that expressed by the formula $K_2O.2U_2O_3.V_2O_5.3H_2O$. The sample contained a considerable quantity of sandstone and a small quantity of lead.

10. *Uranothorite.*—A variety of thorite ($ThSiO_4$) ; a silicate

of thorium and uranium. A clove-brown, translucent specimen. Contained a little lead.

11. *Samarskite.*—A niobate and tantalate of cerium and yttrium. Uranium and iron also present in considerable quantities.

12. *Orangite.*—A variety of thorite. This specimen was a single transparent crystal of a honey-yellow colour. It contained some lead.

13. *Euxenite.*—A niobate and titanate of yttrium, erbium, cerium, and uranium. The sample was of a brownish-black, massive variety.

14. *Thorite.*—A silicate of thorium and uranium ($ThSiO_4$). The sample was of a massive variety, yellowish-brown in colour. It contained a little lead and iron.

15. *Fergusonite.*—Essentially a metaniobate (and tantalate) of yttrium with erbium, cerium, uranium, and iron. The sample was of the variety called "tyrite," and was not completely soluble in hot, concentrated sulphuric acid. The uranium-radium ratio was therefore calculated on the basis of the portion soluble in the acid.

16. *Æschynite* (?).—This specimen was purchased under the name of æschynite, but failed to show the characteristic behaviour of that mineral before the blowpipe. Analysis showed that it was essentially a niobate and titanate of the rare earths, but its further identification was not attempted. In its general appearance it resembled æschynite.

17. *Xenotime.*—Essentially yttrium phosphate containing erbium and cerium. General formula (Y, $Ce,Er)PO_4$. The sample consisted of a single imperfect crystal.

18. *Monazite.*—This substance consists chiefly of cerium phosphate, but also contains notable quantities of lanthanum and didimium. General formula (Ce, $La,Di)PO_4$. The sample was in the form of sand and contained about 5 per cent. of thoria.

19. *Monazite.*—The sample consisted of a single, good-sized crystal. The amount of thoria present was from 7 to 9 per cent.

20. *Monazite.*—This sample was in the form of sand and contained about 5 per cent. of thoria.

21. *Monazite.*—A massive variety from Portland, Conn. Stated to contain about 8 per cent. of thoria.

22. *Allanite.*—A complex silicate, the general composition of which is represented by the formula $(Ca,Fe)_2 . (AlOH)$ $(Al,Cr,Fe)_2 . (SiO_4)_3$. It also contains small quantities of Ce, La, Di, and Er. This material was slightly radioactive, but the emanation which it contained corresponded to only

0·007 per cent. of uranium. Owing to the small proportion
in which the uranium was present, its analytical determination
was not attempted.

The localities from which the specimens were obtained,
the reagents used for decomposing the minerals in the
determination of the emanation, and the percentages of
uranium indicated by the analysis, are given in the table
which follows :—

No.	Substance.	Locality.	Reagent used.	Per cent. Uranium.
1.	Uraninite,	North Carolina...	Dilute HNO_3	74·65
2.	Uraninite,	Colorado	,, HNO_3	69·61
3.	Gummite,	North Carolina...	,, HCl	65·38
4.	Uraninite,	Joachimsthal ...	,, HNO_3	61·74
5.	Uranophane,	North Carolina...	,, HCl	51·68
6.	Uraninite,	Saxony	,, HNO_3	50·64
7.	Uranophane,	North Carolina...	,, HCl	49·84
8.	Thorogummite,	North Carolina...	,, HCl	33·17
9.	Carnotite,	Colorado	,, HNO_3	22·61
10.	Uranothorite,	Norway	,, HCl	11·38
11.	Samarskite,	North Carolina...	Concentrated H_2SO_4	10·44
12.	Orangite,	Norway	Dilute HCl	10·34
13.	Euxenite,	Norway	Conc. H_2SO_4	8·71
14.	Thorite,	Norway	Dilute HCl	7·54
15.	Fergusonite,	Norway	Conc. H_2SO_4	5·57
16.	Æschynite,	Norway	,, H_2SO_4	4·52
17.	Xenotime,	Norway	,, H_2SO_4	0·70
18.	Monazite,	North Carolina...	,, H_2SO_4	0·43
19.	Monazite,	Norway	,, H_2SO_4	0·41
20.	Monazite,	Brazil........	,, H_2SO_4	0.31
21.	Monazite,	Connecticut	,, H_2SO_4	0·30
22.	Allanite,	North Carolina...	,, H_2SO_4 (estimated)	0·007

The following table shows the results which were obtained
in the examination of these minerals. Column I. gives the
activity of the emanation contained in 1 gram of the mineral ;
Column II. the activity of the emanation lost by 1 gram
of the mineral at ordinary temperatures ; Column III. the
sum of the two previous activities (total emanation equivalent
to the radium contained in 1 gram of the mineral) ; Column
IV. the percentage of the total emanation lost by diffusion
from the cold mineral (the emanating power of the mineral
at ordinary temperatures) ; Column V. the weight (in grams)
of the uranium contained in 1 gram of mineral ; Column VI.
the ratio of the radium to the uranium (the number obtained
on dividing the total activity—Column III.—by the quantity
of uranium—Column V.).

From the data given in the table it is evident that
a direct proportionality (within the limits of experimental
error) exists between the quantities of radium and the
quantities of uranium, and the inevitable and only possible

No.	Substance.	I.	II.	III.	IV.	V.	VI.
1.	Uraninite	150·7	19·3	170·0	11·3	0·7465	228
2.	Uraninite	147·1	8·0	155·1	5·2	0·6961	223
3.	Gummite	126·7	20·3	147·0	13·7	0·6538	225
4.	Uraninite	131·8	7·8	139·6	5·6	0·6174	226
5.	Uranophane	108·0	9·7	117·7	8·2	0·5168	228
6.	Uraninite	112·5	3·1	115·6	2·7	0·5064	228
7.	Uranophane	88·8	24·7	113·5	22·8	0·4984	228
8.	Thorogummite	61·1	11·8	72·9	16·2	0·3317	220
9.	Carnotite	41·6	8·1	49·7	16·3	0·2261	220
10.	Uranothorite.........	24·9	0·3	25·2	1·3	0·1138	221
11.	Samarskite	23·2	0·16	23·4	0·7	0·1044	224
12.	Orangite	22·84	0·26	23·1	1·1	0·1034	223
13.	Euxenite	19·8	0·10	19·9	0·5	0·0871	228
14.	Thorite	15·6	1·0	16·6	6·2	0·0754	220
15.	Fergusonite	11·95	0·07	12·0	0·5	0·0557	215
16.	Æschynite	9·98	0·02	10·0	0·2	0·0452	221
17.	Xenotime	1·14	0·40	1·54	26·0	0·0070	220
18.	Monazite	0·88	0·00	0·88	...	0·0043	205
19.	Monazite	0·84	0·01	0·85	1·2	0·0041	207
20.	Monazite	0·76	0·00	0·76	...	0·0034	223
21.	Monazite	0·63	0·00	0·63	...	0·0030	210
22.	Allanite	0·014	0·00	......	...	(0·00007)	

conclusion is that uranium is the parent of radium [*]. The participation by thorium in the production of radium, which has been suggested by some, is entirely excluded, since the radium-uranium ratio in the thorites, containing approximately 50 per cent. of thorium, is the same as that in such minerals as carnotite and uranophane, where the content of thorium is not over a few hundredths of a per cent. at the highest.

The results obtained with monazite are of interest, in view of the recent controversy over the composition of this mineral substance. It was originally stated by Hofmann & Zerban [†] that Brazilian monazite, from which they, at that time, claimed to have separated inactive thorium, contained no uranium. In a later publication, the same authors [‡] credit monazite with containing about 0·1 per cent. of U_3O_8, on the authority

[*] Rutherford, in his book on 'Radioactivity' (p. 334), after pointing out the various reasons for assuming that radium is a disintegration product of some other element, and showing that the requirements of a parent element are more nearly fulfilled by uranium than by any other element, adds the following :—" Taking into account the presence of polonium and actinium in pitchblende, the activity of the best pitchblende is about the same as would be expected if radium were a disintegration product of uranium. If this hypothesis is correct, the amount of radium in any pitchblende should be proportional to the amount of uranium present, provided the radium is not removed by percolating water."

[†] *Berichte d. D. Chem. Gesell.* xxxv. p. 532 (1902).

[‡] *Ibid.* xxxvi. p. 3093 (1903).

of Herzfeld & Korn[*]. A paper by Barker [†] states that several "competent chemists," to whom samples from North Carolina were submitted for analysis, reported that no uranium was present. Zerban [‡] has offered further data on the composition of this mineral, in a paper in which the chemical method employed for the analysis is described. He was able to separate 0·038 per cent. of U_3O_8 from North Carolina monazite, and 0·040 per cent. of U_3O_8 from the Brazilian material.

The analytical method used by Zerban was criticised by the late Clemens Winkler [§]; and the experience of the writer has further confirmed the doubt as to its suitability for the determination in question. It was found, for example, that the addition of 0·4 per cent. of uranium to the original material failed to seriously affect the quantity of uranium ultimately separated when the analytical process involved the preliminary removal of the rare earths by precipitation with oxalic acid. The quantities of uranium obtained by this method varied from mere traces to amounts corresponding to several hundredths of a per cent. of the original mineral, and were to some extent dependent on the quantities of free sulphuric acid contained in the solutions from which the oxalates were precipitated. It seems probable that the greater part of the uranium is precipitated with the oxalates, since a relatively large amount of free mineral acid is required to retain uranium in solution in the presence of considerable quantities of phosphoric acid.

The results given in this paper were obtained by the use of the general method of analysis already described (p. 605), using only 1 gram of the sample; and the identity of the uranium separated was fully established by the well-known reactions with potassium ferrocyanide and hydrogen peroxide. A control analysis conducted with 10 grams of the North Carolina material demonstrated that the final uranium phosphate precipitate did not contain appreciable quantities of elements precipitated by oxalic acid and ammonium oxalate (thorium, zirconium, &c.), and gave uranium equivalent to 0·399 per cent. (corresponding to the ratio 221).

In determining the activity of the emanation it is possible to measure the leak of the electroscope with a fair degree of accuracy to the second place of decimals. When five grams of monazite (North Carolina) were used, the leak observed in two duplicate experiments was equal to 4·40 divisions per

* *Chemie d. selt. Erden.*
† Am. Journ. of Science, xvi. p. 161 (1903).
‡ *Berichte d. D. Chem. Gesell.* xxxvi. p. 3911 (1903).
§ *Ibid.* xxxvii. p. 1655 (1904).

minute*. It is evident that this affords a far more delicate and exact measure than can be obtained by analytical methods of the percentge of the uranium present, which, on the assumption of the value 228 for the constant of the instrument, indicates the following percentages of uranium in the monazites examined :—North Carolina 0·38 per cent. ; Norwegian 0·37 ; Brazilian 0·33 ; Connecticut 0·28. That the uranium is a constituent of the monazite itself and is not due to the admixture of other substances † is indicated by the fairly constant proportion in which it occurs in all of the samples examined. Nos. 18 and 20 were sands containing minerals other than monazite, but No. 19 was a crystalline variety, and No. 21 a massive variety of a high degree of purity.

The presence of these notable quantities of uranium and radium in monazite ‡ affords a plausible explanation of the occurrence of helium in this mineral, without recourse to the unsubstantiated hypothesis of the formation of helium from thorium. The age of monazite from a geological standpoint is evidently extreme, since it occurs as a primary constituent of some of the oldest igneous rocks of the globe. It is, moreover, extremely compact in its structure, and even in the form of fine sand loses at ordinary temperatures less than one thousandth of its radium emanation. It may therefore be safely assumed that it originally contained considerably more uranium than at present, and that the accumulated helium represents the disintegration product of radium for countless ages.

The question of the bearing of the emanating power of the different minerals on the proportion of the various disintegration products contained in them is not without interest. It will be observed from an examination of the numbers in Column IV. of the table on p. 609, that the emanating power of the cold minerals varies from practically zero to 26 per cent. of the total emanation. These numbers are of course considerably influenced by the state of fine division in which the minerals existed when tested, but the more strongly emanating minerals have a very measurable emanating power even when in good-sized fragments. Rutherford has shown § that in all

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* Corrected for the normal leak of the instrument, which was 0·012 division per minute.

† Zerban (*loc. cit.*) attributes the uranium in the North Carolina monazite sand to the presence of samarskite and other uranium minerals.

‡ The presence of radium has been previously observed by Haitinger and Peters (*Sitzungsberichte d. kaiserl. Akademie d. Wissenschaft. Wien,* May 1904).

§ Phil. Trans. Roy. Soc. London, cciv. p. 169 (1904) ; and Phil. Mag. Nov. 1904.

probability both polonium and radio-lead are slow transformation-prodcuts of the radium emanation, and it would therefore be expected that minerals with a high emanating power would contain a smaller proportion of these transformation-products than directly corresponded to the quantities of radium contained in them. Some rough preliminary experiments by the writer have given indications that the highly emanating minerals do actually contain a smaller proportion of both polonium and radio-lead than is present in the minerals having a low emanating power, but the experimental difficulties encountered do not permit of a positive statement to that effect at present. That the loss of emanation is not in any sense proportional to the content of uranium is evident, since one of the most strongly emanating minerals tested was xenotime (No. 17) which contains only 0·7 per cent. of uranium. The loss of emanation is probably dependent on the porosity of the mineral, but it is a curious fact that the two samples which to all appearances were the most porous (Nos. 5 and 9), were not the most highly emanating. Those minerals with a highly compact and vitreous structure (Nos. 11 to 16 and 18 to 21), with the exception of thorite (No. 14), had the lowest emanating powers of any of the samples examined[*].

The possibility of percolating water removing radium from uranium minerals, which has been suggested by Rutherford, is best illustrated in the case of the North Carolina minerals uraninite, gummite, thorogummite, and uranophane. The three latter minerals owe their formation to the action of percolating water on uraninite ; and it is a very common occurrence to find specimens containing a nucleus of undecomposed uraninite, surrounded successively by a layer of gummite and a layer of uranophane. The samples of

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[*] An intimate relation would seem to exist between the emanating power of a mineral and the loss of helium from the same substance, and those minerals with high emanating powers might be expected to contain relatively little helium. Moss (Trans. Roy. Dublin Soc. 1904) has observed that a considerable quantity of helium is set free when pitchblende is finely pulverized *in vacuo*, a phenomenon which Travers (Nature, lxxi. p. 248, 1905) attributes solely to the heating of the substance by attrition. Travers expresses the belief that helium exists in minerals in a state of supersaturated solid solution, and that grinding to an impalpable powder under conditions which precluded a rise in temperature, should result in the evolution of only minute traces of helium. It would seem more probable, however, that the state of the radium emanation during its temporary existence in the mineral is similar to that of the helium, and that the highly emanating minerals in the form of impalpable powders would lose at ordinary temperatures a very considerable proportion of the helium formed within them. It would, of course, be quite impossible to detect the escape of the helium by ordinary methods because of its minute quantity.

uranophane which were used in the investigation described in this paper had been taken from the earth within the last twelve months, and had undoubtedly been subjected to the action of percolating water until their removal. No deficit in the quantity of radium present in these minerals could be detected.

Soddy has described * an experiment undertaken to determine whether radium is formed *directly* from uranium, and the results which he obtained have led him to conclude that such is not the case. Similar experiments covering a period of seven months have been carried out by the writer and point to the same conclusion. It is therefore highly probable, as suggested by Rutherford †, that one or more intermediate changes exist between the uranium atom and the radium atom, although the identity of these intermediate products has not yet been established. The suggestion by Rutherford that actinium may be such a substance is of interest since the position of actinium in the family of radioactive elements has not yet been determined. Thorium appears to be quite out of the question because of the total lack of proportionality between it and the other substances.

The persistent appearance of lead as a constituent of uranium-radium minerals may indicate that this element is one of the final (inactive) products of the disintegration of uranium. Quite appreciable quantities of lead were found present in all of the minerals which were tested for this element (Nos. 1 to 12 and No. 14), with the exception of No. 5, which contained only a very slight trace of elements precipitated in acid solution by hydrogen sulphide. This mineral, No. 5, represents from a geological standpoint the youngest of the compounds examined, and the practical absence of lead from it is only what might be expected if lead were one of the final products. The presence of lead in the secondary pitchblendes from Saxony, Colorado, and Joachimsthal is in itself of little significance, as it is well known that these minerals occur in conjunction with galena and other lead compounds, but the North Carolina uranium minerals are found in a formation (pegmatitic dike) in which no true lead minerals have been identified. The same is true of thorite, which is moreover a well crystallized mineral.

The writer is indebted to Prof. S. L. Penfield of Yale University, Mr. Joseph Pratt of Chapel Hill, North Carolina, and Mr. H. S. Miner, Chemist of the Welsbach Light Co., for a number of the minerals used in this investigation.

139 Orange St., New Haven, Conn., U.S.A.
January 28, 1905.

* Nature, May 19, 1904.
† Bakerian Lecture, Roy. Soc. Lond. 1904.

LV. *On the Dimensions of the Cathode Images.* By N. HEHL [*].

IN vol. iv. p. 608, of the Phil. Mag. for 1902, Mr. H. A. Wilson gives an account of determinations of the current-density at the cathodes of vacuum-tubes when the cathodes are not completely covered with the glow. These determinations lead to the same relation between current-density and pressure as that which I had already discovered and discussed in detail in 1901 in my Erlangen Dissertation [†].

In his experiments, Mr. H. A. Wilson made use of wire cathodes which were placed along the axis of a tube 30 cms. long and only 3 cms. wide. Such narrow tubes may, according to my observations, lead to wrong results, by reason of the fact that there is too little space to allow the glow to spread freely. This accounts for the slight discrepancy between our results, and induces me to publish a supplementary account of the experimental arrangement employed by myself, and of the results obtained in this and in other directions.

Description of Apparatus.

The discharge-tubes employed consisted of large Wulff's bottles 25 cms. high and 15 cms. in diameter, along whose axes were arranged the cathodes to be investigated, which consisted mostly of wire. The length of the image was measured by means of an illuminated scale and telescope, through which the luminous image and the cathode were viewed simultaneously after reflexion in a totally reflecting prism. The relations connecting the area of cathode covered by the glow with the current and the pressure were determined. For full details, reference must be made to the original paper [‡].

Results Obtained.

1. *Relation connecting the area covered by the glow with the current when the pressure is constant.* The measurements showed that *the length of wire covered by the glow was directly proportional to the current,* or, since with a given wire the lengths are proportional to the areas, that

$$\frac{i}{2\pi r l} = \text{constant}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

i being the current, r the radius, and l the length of the wire

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cathode covered by the glow. Corresponding results were obtained with plates.

Thus, so long as the cathode is not fully covered and the pressure is maintained constant, the current-density remains constant.

2. *Relation connecting the area covered by the glow with the pressure when the current is constant.* The measurements were carried out with widely varying currents, and in the case of pure aluminium cathodes in nitrogen led to the result that with a constant current the cathode area covered by the glow is inversely proportional to the pressure, so that

$$2\pi rlp = \text{constant}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where p is the pressure.

3. *Relation connecting current-density and pressure.* From the results expressed by (1) and (2) it follows further that for aluminium cathodes in nitrogen the following proposition holds :—

The current-density at the cathode is directly proportional to the pressure, i. e.,

$$\frac{i}{2\pi rlp} = \text{constant}. \quad . \quad . \quad . \quad . \quad (3)$$

4. *Connexion between current-density and pressure with different electrode materials.*—The materials investigated were aluminium, platinum, and graphite. In the case of platinum, the relation found for aluminium and expressed by (3) was found to hold good, viz., that the current-density is directly proportional to the pressure, the only difference being in the value of the constant, which is 0·47 for aluminium, and 0·33 for platinum, if in equation (3) the current is expressed in milliamperes, the radius r and length l in cms., and p in mm. of mercury. No exact measurements were possible in the case of graphite by reason of the rapid disintegration [*].

5. *Connexion between current-density and pressure in different gases.*—Preliminary observations showed that the relation (3)

$$\frac{i}{2\pi rlp} = \text{constant}$$

only holds for aluminium in nitrogen, but that for any given metal in a given gas we must write

$$\frac{i}{2\pi rlp^n} = \text{constant}, \quad . \quad . \quad . \quad . \quad (4)$$

so that *the current-density varies as the nth power of the pressure.*

[*] Mr. Wilson finds the same value of 0·4 for aluminium and platinum.

For aluminium in hydrogen $n=2$, *i. e.*, in this case *the current-density is proportional to the square of the pressure.*

For surfaces of any shape equation (4) may be written

$$\frac{i}{f \cdot p^n} = \text{constant}, \quad . \quad . \quad . \quad . \quad (5)$$

where f stands for the area covered by the glow.

6. *Relations connecting the thickness of the canal-ray layer, the cathode dark space, and the negative glow layer with the pressure and current-density.*—If the thickness of any of the above layers be denoted by d, and the pressure by p, then, so long as the cathode is not completely covered by the glow,

$$p \cdot d = \text{constant},$$

i. e., *the thickness of the first three layers is inversely proportional to the pressure*[*].

If the cathode is completely covered by the glow, the thickness of the canal-ray layer increases with increasing current, rapidly at first, then more slowly. The thickness of the cathode dark space decreases with increasing current, rapidly at first, then approaching asymptotically a minimum. The drop of potential V at the cathode was, within the range of the present measurements, found to increase nearly in proportion to the current[†].

7. *Relations connecting the luminous intensity of the glow with the current and pressure when the cathode is not completely covered.*—So long as the glow does not completely cover the cathode, and the current remains constant, *the quantity of light sent out by the glow is constant, and entirely independent of the pressure* (the intensity was determined by a photographic method). Since, as shown above, the area covered by the glow is directly proportional to the current, we may also state the above proposition as follows :—

The total intensity of the glow per unit of current remains the same at all pressures, so long as the cathode is not entirely covered.

If, the cathode not being completely covered, the current be varied while the pressure is maintained constant, *the intensity of the glow is directly proportional to the current. Now since the current-density is directly proportional to the pressure, it follows that the intensity of the glow per unit of area is directly proportional to the current-density.*

* In this connexion, reference may be made to the experiments of H. Ebert, Wied. *Ann.* lxix. p. 200, 1899; H. Ebert and P. Ewers, Drude's *Ann.* x. p. 72, 1903, and O. Lehmann, *Verh. des naturwiss. Vereins zu Karlsruhe,* xv. 1902.

† A similar result was found by Herr G. C. Schmidt, *Ann. d. Physik,* i. p. 640, 1900, while recently Herr F. Stark (*Ann. d. Phys.* xii. p. 1, 1903) has found that the relation connecting V and i is not a linear one.

According to all previous determinations[*], the potential drop at the cathode is independent of pressure. If the current is maintained constant while the pressure is varied, the work done at the cathode remains the same. From the experiments it follows that the intensity of the glow is the same. Thus it would appear that the fraction of the energy which is converted into visible glow-radiation is constant. According to E. Goldstein, the glow arises from diffusely scattered cathode rays. If this is correct, it would follow from the above experiments that, independently of the value of the pressure, a constant fraction of the power per unit of current undergoes conversion into the energy of cathode rays, and of this latter a constant fraction into the energy of the glow. This transformation into the energy of the glow takes place at higher pressures in the immediate neighbourhood of the cathode, at lower pressures over a larger volume of the gas.

LVI. *Notices respecting New Books.*

Quadratic Partitions. By Lt.-Col. ALLAN CUNNINGHAM, R.E. Francis Hodgson, London, 1904. Pp. xxiii + 266.

THIS important tabulation of various quadratic partitions will be welcomed by all workers in the theory of numbers. The tables were begun in 1897, and were subsequently laid before the British Association Committee of Section A. The printing and publication were finally carried out with the aid of a grant from the Royal Society's Publication Fund. The scope of the Tables is indicated in the opening sentence of the Introduction. The main table occupies 240 pages, about 8/9ths of the book, and gives all primes (p) below 100,000, the prime factors of ($p-1$), and the terms of the partitions $a^2 + b^2, c^2 + 2d^2, A^2 + 3B^2$, and $\frac{1}{4}(L^2 + 27M^2)$. The same table contains up to the limit $p \not> 25,000$ the terms of the partition $e^2 - 2f^2$, and up to the limit $p \not> 10,000$ the terms of the partitions $a^2 - 5y^2$, $\frac{1}{4}(X^2 - 5Y^2)$, $t^2 + 7u^2$, and $\frac{1}{4}(v^2 + w^2)$. Other partitions with different multipliers of the second quadratic term are given in shorter tables, all cases of the multiplier less than 20 being included with the exception of 4, 8, 9, and 12. Then follows a series of short tables giving up to certain limits solutions of special forms of the Pellian Equation. We have checked the results in many cases, and so far have found no errors. In an Appendix the author indicates to what extent he has been able to utilise previously existing tables by Lambert, Barlow, Jacobi, and others.

Introduction à la Géometrie Générale. By G. LECHALAS. Gauthier-Villars: Paris, 1904.

THIS small pamphlet of 55 pages gives with the lucidity characteristic of French mathematical works a graceful introduction to the mysteries of hypergeometry. The author is one of the Continental authorities in this line of imaginative geometry, and any-

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[*] E. Warburg, Wied. *Ann.* xxxi. p. 579, 1887.

thing from his pen is certain to reward careful reading. There are three chapters. In Chapter I. the conceptions of space symmetry and curvature are considered as they present themselves in Euclidean space. Chapter II. is devoted to Euclidean Geometry of Four Dimensions, the particular aspects discussed being rotation about a plane and the properties of the three-dimensional spheres in this four-dimensional spread. The final chapter is taken up with a brief account of certain fundamental properties of spaces with negative curvature. The outlook throughout is purely geometrical, and the reader has for the most part to supply mentally the geometrical figures which form the bases of the arguments.

Traité Théorique et Pratique d'Electricité. Par H. Pécheux, *Professeur de Physique et de Chimie à l'École Nationale d'Arts et Métiers d'Aix.* Avec Notes Additionnelles de J. Blondin et E. Néculcea. Préface de J. Violle. Paris : Ch. Delagrave. Pp. xx + 720.

The subjects dealt with in this work cover an enormous range—including not only the theoretical treatment of magnetism and electricity, but also the more important practical applications of these sciences. As a record of the course of instruction provided at the École Nationale d'Arts et Métiers the book is interesting, but the task attempted by the author is a very formidable one, and there are various blemishes in the book which call for criticism.

On the very first page we read :—". . . en 1600, Gilbert, médecin anglais, reconnut que d'autres substances : la résine, le caoutchouc, la gutta-percha . . . s'électrisent." This, if true, is very interesting information indeed from a historical point of view ; but the author does not explain how Gilbert came into possession of india-rubber and gutta-percha. We were always under the impression that india-rubber was unknown to the civilized world until the eighteenth century, and that the introduction of gutta-percha did not take place until the nineteenth century.

In dealing with magnetism, the author repeatedly uses the term " coercive force " in its old-fashioned and now generally discarded meaning, and is evidently unaware of the new meaning given to this term by Hopkinson.

The various diagrams of magnetic fields given in the section on dynamos are hopelessly wrong, not the slightest attempt being made to indicate the refraction of the lines of induction as they pass from air into iron, or *vice versa*.

In deducing the formula for the E.M.F. of a Gramme ring, the author uses a method which we can only characterize as inconceivably ponderous.

The descriptions of dynamos, alternators, &c. given by the author are far from satisfactory, and include a number of types now completely obsolete, and fit only for the museum. The author himself, indeed, seems to be aware of this fact ; for in one instance, at least, after describing a machine, of which an illustration is given, he states that it is no longer used. Then why describe it, and deliberately waste valuable space ?

We cannot criticise the book in detail, but, apart from minor blemishes such as those we have just mentioned, it seems to us that, in view of the extremely elementary nature of the work, there is a quite unnecessary display of mathematics, the author using the infinitesimal calculus in many cases where a much simpler mode of treatment would have led to the desired result. The space at the disposal of the author might, we think, have been utilised much better, and many of the poorly reproduced pictures of the trade-catalogue order might with advantage be omitted, and diagrams or drawings of real value substituted for them.

Die Schule der Chemie. Erste Einführung in die Chemie für Jedermann. Von W. OSTWALD, *O. Professor der Chemie an der Universität Leipzig.* Zweiter Theil. Braunschweig: F. Vieweg und Sohn. 1904. Pp. viii + 292.

NOT much more than a year has elapsed since we had occasion to review Part I. of this excellent work, and now Part II., which completes the book, is before us. We are glad to note that an English translation is in active preparation, so that English readers will soon have an opportunity of becoming acquainted with a work of exceptional interest and educational value.

Part II. deals with the chemistry of the more important elements and their compounds. After an account of the mode of preparation and properties of chlorine, we are in Lesson 3 introduced to the subject of acids and bases, and the quantitative relations involved in the neutralization of an acid by a base pave the way for the consideration of the chemical elements, their combining weights, the law of multiple proportions, the atomic theory, and the volume law of gaseous combination. The next three Lessons deal with the phenomena of electrolysis, acids and salts, and then the study of the various elements is resumed and continued to the end of the book.

The question-and-answer form of exposition used in Part I. is also retained in Part II., and adds not a little human interest to the book, which is worthy of the closest study by student and teacher alike, for—as the author himself candidly admits in his preface—it was not written solely for the benefit of beginners.

LVII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

February 22nd, 1905.—J. E. Marr, Sc.D., F.R.S., President, in the Chair.

THE following communications were read:—

1. 'On the Order of Succession of the Manx Slates in their Northern Half, and its Bearing on the Origin of the Schistose Breccia associated therewith.' By the Rev. John Frederick Blake, M.A., F.G.S.

The author first describes a section where the Barrule Slate,

the Snaefell Laminated Slate, and the Agneash Grit follow in descending order, dipping north-westward. The last-named is markedly laminated. The ascending series, by dip, is again repeated after a fault, and at Tholt-y-will the Schistose Breccia appears near or at the top. The best sections of the latter for stratigraphy are in Glen Auldyn and Sulby Glen. The Schistose Breccia is therefore taken as an additional upper member of the series.

General Distribution of the Manx Slates.—The Agneash Grit in Maughold Head is nearly vertical, and is followed by the Snaefell Laminated Slates. The height to which the Barrule Slates attain diminishes from east to west. The Schistose Breccia depends on the presence of those slates; its occurrence is noted in four areas to the west of the Barrule Slates, which are repeated by faults.

The Position of the Lonan and Niarbyl Flags is indicated by two colours on the map. The Lonan Flags, etc. are never seen underlying the Manx Slates, but they overlie the earlier Manx Slates in the lower country near Maughold, near Sulby Glen, and north of Peel. They may, therefore, be called the Sulby-Glen Series.

Characters and Origin of the Schistose Breccia.—Prof. C. R. Van Hise gives three criteria for autoclastic rocks, with none of which do the Schistose Breccias agree, namely, the fragments of an autoclastic rock must be derived from the adjacent material; whether they be below or above; and the rock itself may be traced into an ordinary brecciated rock. The fragments examined do not agree with this statement. The rocks are not truly conglomeratic—some are too soft, or others too hard.

Examples at Ballaneary and elsewhere occur, of true autoclastic rocks, showing contrast.

2. 'On the Wash-outs in the Middle Coal-Measures of South Yorkshire.' By Francis Edward Middleton, F.G.S.

In the neighbourhood of Aldwarke and Thrybergh extensive workings have proved that wash-outs have occurred at various horizons in the Middle and Lower Coal-Measures, and the limits of certain of these have been accurately laid down on plans.

Denudation in the Barnsley Seam has been found over an area 1700 yards in length from east to west, and in the Parkgate Seam (240 yards below) over an area 2600 yards long from north to south. In neither case was the wash-out completely crossed, but its width cannot be less than 600 yards. The Swallow-Wood Seam, lying 60 yards below the Barnsley and above the Parkgate Seam, has been partly worked under the same area; but no signs of a wash-out have been found.

The opinion of the author is that the wash-outs occupy the sites of winding streams, meandering through the alluvial tracts in which the coal-seams were being formed.

Fig. 1.
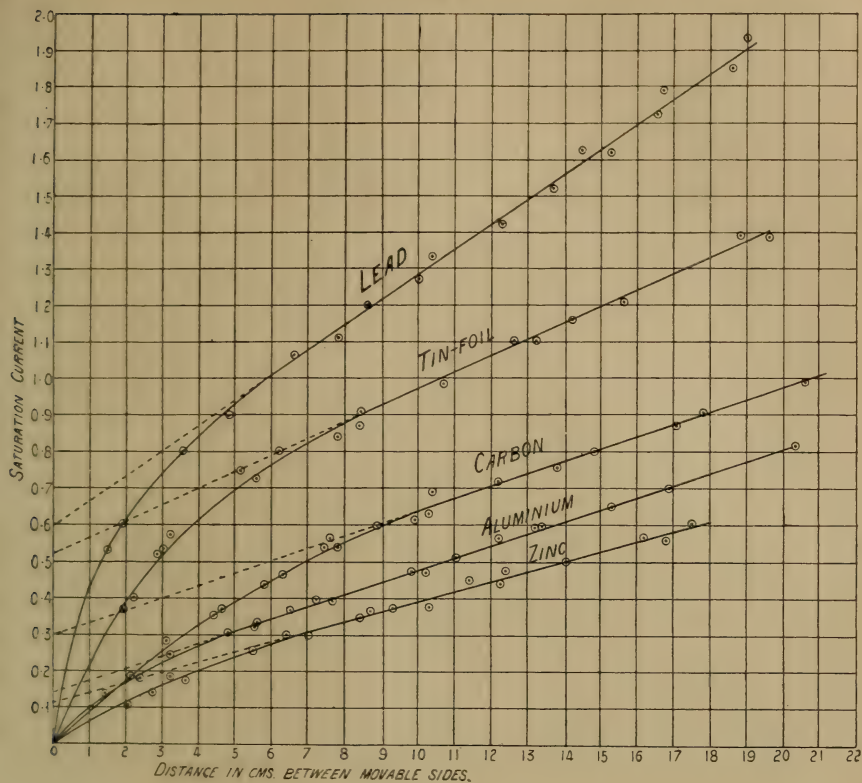
Fig. 2.
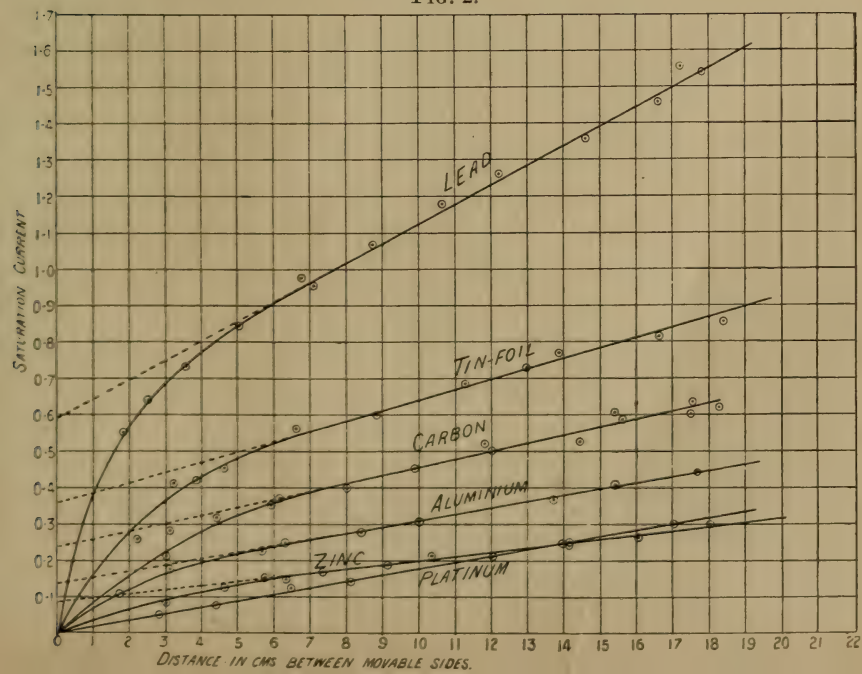
THE

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JOURNAL OF SCIENCE.

[SIXTH SERIES.]

M A Y 1905.

LVIII. *Heating Effect of the* γ *Rays from Radium. By*
E. RUTHERFORD, *F.R.S., and* H. T. BARNES, *D.Sc., Pro-
fessors of Physics, McGill University, Montreal**.

BY the experiments of Curie and Laborde † and others, it
is now well established that one gram of radium in radio-
active equilibrium continuously emits heat at the rate of
about 100 gram-calories per hour. Rutherford and Barnes ‡
showed that the emission of heat from radium was intimately
connected with its radioactivity, and was divided between
the different radioactive products of radium in proportion to
their α ray activity. These results indicated that the heating
effect of radium and its radioactive products was mainly due
to the bombardment of the active matter (on the vessel con-
taining it) by the stream of α particles which are continuously
thrown off. One of us has calculated that the energy of the
β particles emitted from radium is only a small percentage
of the energy of the α particles. This conclusion is supported
by the experiments of P. Curie, who found that only a small
increase of the heating effect of radium was observed when
the radium was surrounded by one millimetre of lead—a
thickness sufficient to absorb a large proportion of the β
particles. The total ionization which results from a complete
absorption of the β and γ rays in the gas, is in each case only

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* Communicated by the Authors. A preliminary account of the results
of this investigation was communicated in a letter to ' Nature,' p. 151,
Dec. 15, 1904.
   † Curie and Laborde, *Comptes Rendus*, cxxxvi. p. 673 (1903).
   ‡ Rutherford and Barnes, Phil. Mag. Feb. 1904.

about one or two per cent. of the total ionization due to the α rays. If the relative number of ions produced by complete absorption of the different types of radiation may be taken as a comparative measure of the energy of the radiations, it is seen that the heating effect of the γ rays from radium should not be more than a few per cent. of that resulting from the α rays.

Paschen *, however, in a recent paper concluded that radium, when surrounded by a sufficient thickness of metal to absorb all the γ rays, gave out heat at the rate of 226 gram-calories per gram per hour, or, in other words, the heating effect of the γ rays was greater than that of the α rays.

Such a conclusion followed naturally from the hypothesis of the nature of the γ rays which Paschen had advocated in a series of papers. He considered that the γ rays consisted of negatively charged particles (electrons) projected with a velocity exceedingly close to the velocity of light. Since, according to the theory, the apparent mass of the electron increases with the speed as the velocity of light is approached, these electrons, if projected with very nearly the velocity of light, would have a considerable mass.

In order to account for the absence of any appreciable deflexion of the γ rays in a magnetic field, it was necessary to suppose that the apparent mass of the electron was at least forty times as great as that of the hydrogen atom.

Even if the number of such high-speed electrons were only a small fraction of the total number of electrons expelled from radium, yet, on account of their great energy of motion, they should produce considerable heating effect in bodies in which they were absorbed. If this theory of the nature of the γ rays were correct, a large heating effect of the rays due to these electrons might be reasonably expected.

Before describing the experiments we have made in this subject, a brief account will be given of the experimental arrangement employed by Paschen. Recourse was had to a form of Bunsen ice calorimeter, which had already been used for the determination of the heating effect of radium by Curie and Laborde. Paschen used 50 milligrams of pure radium bromide, which was enclosed in a glass tube 17 mm. long with walls ·3 mm. thick. This was inserted in a small ice calorimeter which was placed inside the vessel of a large ice calorimeter. The amount of mercury drawn in to compensate for the melted ice was obtained by weighing. Several observations were made which agreed

* Paschen, *Phys. Zeit.* v. p. 563 (1904).

very well, giving for the heating effect of the pure radium 98·5 gram-calories per hour. (Rd = 258.) On surrounding the radium tube with ·7 of a millimetre of lead in the same apparatus, no appreciable change in the heating effect was obtained.

In order to surround the radium bromide with a thickness of lead sufficient to absorb a large proportion of the γ rays, it was found necessary to use a calorimeter of much larger size. As the increase in the size of the calorimeter caused a corresponding increase in the errors involved from the ingress of heat, two exactly similar calorimeters were used. Similar vessels containing mercury were placed respectively on the opposite pans of a balance and supplied equal quantities of mercury to each calorimeter in order to compensate for the increase in the melting of the ice from outside sources. The calorimeters were made with an inner tube 4 centimetres wide, and an outer mantle 8 centimetres wide. Each calorimeter was placed in a stand and surrounded by an air mantle, and each stood in an accumulator vessel. Both vessels were placed side by side in a box. Fresh cracked ice was placed in the vessels which were connected by a siphon tube. After waiting from two to three hours for the temperature to become steady, readings were commenced. After ten hours a fresh filling of ice had to be made. The radium bromide tube was surrounded by a lead cylinder which stood in turpentine-oil in the calorimeter. The γ rays had to pass through a thickness of 1·92 cm. of lead. The balance between the calorimeters in the differential arrangement was kept in general under 5 milligrams per hour without the radium bromide tube. When this tube was inserted in the lead cylinder, the difference in melting amounted in three experiments to 89·6, 90·5, and 90·3 milligrams per hour. Other results were obtained under less favorable circumstances, which showed wider divergences, but which gave a mean of 94·9 mg. per hour. Taking 90 mg. per hour as the true result, this makes the total heat emission of the 50 mg. radium bromide 5·829 gram-calories per hour, whereas in the previous experiment only 3·04 gram-calories were obtained, leaving a margin of 2·79 gram-calories to be accounted for, apparently, by the γ rays. This result was so unexpected, and was of so much importance in considering the nature of the γ rays, that we decided to verify the experiments if possible by a different method. The difficulty of obtaining steady readings with the ice calorimeter is well known, and it was thought that more rapid and accurate measurements could be made by means of the differential air calorimeter which we had

previously employed to measure the heating effect of the radium emanation.

It was necessary to use larger flasks than were previously employed *, in order to admit the lead cylinders surrounding the radium bromide tube.

Our method was to surround the two glass tubes, into which the small radium bromide capsule was lowered, by exactly similar lead cylinders, 3 cm. high and 3 cm. in diameter. A hole was bored to the centre of the lead cylinder, into which the glass tube was fixed. The leads, supported by the glass tube, were accurately centred in the flasks. A second pair of cylinders, 5 cm. high, were cast and fixed in exactly similar glass tubes. The surface of the leads was wrapped around with a thin sheet of polished aluminium, to present the same radiation surface in all cases.

A pair of aluminium cylinders were cast in the same way around similar glass tubes, and the surfaces of the cylinders wrapped with polished aluminium. These cylinders were exactly equal in size to the first pair of leads.

It was a matter of a very few minutes to change the leads for the aluminiums and *vice versa,* and the greatest care was taken to have the cylinders always at the same level in the flasks. Great reliance can be placed on comparative measurements where a change of conditions can be made quickly. We think that the weak point of the differential ice-calorimeter measurements lay in the fact that apparently only one side was used for the lead, as the conditions could not easily be reversed.

In our experiments the radium capsule was placed in each flask alternately, and a change from the lead to aluminium cylinders gave a direct comparison of the relative absorption in the two metals. It can readily be shown that at least 50 per cent. more of the γ rays must have been absorbed in the leads than in the aluminiums. The heating effect of the radium could also be obtained without a metal envelope, as in our original experiments. In order to be independent of such difficulties as are introduced by the difference in thermal conductivity of the two metals, and the relative temperature of the two surfaces, which seems to exert the greatest influence in raising the temperature of the air in the flasks, we used our original heating-coil to calibrate the readings in every case. This consisted of 65 ohms of fine silk-covered manganin wire, wound in a coil of approximately the same area as the radium, and the heating current was adjusted so as to give approximately the same heat emission as the radium bromide used.

* See previous description, *loc. cit.*

The two flasks were connected with a U-tube containing xylene, and were immersed in a large water-bath. The whole was placed in a constant-temperature room in the basement of the building and the xylene level was read by means of a Pye microscope with a micrometer eyepiece. The difference between the levels in the two limbs of the U-tube was obtained each time. Systematic time-interval readings were made of one hour each so as to allow ample time for the metals to adjust themselves to the various temperatures.

We were surprised to find how sensitive the arrangement really was, and how constant the readings were for such minute sources of heat.

The method of making an experiment was very simple. The radium bromide was carefully weighed, and enclosed in a small brass capsule, closed by an accurately screwed cover, to which a fine thread was attached. The capsule was not more than 3 mm. wide and 15 mm. long, and enclosed 23·7 mg. of pure radium bromide. Having closed the flasks at atmospheric pressure, the radium tube was inserted in the lead-encased glass tube on one side, and after half an hour the first reading of the xylene levels was made. The radium tube was then transferred to the corresponding lead-covered glass tube in the other flask, and after exactly one hour the second reading was made. Then followed alternate readings accurately at hour intervals, until a sufficient number of readings were obtained. Usually a whole day was devoted to one set of readings.

Having found the difference in level of the xylene columns in each case after transferring the radium tube, the sum of these gave the total effect. The observations with the heating-coil were carried out in a similar manner, and gave the necessary data for calculating the heat emission. A constant heating current of ·0068 ampere was used throughout, and hence, by taking the ratio of the reading with the radium and with the coil, it could be seen at once whether any increase in heat emission had taken place when the aluminium cylinders were replaced by the lead.

In the case of the aluminium cylinders, the following readings were obtained from radium bromide :—

Date.		Number of observations.	Mean reading in divisions of microscope.
November 9th	. . . . .	4	9·40
,, 11th	. . . . .	6	8·97
,, 14th	. . . . .	4	9·27
,, 16th	. . . . .	6	8·97
	Average value		9·15

For the heating-coil the following readings were obtained :—

Date.	Number of observations.	Mean.
November 9th	3	16·1 div.
,, 10th	6	16·36 ,,
,, 15th	6	16·1 ,,
Average		16·19

The ratio of heating-coil to radium gives 1·769. Taking the coil as 65·2 ohms, and the current as ·0068 ampere, the total heating effect of the radium (Rd = 225) comes 106 gram-calories per hour.

Experiments with the leads gave for the radium on Nov. 19th as a mean of seven readings 11·94 divisions. These readings were as follows :—

Time.	Scale.
11.30	11·8 div.
12.30	11·7 ,,
1.30	12·0 ,,
2.30	12·7 ,,
3.30	11·2 ,,
4.31	13·4 ,,
5.30	10·8 ,,

Using the heating-coil, the following readings were obtained on Nov. 21st, the current remaining as before :—

Time.	Scale.
2.00	20·5 div.
3.00	20·55 ,,
4.05	20·65 ,,
5.00	20·5 ,,

This gives 20·6 as the average. The ratio of 20·6 to 11·94 comes 1·725, and the heat emission of the pure radium comes 108 gram-calories per hour. This showed us conclusively that no appreciable increase in heating effect was brought about by absorbing the γ rays. Had Paschen's results been correct, an increase of at least 50 per cent. should have been observed.

We decided to verify these results by making slightly larger lead cylinders. These were 5 cm. long and gave an absorbing mass 11 mm. thick (diameter 3 cm.). For the radium we obtained the following readings:—

Date.	Number of observations.	Mean.
November 30th	3	12·5 div.
December 1st	6	12·33 ,,

This gives an average of 12·42.

From the heating-coil we obtained as a result of six observations on Dec. 5th, 21·4 divisions with the same current as before. The ratio of these numbers comes 1·723, which is identical with the previous lead results, and agrees with the aluminium to within 3 per cent.

As a matter of interest, we removed the lead coverings and used the uncovered glass tubes. The following observations were obtained for radium:—

Date.					Number of observations.	Mean.
November 22nd	.	.	.	.	4	11·2
,, 23rd	.	.	.	.	10	11·34
,, 28th	.	.	.	.	5	11·8

This gives an average of 11·45.
With the heating-coil we obtained :—

Date.					Number of observations.	Mean.
November 24th	.	.	.	.	5	18·66
,, 26th	.	.	.	.	5	18·48

This gives an average of 18·57 and a ratio to the radium of 1·622, or a total heat emission from pure radium of 115 gram-calories per hour. The radical change in the conditions of the experiment on removing the comparatively large surface for radiation in the shape of the metal coverings, probably accounts for this value being different from the others by 7 per cent.

The crucial test is, we think, the comparison of the heat emission from the cylinders of aluminium and lead.

It is a matter of interest to compare the readings obtained for the different metals. The thermal conductivity of the aluminium is about four times as great as that of lead, and hence the average temperature of the former would be less than for the latter. The air in the flasks is heated chiefly at the metal surfaces, and the variation of the readings in the case of the lead and aluminium cylinders is probably due to the differences in the amount of heat escaping from the flat ends of the cylinders. In all cases the readings from the heating-coil showed a corresponding change, showing conclusively that this increase had nothing to do with the absorption of the γ rays. Moreover the observations show a small diminution in heat emission from the leads over the plain glass where only a small fraction of the β and γ rays were absorbed. We must conclude from our experiments that the γ rays contribute but a very small fraction of the total heat emission of radium, and that the apparent increase observed by Paschen must be explained by other causes.

The small heating effect of the β and γ rays compared with that due to the α rays is thus in agreement with calculation based on the relative amount of ionization produced by the rays. The results of this investigation combined with those given in our previous paper show that the heating effect of radium is largely due to the bombardment by the α particles expelled from its own mass.

McGill University, Montreal.
          Feb. 22nd, 1905.

Note.—Since the above paper was sent off for publication, a paper by Paschen has appeared (*Phys. Zeit.* vi. p. 97, Feb. 1905), in which he states that the conclusions advanced in his earlier paper have not been confirmed by later work. The ice calorimeter was found to be quite unsuitable to measure such small quantities of heat with accuracy.

LIX. *On the Excited Activity of Thorium.   By* Miss J. M. W. Slater, *B.Sc. (Lond.), Bathurst Student, Newnham College, Cambridge***.

[Plate VIII.]

Introduction.

THE following paper consists of two parts, giving an account of an investigation into the effect on thorium-excited activity, first of the cathode-ray discharge, and secondly of heat. The work was undertaken at the suggestion of Prof. J. J. Thomson. It has been shown that the activity imparted to a surface which has been in contact with the thorium emanation is probably due to a material deposit. The theory of successive changes as the cause of radioactivity has been worked out by Prof. Rutherford and Mr. Soddy†, who show that the disintegration of the thorium atom gives rise in succession to thorium X, emanation, and an active deposit, the latter being obtained on surfaces in contact with the emanation, and imparting to them a temporary activity. This decays to half value in about eleven hours, with the final production of inactive substances in quantities too small to be detected. The active deposit can be dissolved off from the surface by various acids &c., and separated again from the solution by electrolytic and other

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* Communicated by Prof. J. J. Thomson.
          † Phil. Mag. May 1903.

methods (von Lerch[*], Pegram[†]); it can be volatilized at a white heat (Gates)[‡] : but no methods hitherto employed have been able to destroy the activity, or to alter its rate of decay; under nearly all conditions this follows an exponential law, falling to half value in about eleven hours. In a few cases, where electrolytic and chemical methods have given substances with a quicker rate of decay, this can best be explained (as will be shown later) by assuming that there has been a separation of different stages in the decomposition of the active deposit, having different radioactive constants.

I. *Influence of the Cathode Ray Discharge.*

It was thought possible that cathode rays, which penetrate the molecules of substances on which they fall, might be able to affect the internal changes going on in the active molecule, and so alter its rate of decay.

Method of Investigation.

The following method was employed. A quantity of thorium hydroxide was placed in an earthed metal box, and the substance to be made active was fixed on an insulating support over the thorium, and connected to the negative pole of a battery of small storage-cells, the positive pole being earthed. The exposures used were generally about 24 hours ; even after this time there was usually a slight initial rise in the activity of the surface after its removal from the thorium, and the active matter was not exposed to the cathode rays until the period of regular fall (half value in 11 hours) had set in.

The activity was tested by observing the rate of deflexion of a quadrant electrometer in the usual way. The active metal was suspended inside a metal cylinder, from which it was insulated by ebonite slabs and guard-ring, and was connected to one pair of quadrants of the electrometer, the other pair being permanently earthed. The cylinder was charged to a potential high enough to give a saturation current across the enclosed air space (200 volts was generally used). In making an observation a key was raised so as to break the contact between the earth and the pair of quadrants connected to the active metal, and the time taken for a deflexion across a given length of the scale was observed by means of a stop-watch.

Two similar active surfaces which had been exposed to

* Drude's *Annalen*, Nov. 1903.   † Phys. Review, Dec. 1903, p. 424.
‡ Phys. Review, p. 300 (1903).

thorium for the same time and at the same potential were
used in nearly all cases, so that the activities before and after
the exposure of one of them to the cathode rays could be
compared.   The sensitiveness of the electrometer varied some-
what in the course of an experiment, as the charge on the
needle leaked away ; the effect of this was to make the ap-
parent value of the rate of decay rather too large.   When
comparative values were taken it was not generally necessary
to correct for this ; when the absolute constant of decay was
required, the electrometer was standardized at intervals by
connecting a uranium cell to the insulated quadrants.   The
saturation current across such a cell is constant, and therefore
the rate of deflexion is directly proportional to the sensitiveness
of the electrometer.

The substance made active was in nearly all cases alumi-
nium.   This is particularly suitable as it neither oxidizes nor
sputters to any appreciable extent in a discharge-tube.   In
the earlier experiments long strips of aluminium-foil were
used, and were coiled round in the tube so as to expose as
large a surface as possible to the cathode rays ; later on it
was found better to concentrate the activity on a small area
in the centre of a disk of the metal.   The discharge-tube
used with these disks is shown in fig. 1.

Fig. 1.
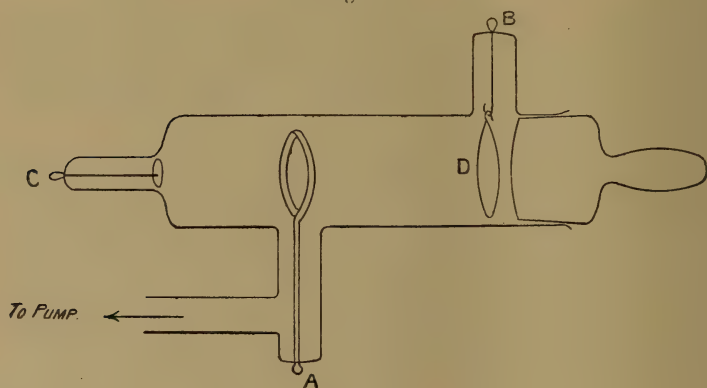
The cathode C was about 10 cm. from the active disk D,
the anode in the form of a ring being placed between them,
so as to minimize the risk of sputtering from the cathode to
the active surface.   The disk could be earthed, or connected
to either electrode, by means of the terminal B.   The tube
was closed by a ground-glass stopper, so that the active disks
could be put in and taken out without the necessity for sealing

and cutting the glass each time. The discharge was generally obtained from an induction-coil, and the times of exposure were usually from half an hour to an hour.

Effect of the Discharge.

In the earlier qualitative experiments, a fall in activity of from 10 per cent. to 20 per cent. was obtained in almost all cases after exposure to the rays, the curves of decay being similar to that of diagram 1, where the surface A was exposed to the discharge for $\frac{3}{4}$ hour, the control-piece B being tested while A was in the discharge-tube.

The effect was then examined in more detail, so as to determine as far as possible the exact nature of the action. The first point which it was important to settle was whether the activity lost by the metal disks was destroyed—*i. e.*, the rate of decay increased during the discharge—or whether it was merely driven off, in which case it should be found on the walls of the tube. To decide this point, a very small gold-leaf electroscope was made, consisting of a little brass plate and gold-leaf supported on an insulating stem of sealing-wax. This could be placed inside the tube, while the rate of collapse of the leaf was observed through a reading-microscope outside, both before and after the exposure of the active disk to the discharge.

The results obtained could hardly be regarded as quantitative, as no special precautions were taken to prevent leakage from the leaf along the insulating stand, and the capacity also varied somewhat according to the exact position of the electroscope relative to the metal in the tube ; the nature of the change occurring was, however, clearly shown. The electroscope readings could be roughly standardized by observing the increase in the rate of fall of the leaf when a disk of known activity was introduced into the tube. In these experiments, the observed deflexions of the electrometer were corrected for change of sensitiveness by means of the uranium cell. The observations of January 28th will serve as an example.

Rate of collapse of gold-leaf in tube before discharge .... $\Big\} = \cdot 36$ scale-div. per min.

Rate of collapse with disk C in tube $= 3 \cdot 24$ div. per min.

Therefore increase due to $C = 2 \cdot 88$ per min.

C gave an electrometer leak of 263 scale-divisions per minute, the leak with the uranium cell being $27 \cdot 75$ div. per min.

Therefore 1 div. per min. electroscope corresponds to about 88 per min. electrometer. On another occasion, 1 div. per min. electroscope corresponded to 68 per min. electrometer (corrected to uranium leak $= 27 \cdot 75$); therefore mean value of 1 div. per min. electroscope $= 78$ per min. electrometer.

On January 27th the disk C was exposed to the discharge for $\frac{3}{4}$ hour, and on removal from the tube the activity was found to have fallen to 85 per cent. of its original value (allowing for the regular decay); C now gave an electrometer leak of 429 div. per min. (uranium leak $= 20 \cdot 6$). The 15 per cent. of activity lost should therefore give an electrometer leak of $75 \cdot 8$ per min. ($\mathrm{Ur} = 20 \cdot 6$), or 102 per min. ($\mathrm{Ur} = 27 \cdot 75$); this corresponds to about $1 \cdot 3$ per min. electroscope leak. The observed rate of fall of the gold-leaf in the tube was $\cdot 17$ div. per min. before the exposure of disk C to the rays, $\cdot 92$ per min. after; the leak is therefore increased by $\cdot 75$ div. per min. A large proportion of the activity lost by the disk is thus evidently present on the walls of the tube, and in view of the rough nature of the experiment it is extremely probable that all the activity lost is volatilized unchanged. Again, on January 29th, disk A was exposed for one hour to the discharge, and lost 52 per cent. of its activity. The ratio of the electroscope leak with A in the tube to that without A, after exposure, was found to be $2 \cdot 3$; the greater part of the activity lost is thus accounted for. The walls of the tube retained their activity for several hours, and drawing a current of air through did not diminish it.

It is therefore evident that the active matter is volatilized, not destroyed, during the passage of the discharge; and later experiments made with a rather different arrangement, by means of which the activity driven off could be separately examined, confirmed this conclusion.

Cause of Loss of Activity.

It was not easy to determine the immediate cause of the effect observed, as during the cathode-ray discharge a number of secondary actions are possible. With the ordinary arrangement of the tube, the disk being at D, C cathode and A anode, a loss of activity was always obtained, whether the disk was insulated, earthed, or connected to the anode. The discharge was sometimes obtained from an induction-coil and sometimes from a large Wimshurst machine, the latter being much steadier in its action; the results, however, were very similar. The loss of activity varied in amount from 20 per cent. to 60 per cent. for exposures of $\frac{1}{2}$ hour to 1 hour; but the variation seemed to depend more on the air-pressure in

the tube and the intensity of the discharge than on its source, or the connexions of the disk. When, however, the disk was made the cathode, the effect was much greater, being as much as 90 per cent. in 10 minutes for the coil discharge, and 55 to 60 per cent. in 10 minutes for the Wimshurst discharge.

The direct incidence of cathode rays is known to cause volatilization with some metals, as found by Dr. H. A. Wilson, who observed that a thin silver film was removed from a glass surface when a strong beam of cathode rays fell upon it. This, however, is not a sufficient explanation of the present action, for in some cases the active side of the disk was turned away from the cathode, the inactive side being exposed to the rays ; in others the rays were deflected off the disk by means of a strong magnetic field. In the former case the loss of activity was undiminished, being often as much as 50 or 60 per cent. in $\frac{3}{4}$ hour ; in the latter ths loss, though less, was not prevented, being from 10 to 20 per cent. Also when the pressure was so high that only very slight fluorescence appeared on the glass walls, a loss of activity still occurred.

The most probable cause which next suggests itself, is an action analogous to the sputtering of a metallic cathode during the passage of the discharge, and this is supported by the large effect obtained with the active disk as cathode. To test this hypothesis, a polished disk of cadmium (a metal which gives considerable sputtering) was substituted for the active disks in the discharge-tube. When this was connected to the anode for the coil-discharge, a fairly thick film was obtained on the walls of the tube in 1 hour, the metal being sputtered off both towards the cathode and back on to the glass stopper. With the cadmium disk as cathode, a noticeable deposit was obtained in 10 minutes, though considerably less than before. With the Wimshurst discharge no deposit was obtained from the anode in 35 minutes, and only a very slight one from the cathode after 25 minutes. The effect cannot, therefore, be regarded as corresponding at all closely to the loss of activity from the active disks. At the same time, it is very probable that in the latter case more than one cause may be operating, and that when the active disk is made cathode the sputtering either of the active matter itself or of the metal on which it is deposited is the predominant factor, while in other cases some other cause is more important. The very small amount of sputtering obtained by various observers with aluminium, even as cathode for high potential discharges, makes it more probable that it is the active matter only which goes off.

This conclusion is strengthened by the fact that when the surface made active was mica instead of aluminium, the results obtained were very similar.

The other possible agents which remain to be considered are (*a*) low pressure, (*b*) strong local heating caused by the incidence of cathode rays, (*c*) ultra-violet light, (*d*) chemical action of the gases in the tube on the surface made active, (*e*) chemical action on the active deposit itself, (*f*) action of free ions formed during the discharge, (*g*) a secondary radiation of the nature of soft Röntgen rays.

(*a*) Low pressure alone, without the passage of a discharge, has been found by previous investigators to have no effect on the excited activity, and this was confirmed with the present apparatus.

(*b*) The heating effect may also be safely excluded, for the temperature of volatilization of thorium-excited activity has been found to be over 600° C., while the temperature in the present case was probably always under 100°, and usually the tube only became slightly warm. The surface of the aluminium remained bright, and showed no sign of any approach to the melting-point.

(*c*) The effect of ultra-violet light was tested by placing an active disk about 1 cm. from aluminium points, between which a discharge from a coil and leyden-jar passed, giving a strong blue light rich in ultra-violet rays. A quartz plate was placed in front of the disk, to prevent any spurious effect due to sputtering from the points, and the disk was connected to the negative pole of a battery giving a potential of -280 volts. No loss of activity occurred after an hour, and the experiment was therefore repeated with the disk in air at a low pressure. This was obtained by placing it in a tube with a quartz window, which was made air-tight and exhausted by means of a water-pump to a pressure of about $1\frac{1}{2}$ cm. The disk was connected to a potential of -500 volts, the positive pole of the battery and a metal ring in front of the disk being earthed. Exposure to the discharge still caused no change of activity, and it may therefore be concluded that the ultra-violet light present during the cathode-ray discharge is without effect.

(*d*) The mica disks mentioned above were used primarily in order to eliminate any chemical or other effect due to the nature of the active surface. The activity lost was just about the same as when aluminium disks were used, varying from 30 to 60 per cent.; and here again the effect was quite as large when the inactive side was exposed to the rays.

(*e*) A direct action of the gas on the active deposit itself,

with the formation of a volatile compound, is still possible. This was first tested by replacing the air in the discharge-tube by hydrogen; the loss of activity was again from 30 to 55 per cent. A tube of a different shape was then used (fig. 2). The discharge passed between the terminals A and

Fig. 2.
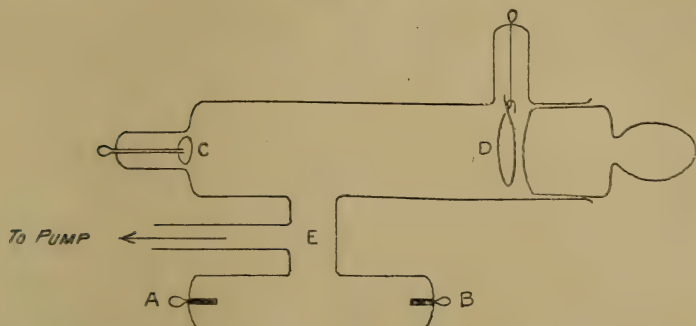
B, so that the active disk D was out of the direct line of discharge, while the connecting tube E was wide enough to allow free intermixture of gases between the lower branch and D; with this arrangement there was no loss of activity. When C was made anode, A or B being cathode and D insulated, about 10 per cent. of the activity was lost; this may easily be due to the anode acting temporarily as cathode, for a considerable amount of reversal always occurs with a coil-discharge. To test this point, C was made anode and B cathode for the Wimshurst discharge; in this case there was no loss of activity. With D as anode and A cathode, there was about 20 per cent. reduction. This is considerably less than the average value with D anode and C cathode, and this (together with the low values for the deflected discharge) gives support to the view that some part of the effect is due to a direct action of the rays. It was also observed that when the discharge took place at a pressure which was too high to give any cathode rays, but would not be likely to alter any chemical actions occurring, there was no loss of activity. All these experiments show that the action cannot be a chemical one.

(*f*) That the effect is due to the presence of free ions in large numbers near the active surface is also improbable, in view of these results. This point was further tested by negatively electrifying an active disk close to a Bunsen flame: the flame-gases, though strongly ionized, caused no diminution in the activity.

(*g*) The effect of a secondary radiation was tested by enclosing the active disk while in the discharge-tube in a little cylindrical box of aluminium, with a closely-fitting lid in which was a window of very thin aluminium-foil; the active surface was immediately behind the window, on to which a strong beam of cathode rays fell. There was no loss of activity in this case. The effect of Röntgen rays was also directly tested by placing the active disk just outside a Röntgen-ray bulb, where a fluorescent screen showed that the rays were strongest: an exposure of $\frac{3}{4}$ hour caused no change.

The only causes left capable of explaining the observed effect are, therefore, the variations of potential occurring during the discharge, together with some direct action of the cathode rays. To obtain, if possible, further light on this point, a number of experiments were made in which the active disks were enclosed in earthed metal boxes. The lid of the box was generally of fine copper gauze. It was thought at first that this arrangement would prevent any considerable variation of potential on the enclosed disk, but in practice it is hardly possible to get the earth connexions good enough to prevent changes of potential with the very rapid variations occurring with a coil-discharge, especially when (as was generally the case) the gauze lid faced the cathode, and the cathode rays were able to enter the box. When the metal bottom was towards the cathode, the loss of activity did not occur. When the box was made cathode, the enclosed disk lost 15 to 20 per cent. in a short time.

The similarity of the effects obtained with rays falling on the active and inactive sides of the disks is rather curious. The incidence of cathode rays must cause very large momentary variations of potential, which will spread below the surface struck to a depth varying inversely as the conductivity of the material. Thus with thin mica disks, it is not surprising that there should be sputtering of the active matter from the back as well as from the front. Aluminium also is not one of the metals of highest conductivity, and the thin disks generally used (·12 mm. in thickness) may allow of considerable variations of potential reaching the side away from the cathode. To test this point, disks ·34 mm. thick were used. When cathode rays fell directly on the inactive side, these disks also lost their activity, though not to the same extent as before, the diminution observed being about 10 per cent. In some cases, it was evident that the far side of the disk (on which was the active matter) was behaving as a secondary cathode, as the glass stopper behind it glowed

green all over. When these thick disks were fitted into the end of a metal cylinder (the rays falling again on the inactive side) they did not lose any appreciable amount of activity, while the thin disks in the same circumstances lost 15 per cent. Here the disturbances would tend to spread over the surface of the cylinder rather than through the metal, and the difference caused by the extra thickness of aluminium becomes more noticeable. When thin copper disks were used, there was less loss of activity from the back than was the case with aluminium of about the same thickness.

Separation of two Stages in the Radioactive Change.

The majority of the experiments, and especially those in which the active disk was enclosed in a metal box, were more useful in showing the nature than the cause of the change occurring, which was found to be less simple than was at first supposed. When the activity of a disk was tested for some time after its exposure in the tube, it was nearly always found that the rate of decay was diminished, so that after an hour or two a part of the activity lost had been recovered. In many cases there was an actual increase of activity for the first hour or so, but this was only temporary, and the rate of decay always returned eventually to its normal value. Curve I. (diagram 2, Pl. VIII.) is typical of the larger rises; there is a maximum about two hours after exposure, the activity then diminishes, and after about four hours its rate of decay is the same as that given by the unaltered disk (Curve IV.). In other cases (Curve II.) there is a less marked rise, and the maximum is sooner reached; Curve III. shows hardly any perceptible rise, but the rate of decay is at first abnormally slow. In the experiments made with the disk enclosed in a metal box, a large proportion of the activity lost by the disk was generally found on the box, and its rate of decay could be separately tested. This was found in nearly all cases to be *greater* than the normal value, but gradually diminished so that any activity remaining after three or four hours had the normal rate of decay. Diagram 3 shows a typical curve; the activity of the box is plotted on a larger scale, to show the shape of the curve clearly.

Similar results were obtained by placing a second (inactive) disk in the discharge-tube close to the active surface; part of the activity was transferred from one disk to the other, the subsequent rate of decay of the first being smaller, and the second greater, than the normal.

The general similarity of the curve obtained after exposure

of the disk to the discharge (diagram 2, Curve I. second branch) to that given by a surface exposed for a *short* time to the thorium emanation * will be at once observed; and the explanation of the rise is undoubtedly the same in the two cases. Two stages in the decomposition of the active deposit are present simultaneously ; they have different rates of decay, and are not at first in the equilibrium proportions which are reached after a few hours.

Prof. Rutherford has shown† from the curve for short exposures that in this case there are two stages present, the first of which changes into the second without giving off rays, while the second in decomposing gives rise to the radiation which is responsible for the excited activity. Each change may be expressed by an equation of the form

$$I_t = I_0 e^{-\lambda t},$$

where I_0, I_t, are proportional to the number of particles of a given kind changing per second at the beginning and end of the time t, e is the base of the natural logarithms, and λ is a constant. The values of λ for the two substances are ·75 and ·063 (if the time is measured in hours), giving decay to half value in 55 minutes and 11 hours respectively. The same constants agree very well with the curves obtained in the present work, using the general equation for the activity (I) at any time t,

$$\frac{I_t}{I_0} = e^{-\lambda_1 t} + \frac{\kappa \lambda_2}{\lambda_1 - \lambda_2} (e^{-\lambda_2 t} - e^{-\lambda_1 t}),$$

so that the same two substances are evidently present.

One alteration must, however, be made in Rutherford's theory. He assumes that the first change is the rapid one, so that the regular exponential curve is obtained when the first stage has almost entirely disappeared. But the separation of activity obtained in the discharge-tube always occurred after the stage of regular decay had set in ; in one case the active disk was exposed to the discharge 30 hours after its removal from the thorium, and a separation of activities was still obtained. This shows that the two stages are always present simultaneously, and it is evident that the *first* change (occurring without emission of rays) is the *slow* one, and the second the rapid one. This is confirmed by the possibility of obtaining the rapidly-decaying activity almost pure on the box ; in some cases this transferred activity fell to half value in about an hour, and after two or three hours the box had become almost inactive. This assumption agrees just as well as the other with Rutherford's data, for it only necessitates

* Rutherford, ' Radioactivity,' p. 260.   † *Loc. cit.* p. 269.

the interchange of λ_1 and λ_2 in the equations, and therefore the alteration of a constant, and it is only relative values that Rutherford has obtained. It will be convenient to distinguish the two stages as "thorium A" and "thorium B," using the same terminology as that applied by Rutherford to the excited activity of radium [*].

The amount of separation effected in any given case can easily be calculated by a comparison of the curve for the exposed disk (which we may call α) with that for the control disk, β. The ratio of the activities of the two would remain constant but for the diminution caused in one of them by the discharge. As the activity observed is always due to the second change (the first giving no rays), the fall in the ratio of activities $\alpha : \beta$ immediately after exposure gives the amount of thorium B driven off from α. After three or four hours, when the normal rate of decay is recovered, the activity of α has become proportional to the amount of the slower-decaying substance present, so that the ratio of α to β at this stage gives the amount of thorium A driven off in the discharge-tube. For example, the ratio of α to β immediately after the exposure of α to the discharge was found on one occasion to be ·5 (taking the initial ratio as unity), and the constant ratio obtained four or five hours later was ·7, so that 50 per cent. of thorium B was removed, and 30 per cent of thorium A. When the active disk was made cathode, and a large proportion of the activity driven off, there was often very little separation; on one occasion 53 per cent. of A and 58 per cent. of B were removed, on another 90 per cent. of A, and 92 per cent. of B. In other cases the separation effected was generally greater; and where the activity driven off was also measured, it was found that the greater the proportion of B to A in the active matter lost by the first disk, the greater was the rate of decay of the separated activity.

Percentage of the two active substances removed from the disk.		Time of decay to half value of the separated activity.
Thorium A.	Thorium B.	
3 per cent.	13 per cent.	1 hour.
10 ,,	18 ,,	1 hour 40 min.
28 ,,	38 ,,	3 hours.
90 ,,	92 ,,	11 hours.

[*] Since the above was written, Prof. Rutherford has himself made the change suggested (Phil. Trans. A. 376).

II. *Influence of Heat on the Excited Activity.*

In view of the above results, it was thought interesting to investigate the effect of heat on the excited activity in more detail than had previously been attempted, in order to obtain if possible a further separation of the two stages by a kind of fractional distillation. Von Lerch[*] has investigated the amount of activity removed by heating for a short time at different temperatures. He found no loss of activity at 800°, but 16 per cent. was removed by heating for half a minute at 1020°, and the rate of volatilization increased as the temperature was raised to 1460°, at which point 99 per cent. was removed after $1\frac{1}{2}$ minutes total heating. He does not seem to have observed the rate of decay of the activity remaining on the wire after heating at the lower temperatures, so that his results give no indication as to the possibility of a separation by this means.

Method of Investigation.

The method used was as follows:—A platinum wire was made active by negatively electrifying over thorium in the usual way ; its activity was tested by the same apparatus as that used for the active disks in the previous investigation. The wire was heated by an alternating current obtained from a transformer coil. The temperature could be regulated by varying the number of turns of wire on the transformer, while for fine adjustments a rheostat of stout german-silver wire dipping in mercury was used. The platinum wire was placed along the axis of a glass tube, to protect it from air currents, and its temperature was measured by a thermocouple of fine platinum and platinum-rhodium wires. The galvanometer deflexions were standardized by observing the melting-point of very small crystals of pure sodium sulphate placed on the hot wire close to the junction, and then using Callendar's curve of corrections[†]. As it was not possible to have the junction welded to the wire, the readings may have varied somewhat, according to the way in which the junction lay on the wire. The junction was held down by a spring, and care was taken that its position should be as nearly as possible the same in each experiment. In standardizing the galvanometer readings a number of independent observations were taken, in which the position of the junction and the tension of the spring varied in the same way as in the experiments with the active wire. The deflexions corresponding

* Drude's *Annalen*, Nov. 1903.
† Phil. Mag. vol. xlviii. 1899, p. 533.

to the melting-point of the crystals generally differed by only a few millimetres, the greatest variation being about 1 cm., which corresponds to about 25° ; so that the temperature to which the wire was heated may be considered to be known within these limits, and a greater accuracy was not aimed at. The object of these experiments was to find the general nature of the effect, rather than to get very exact determinations of temperature.

When it was desired to investigate the activity driven off from the wire, a sheet of lead-foil rolled into a cylinder was placed inside the glass tube, and a large proportion of the activity lost by the wire was found deposited on this cylinder. After heating, the wire and lead were removed, the inner (active) surface of the cylinder was turned outwards, and its activity was tested in the same way as that of the wire. Fig. 3 shows the arrangements for heating.

Fig. 3.
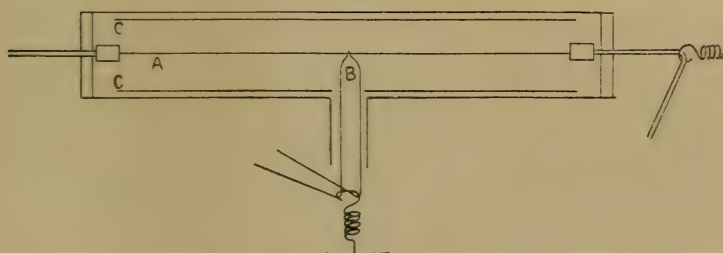
A. Active platinum wire.   B. Thermocouple.   C, C. Lead cylinder.

Results.

As a preliminary experiment, an active wire was heated for 1 minute to about 1100°, and at once tested. Its activity was found to have diminished by about 40 per cent., but there was a much more striking change in the rate of decay; for the activity decreased rapidly and at a steady rate, falling to half value in just over an hour (diagram 4, Pl. VIII.). The experiment was repeated, and the same result again obtained. It was thus at once evident that there had been a very considerable separation of the two active substances, thorium A and thorium B. The effect of heat is, however, the reverse of that obtained with cathode rays in that, while in the latter case the thorium B is the more volatile, in the former it is the thorium A which is driven off in greatest amount.

To confirm this result, the activity driven off was collected

on a lead cylinder, as explained above, and its rate of decay was also tested. The curves obtained are shown in diagram 5, and they are in complete agreement with the above deductions. The activity of the lead is at first very small, showing that there is very little thorium B present. It rises for about 3 hours, when it reaches a maximum many times greater than the value given by the first readings, and then begins to fall off. The activity of the wire as before falls rapidly at first; after some 4 hours the rates of decay of the two fractions become the same, both having the normal value of half in 11 hours. The activity of the lead is only a part of that lost by the wire; it is plotted on a larger scale in the diagram. In this case no thorium B has been lost by the wire during the heating; the active matter on the lead is at first entirely thorium A, and the curve it gives is similar to that of a surface exposed for a *short* time to thorium.

Temperature of Separation.

The temperature at which volatilization of each constituent occurs was then examined in more detail. The percentage of each substance removed was calculated in the same way as in the previous series of experiments. The loss of thorium B was measured by the immediate reduction of activity, that of thorium A by the difference between the value observed for the activity after 3 or 4 hours, and the value it would have had if the rate of decay had been normal.

It was found that thorium A begins to distil at a considerably lower temperature than had previously been noticed. At quite a dull red heat it begins to volatilize, and below 700° some thorium B is also removed. 6 minutes' heating at 1050° is enough to remove all the thorium A and the greater part of the thorium B, while 2 or 3 minutes' heating at 1100° or 1200° leaves the wire practically inactive. Some typical curves are shown in diagram 6, where the scales have been so arranged that the activity immediately before heating has the same value (50) in each case. The radioactive constants for these curves agree well with those given by Prof. Rutherford. At the higher temperatures, where the thorium A has been almost entirely driven off, the activity falls to half value in very little more than the 55 minutes required by pure thorium B.

The following table gives a summary of the chief results obtained by heating to different temperatures :—

Time of heating in minutes.	Temp.	Percentage removed.	
		Thorium A.	Thorium B.
3	630°	0	0
3	640°	24	0
4	680°	40	0
3	730°	74	10
1½	830°	57	13
3	955°	95	74
6	1050°	98	77
2	1090°	99	91
3	1280°	100	99

This method is thus a very convenient one for obtaining pure thorium B. The separation of the two substances also gives the answer to another question which has been hitherto left undetermined. It shows that the loss of activity of the wire is not due to its own disintegration, but that the process is a true distillation of the active substances, each of which volatilizes at a definite temperature.

Results of Chemical Separations.

This theory of the simultaneous presence, on any surface made active by exposure to thorium, of two substances having different properties and different rates of decay, gives a simple explanation of most of the results of investigations by chemical and electrolytic methods, which would otherwise seem to show the existence of a very large number of different substances, varying in their rates of decay. An examination of the numbers given by v. Lerch[*] shows that in the majority of cases the decay of activity (when it departs from the normal rate) does not follow an exponential law at all. For instance, the active $BaSO_4$ precipitate obtained from a solution of the active deposit in hydrochloric acid rose 8 per cent. in activity during the first 3 hours, then fell off, very slowly at first, and 8 hours after its separation was decaying at the normal rate of half value in 11 hours. Here there is, therefore, at first an excess of thorium A. Again, the activity obtained on a platinum cathode during the electrolysis of an active hydrochloric acid solution was reduced to one-fourth of its initial value in the first 3 hours, while the mean rate of decay over the next 20 hours was less than one-third of this. In this case there was, therefore, a large excess of thorium B. There is only one case in all v. Lerch's results to which this

[*] Drude's *Annalen*, Nov. 1903.

explanation does not appear to apply. The electrolysis of active hydrochloric acid solutions with electromotive forces too small to decompose the water, gave an active deposit which decayed according to an exponential law, and fell to half value in about $4\frac{1}{2}$ hours. This would therefore appear to be a third stage in the changes gone through by the active deposit, though other investigations have given no evidence of such a substance.

Pegram*, by electrolytic methods, has also obtained products with various rates of decay, but these again are never exponential, and the substances are probably in all cases mixtures.

Conclusions.

The conclusions drawn from the above investigations may be summarized as follows:—

The two stages in the disintegration of thorium-excited activity, whose existence has been already indicated by theoretical considerations, are always present simultaneously on a surface which has been exposed to thorium emanation. The constituent with the slower rate of decay (half value in 11 hours) is the one first formed, and may be called thorium A. The thorium B formed from it, which is responsible for the radiation emitted, is the substance whose rate of decay is the more rapid one (half in 55 minutes).

These two substances show differences in their properties by means of which they may be separated. They are both partly volatilized on exposure to the cathode-ray discharge, thorium B being more volatile than thorium A under these circumstances. The cause of the volatilization is an electrical one, being partly a direct action of the cathode rays, and partly a sputtering of the active matter, somewhat similar to that observed with metallic cathodes.

Under the influence of heat, thorium A is more volatile than thorium B. The former constituent can be almost entirely removed by keeping an active wire for a few minutes at a dull red heat. Nearly pure thorium B is then left on the wire, the activity of which decays to half value in about 55 minutes.

In conclusion, I wish to express my best thanks to Prof. J. J. Thomson for his very kind interest and valuable advice during the course of these experiments, which were carried out at the Cavendish Laboratory.

* Phys. Review, Dec. 1903.

LX. *On the Polarization at a Metallic Anode.* *By* S. R. Milner, *D.Sc., Lecturer in Physics, University College, Sheffield* [*].

THE theory of the electromotive force of concentration-cells developed by Nernst affords a simple explanation of the polarization which results in cells or voltameters from the passage of a current through them. In his *Theoretische Chemie*, p. 676, Nernst points out that the current will produce alterations, either in the concentrations at the surfaces of the two electrodes of the metallic ions present in the solution, or in that of the hydrogen or oxygen occluded in the two electrodes. In either case a "concentration-cell" is produced : its electromotive force, which is always in the opposite direction to that which produces the polarizing current, constitutes the polarization.

While this theory is in excellent qualitative agreement with most of what is known about polarization, so far as I am aware no direct quantitative comparisons of the theory with experiment have been carried out. Such a comparison is of value since, in certain simplified types of cells, the polarization is calculable on the theory entirely in terms of known quantities. so that a comparison of absolute values is possible. It is also of interest that the mathematical development of the theory leads to a simple expression, which may be experimentally tested, for the polarizations produced not only by a constant current, but by any arbitrarily varied current however complex ; in other words, for the polarizations as dependent on the whole previous history of the cell with regard to the current which has passed through it. The calculation of the polarization developed in a certain type of cell by an arbitrarily varied current, and a comparison of the results with those of experiment. form the subject of the present communication.

The kind of cell considered is one in which the anode (A. fig. 1. p. 646) is a metal plate surrounded by a vertical glass tube of the same sectional area as the plate, and immersed in a solution which contains the metal ions. For the sake of definiteness suppose the plate to be silver, and the solution to consist of nitric acid containing a very small quantity of silver nitrate mixed with it. The cathode C is a large silver plate suspended in a similar solution in another beaker. the solution in the two beakers being connected by a siphon. The observed polarization of such a cell when a current is passed through it from A to C may be looked upon as being

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made up of two parts : the polarizations at the anode and at the cathode respectively. Experimentally, either of these may be measured separately, by observing, not the total back-electromotive force of the cell, but the electromotive force

Fig. 1.
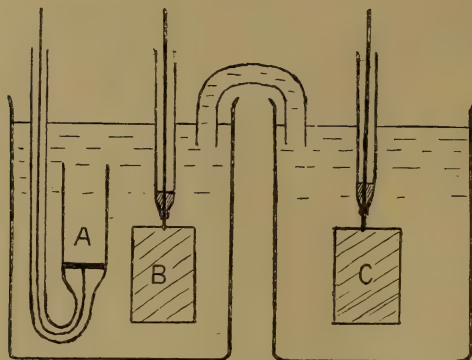
between the corresponding electrode and a " free " silver electrode B suspended in the solution. In a cell of this type, the anodic and the cathodic polarizations are practically independent of each other, and either may be destroyed without affecting the other, which then becomes the whole polarization of the cell. Thus in the cell of fig. 1, the cathode may be rendered practically unpolarizable by adding strong silver nitrate to the solution in the right-hand beaker. It greatly simplifies matters to deal with the two polarizations separately ; and in this paper the anodic polarization of the cell only is considered.

When a current is passed through the cell in the direction from A to C, the passage of each unit of charge is associated with the solution of an equivalent amount of silver at the anode A ; the concentration of the silver ions at the surface of A is thereby increased above its initial value, while that outside the tube in the neighbourhood of the free electrode B remains unchanged. The actual increase produced at A does not, however, depend solely on the charge passed through the cell, for the excess of concentration tends to be removed by the motion of the silver ions up the tube away from the surface—a process which is partly an ordinary diffusion of the ions due to the variation in their concentration which has resulted, and partly a motion due to the electrical forces acting on them. The concentration $c_{x=0}$ (x representing distance up the tube from A) of silver ions which is ultimately established at the surface of A will thus depend on the current, the time, and

the mobility of the silver ions. The polarization, V, or the potential-difference between A and B excluding the current-resistance fall of potential between them, may, by the theory of the concentration-cell, be expressed in terms of this concentration and that, c_1, at the surface of B. When the nitric acid is in considerable excess, its effect is to practically annul any potential-difference in the solution itself due to the different concentrations of the silver ions at A and B, and the polarization will consequently be simply the difference in the two potential steps which exist at the surfaces of the two electrodes. On the theory of Nernst it will be given by the expression

$$V = \frac{RT}{\epsilon} \log \frac{c_{x=0}}{c_1}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

in which R is the gas-constant, T the absolute temperature, and ϵ the charge carried by a gramme-ion of silver.

If the current passing through the cell be C, which may be any arbitrary function of the time, the number of gramme-ions of silver dissolved per second at the anode will be C/ϵ. Putting A for the sectional area of the tube and of the electrode, and $\dot{c}$ for the number of gramme-ions of silver passing upwards per second through a square centimetre of the cross section of the tube, we must consequently have

$$\dot{c}_{x=0} = \frac{C}{\epsilon A}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

for the whole of the silver ions dissolved must pass immediately upwards through a cross section of the tube imagined indefinitely close to the electrode. $\dot{c}_{x=0}$ is thus a known function of the time. Before the current is put on, the concentration of the silver ions is everywhere c_1, and this value is maintained permanently at the end of the tube by reason of the large volume of solution in which the tube is immersed. The problem of the polarization is consequently reducible to the determination of $c_{x=0}$, the concentration of the silver ions at the surface of the anode, in a process of diffusion of the ions up the tube which is subjected to the above conditions.

The differential equation of the motion of the silver ions is not, however, strictly that expressing the ordinary law of diffusion, for in addition to the diffusional process the ions are urged upwards by the electrical forces in the electrolyte which are called into play. If we make the usual assumption that the silver nitrate is completely dissociated, and that the osmotic pressure of its ions obeys the gas law—a supposition

which, while only approximately correct, does not seem to lead to any serious error in its results—the number of silver ions passing upwards through a square centimetre of any cross section of the tube per second may be expressed on the theory developed by Nernst and Planck by the equation

$$\dot{c} = \frac{RTU}{\epsilon}\left(-\frac{dc}{dx}\right) + U c\left(-\frac{d\phi}{dx}\right), \quad . \quad . \quad . \quad (3)$$

where U is the ionic velocity of the silver ions, and $\left(-\dfrac{dc}{dx}\right)$ and $\left(-\dfrac{d\phi}{dx}\right)$ are the rates of decrease of the concentration and the potential respectively in going up the tube. Equation (3), which in the general case of a mixture of nitric acid and silver nitrate leads to very complex results, degenerates in two cases into the ordinary law of diffusion : namely, when the concentration of the nitric acid is zero, and when it is very large compared with that of the silver nitrate. In the latter case, which is that realized in the present experiments, practically no potential-difference is set up in the electrolyte due to the variation in the concentration of the silver ions, and the actual slope of potential becomes simply that given in terms of the current by Ohm's law, *i. e.*

$$-\frac{d\phi}{dx} = \frac{C\rho}{A},$$

where ρ is the specific resistance of the solution. Consequently the second term on the right of (3) may be made as small as we please and negligible in comparison with the first by sufficiently increasing the concentration of the nitric acid, and thereby diminishing the specific resistance of the solution. To avoid complexity we need only consider the case in which the concentration of the nitric acid is sufficiently great for this approximation to apply, and write (3) in the form

$$\dot{c} = \frac{RTU}{\epsilon}\left(-\frac{dc}{dx}\right). \quad . \quad . \quad . \quad . \quad (4)$$

This expresses the ordinary law of diffusion, the coefficient of diffusion of the silver ions being

$$D = \frac{RTU}{\epsilon}. \quad . \quad . \quad . \quad . \quad (5)$$

By substituting in (2) the value of $\dot{c}_{x=0}$ given by (4) we have

$$\left(-\frac{dc}{dx}\right)_{x=0} = \frac{C}{ARTU}, \quad . \quad . \quad . \quad (6)$$

that is, $\left(-\dfrac{dc}{dx}\right)_{x=0}$ is an arbitrary function of the time, say $F(t)$.

By differentiating (4) with respect to x and making use of the identity

$$-\frac{d\dot{c}}{dx} - \frac{dc}{dt}$$

we obtain the differential equation

$$\frac{d\dot{c}}{dt} = D\frac{d^2c}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and we require the solution of (7), subject to the conditions

$$c = c_1 \text{ throughout, when } t \leq 0, \quad . \quad . \quad . \quad (8)$$

$$-\frac{dc}{dx} = F(t), \text{ when } x=0, \text{ and } t>0, \quad . \quad . \quad (9)$$

$$c = c_1 \text{ at all times, when } x=l, \quad . \quad . \quad (10)$$

l being the length of the tube.

The solution of (7) under these conditions may be most simply derived from the solution for the case in which $F(t)$ is a constant, say F. When

$$\left(-\frac{dc}{dx}\right)_{x=0} = F, \quad . \quad . \quad . \quad . \quad (11)$$

the concentration c at any point in the tube is given by the expression

$$c = c_1 + (l-x)F - \Sigma B_m e^{-\left(\frac{m\pi}{l}\right)^2 Dt} \cos\frac{m\pi x}{l}, \quad . \quad (12)$$

provided that

$$\Sigma B_m \cos\frac{m\pi x}{l} = (l-x)F, \quad . \quad . \quad . \quad (13)$$

where the summation includes all values of m except $m=0$; for (12) along with (13) is a solution of the differential equation which inspection shows satisfies the conditions (8), (10), and (11). Equation (13) shows that $(l-x)F$ may not be expanded as an ordinary Fourier Series in which m has the values 0, 1, 2, &c., for the term in which $m=0$ is excluded by the conditions. But it may be expressed as a series of cosines in which m has the values $\dfrac{1}{2}, \dfrac{3}{2}, \dfrac{5}{2} \cdots \dfrac{2n+1}{2}$, and which is equal to it for all values of the variable including

$x=0$ and $x=l$. If this be done we obtain

$$(l-x)\mathrm{F} = \frac{8l}{\pi^2}\mathrm{F}\left\{\cos\frac{1}{2}\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3}{2}\frac{\pi x}{l} + \dots\right\}$$

$$= \frac{8l}{\pi^2}\mathrm{F}\sum_0^\infty \frac{1}{(2n+1)^2}\cos\frac{2n+1}{2}\frac{\pi x}{l} \quad . \quad . \quad . \quad (14)$$

Substituting the corresponding values of m and B_m given by (14) in the summation of (12) we get

$$c = c_1 + l\mathrm{F}\left\{1 - \frac{x}{l} - \frac{8}{\pi^2}\sum_{n=0}^\infty \frac{e^{-\left(\frac{2n+1}{2}\frac{\pi}{l}\right)^2\mathrm{D}t}\cos\frac{2n+1}{2}\frac{\pi x}{l}}{(2n+1)^2}\right\} \quad (15)$$

as the solution for the case in which $\mathrm{F}(t) = \mathrm{F}$ and is constant.

When $\mathrm{F}(t)$ varies with the time, let us suppose first that it is a discontinuous function, composed of a series of suddenly occurring increases or decreases during the intervals between which the function maintains constant values. While $\mathrm{F}(t)$ is zero before the instant $t=0$, let it suddenly take up the value $\delta_0\mathrm{F}$ at the time $t=0$, undergo the further sudden increase $\delta_1\mathrm{F}$ at the time $t=t_1$, $\delta_2\mathrm{F}$ at $t=t_2$, &c., so that we may write

$$\mathrm{F}(t) = \sum_{r=0}^r \delta_r\mathrm{F} \text{ when } t>t_r \text{ and } <t_{r+1}.$$

Now the expression

$$c = c_1 + l\delta_r\mathrm{F}\left\{1 - \frac{x}{l} - \frac{8}{\pi^2}\sum_{n=0}^\infty \frac{e^{-\left(\frac{2n+1}{2}\frac{\pi}{l}\right)^2\mathrm{D}(t-t_r)}\cos\frac{2n+1}{2}\frac{\pi x}{l}}{(2n+1)^2}\right\} \quad (16)$$

that is (15) with $\delta_r\mathrm{F}$ written for F, and $t-t_r$ for t, is a solution of (7) subject to the conditions (8) and (10) and possessing the property that (instead of (11))

$$-\frac{dc}{dt} = \delta_r\mathrm{F} \text{ when } x=0 \text{ and } t>t_r.$$

Also the sum of any number of expressions similar to (16) is a solution of the differential equation, consequently the expression

$$c = c_1 + \sum_{r=0}^r l\delta_r\mathrm{F}\left\{1 - \frac{x}{l} - \frac{8}{\pi^2}\sum_{n=0}^\infty \frac{e^{-\left(\frac{2n+1}{2}\frac{\pi}{l}\right)^2\mathrm{D}(t-t_r)}\cos\frac{2n+1}{2}\frac{\pi x}{l}}{(2n+1)^2}\right\} \quad (17)$$

is a solution which satisfies the conditions (8) and (10) and has the property that

$$-\frac{dc}{dt} = \sum_{r=0}^{r} \delta_r \mathrm{F}, \text{ when } x=0, \text{ and } t>t_r, \text{ and } <t_{r+1}.$$

Thus

$$\left(-\frac{dc}{dt}\right)_{x=0} = \mathrm{F}(t)$$

at all times, and (17) is consequently the required solution.

For our purposes it may be considerably simplified. In the first place we want only the concentration at the surface of the electrode $c_{x=0}$, and putting $x=0$ in (17) we have

$$c_{x=0} = c_1 + \sum_{r=0}^{r} l \cdot \delta_r \mathrm{F} \left\{ 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{e^{-\left(\frac{2n+1}{2}\frac{\pi}{l}\right)^2 \mathrm{D}(t-t_r)}}{(2n+1)^2} \right\}. \quad (18)$$

The second series in (18) converges very slowly and is impracticable for calculation unless $\left(\frac{\pi}{l}\right)^2 \mathrm{D}(t-t_r)$ is comparable with unity. When l is (say) two or three centimetres this is not the case unless $t-t_r$ is very great (several days). For all smaller times a tube a few centimetres long is equivalent to one of infinite length, so far as the effect on the diffusion is concerned; and we may replace (18) by the limiting value to which it approaches when l becomes infinite. To do this take the 1 in (18) inside the summation in n by writing for it its identical value

$$\frac{8}{\pi^2} \sum_{0}^{\infty} \frac{1}{(2n+1)^2} = 1,$$

and (18) becomes

$$c_{x=0} = c_1 + \sum_{r=0}^{\infty} \delta_r \mathrm{F} \cdot \frac{8}{\pi^2} l \sum_{n=0}^{\infty} \frac{1 - e^{-\left(\frac{2n+1}{2}\frac{\pi}{l}\right)^2 \mathrm{D}(t-t_r)}}{(2n+1)^2}. \quad (19)$$

When $l=\infty$, the second series becomes now the sum of an infinite number of infinitely small terms, and degenerates into the definite integral

$$\int_0^\infty \frac{1 - e^{-\left(\frac{2n+1}{2}\frac{\pi}{l}\right)^2 \mathrm{D}(t-t_r)}}{(2n+1)^2} dn.$$

Substituting

$$m = \frac{2n+1}{2}$$

and

$$z = \left(\frac{\pi}{l}\right)^2 \mathrm{D}(t-t_r),$$

this becomes

$$\frac{1}{4} \int_{\frac{1}{2}}^\infty \frac{1 - e^{-m^2 z}}{m^2} dm.$$

Since z is an indefinitely small quantity, the value of this integral will not be altered by altering its lower limit from $\frac{1}{2}$ to 0, and since

$$\int_0^\infty \frac{1-e^{-m^2z}}{m^2}\,dm = \sqrt{\pi z},$$

we have

$$\lim_{l=\infty} l \sum_{n=0}^\infty \frac{1-e^{-\left(\frac{2n+1}{2}\frac{\pi}{l}\right)^2 D(t-t_r)}}{(2n+1)^2} = \frac{1}{4}l\sqrt{\pi \cdot \left(\frac{\pi}{l}\right)^2 D(t-t_r)}$$

$$= \frac{1}{4}\sqrt{\pi^3 D(t-t_r)}.$$

Introducing this into (19) we obtain

$$c_{x=0} = c_1 + 2\sqrt{\frac{D}{\pi}} \sum_{r=0}^\infty \delta_r F \sqrt{t-t_r} \quad . \quad . \quad . \quad (20)$$

(20) is thus the solution for the case in which $F(t)$ consists of sudden finite variations separated by intervals during which the function remains constant. But it is clear that the result will hold however small the variations and the intervals; it will consequently hold in the limit when $F(t)$ undergoes a continuous variation. Thus let $F(t)$ be a continuous function, the value of which is zero before the instant $t=0$. Expressing it in the form

$$F(t) = \int_0^t F'(\theta)d\theta,$$

we may replace in (20)

t_r by θ, $\delta_r F$ by $F'(\theta)d\theta$, and $\sum_{r=0}^r$ by $\int_{\theta=0}^t$.

We thus obtain

$$c_{x=0} = c_1 + 2\sqrt{\frac{D}{\pi}} \int_0^t \sqrt{t-\theta} \cdot F'(\theta)d\theta, \quad . \quad (21)$$

and from this expression numerical values may at once be obtained when the form of the function $F(t)$ is known. This form of expression may be considered to include the case in which the function undergoes sudden finite changes of value as well as a continuous variation; for if a sudden variation occur at the instant θ, $F'(\theta)$ is infinite, and $F'(\theta)d\theta$ may be put equal to δF the finite change in the value of the function; the corresponding part of the integral then simply gives rise to a term

$$2\sqrt{\frac{D}{\pi}}\delta F \sqrt{t-\theta}$$

in the expression for the concentration, which is in agreement with the corresponding term obtained from (20).

If we now substitute in (20) for F(t) its value in terms of the polarizing current given by (6), and the value of D given in (5), we get

$$c_{x=0} = c_1 \left\{ 1 + \frac{2}{Ac_1} \frac{\sum\limits_{r=0}^{r} \delta_r C \sqrt{t-t_r}}{\sqrt{\pi\epsilon RTU}} \right\},$$

where $\delta_r C$ is the sudden increase in the current at the time t_r, and the summation includes all the variations which have occurred since just before the instant $t=0$. Substituting this value of $c_{x=0}$ in (1), we obtain finally for the polarization

$$V = \frac{RT}{\epsilon} \log \left\{ 1 + \frac{2}{Ac_1} \frac{\sum\limits_{r=0}^{r} \delta_r C \sqrt{t-t_r}}{\sqrt{\pi\epsilon RTU}} \right\} \quad . \quad (22)$$

In the case in which the current undergoes a continuous variation as well as sudden ones we may write

$$C = f(t) = \int_0^t f'(\theta)d\theta,$$

and using (21) instead of (20), we get

$$V = \frac{RT}{\epsilon} \log \left\{ 1 + \frac{2}{Ac_1} \cdot \frac{\int_0^t \sqrt{t-\theta} f'(\theta) d\theta}{\sqrt{\pi\epsilon RTU}} \right\} \quad . \quad (23)$$

Both (22) and (23) are useful forms, and from them the polarization may be calculated absolutely in any given case.

To obtain numerical values we must express all the quantities involved in terms of C.G.S. units. We have

$$R = 8\cdot32 \times 10^7 \text{ ergs per degree C.}$$
$$T = 291 \text{ at } 18^\circ \text{ C.}$$
$$\epsilon = 9654 \text{ electromagnetic units per gramme-ion}$$
for a monovalent ion such as silver.

Hence
$$\frac{RT}{\epsilon} \log_e x = \frac{10^{-8}}{\cdot4343} \frac{RT}{\epsilon} \log_{10} x \text{ volts.}$$
$$= \cdot0578 \log_{10} x \text{ volts.}$$

An accurate estimation of the value of U, the effective ionic velocity of the silver in the nitric acid, offers some difficulty, but for our present purposes it is simplest to neglect the possible effect on its value of incomplete dissociation of the silver nitrate and to take it as the same as that in a very dilute solution of a silver salt in water. Kohlrausch and Holborn (*Leitvermögen der Electrolyte*, p. 200) give for

the ionic mobility of silver at infinite dilution (*i. e.*, that part of the molecular conductivity of a silver salt which depends on the silver ion only) the value 55·7 at 18° C. This is equivalent to

$$U = 5·77 \times 10^{-12} \text{ cm. per second for a slope of potential}$$
$$\text{of one electromagnetic unit per cm.}$$

With this value

$$\frac{2}{\sqrt{\pi \epsilon RTU}} = ·03072,$$

and consequently

$$V = ·0578 \log_{10} \left\{ 1 + \frac{·03072}{Ac_1} \Sigma \delta_r C \sqrt{t - t_r} \right\}$$

where the concentration must be expressed in gramme-molecules per c.c., the current in E.M. units, and the time in seconds. When the concentration is expressed in gramme-molecules per litre, the current in amperes, and the time in minutes, the factor ·03072 must be replaced by

$$·03072 \times 1000 \times \frac{1}{10} \times \sqrt{60}, \text{ or } 23·80,$$

and

$$V = ·0578 \log_{10} \left\{ 1 + \frac{23·80}{Ac_1} \Sigma \delta_r C \sqrt{t - t_r} \right\} \text{ volts.} \quad (24)$$

Since the absolute temperature varies ·34 per cent. per degree at 18° C., and the ionic velocity of silver varies approximately 2 per cent. per degree, the factor ·0578 is subject to a variation of $+·34$ per cent., and the factor 23·8 to a variation of $-1·17$ per cent. for each degree above 18° C.

Results with Silver-Nitrate Cells.

In making the experiments a cell similar to that shown in fig. 1 was used. The beakers were suspended in a large water-bath which stood on a slate shelf free from the floor. The bath, although its temperature varied when measured over long periods, showed no appreciable variation in the period of a few hours required for an experiment; and con-vection effects are likely to be more completely eliminated in this way than when a thermostat is used, in which case slight but quick fluctuations of temperature occur. The electrodes were of fine silver containing 99·8 per cent. of the metal, and had shown themselves satisfactory in previous concentration-cell experiments. The anode A was cast in the form of a small button and turned down so as to exactly fit the surrounding glass tube, into which it was cemented

with sealing-wax. Two anodes and tubes, (1) & (2), of different areas were used.

When the current passing through the cell is produced by an applied E.M.F. of only a few volts, it rapidly falls off, due to the back-E.M.F. of polarization set up. In most cases it was desirable to keep the current constant, so as to simplify the arithmetical calculations from equation (24) : to effect this a battery of 64 secondary cells, applied to the polarizable cell through a resistance of $60,000 + 500$ ohms, was used to produce the current. The current was measured by observing on a direct-reading potentiometer the potential-difference between the ends of the 500 ohms. The potential-difference between the anode A and the free electrode B of the cell was measured on the same potentiometer in terms of the mean of two Clark cells kept in a water-bath. Care was of course taken to avoid affecting the polarization by allowing any appreciable current to pass between A and B in taking the readings. To avoid this, a resistance of 10,000 ohms was kept permanently in the galvanometer circuit, and in addition only momentary contacts of the potentiometer key were made.

During the period of recovery of the cell, when the polarizing current is no longer on, the true polarization V many be thus measured direct. But while the current is being applied, the observed potential-differences between the anode A and the free electrode B give the quantity $V + CR_{AB}$, where C is the current, and R_{AB} the resistance between these two points ; and to obtain the true polarization a knowledge of this resistance is necessary. I found it was impossible to measure this accurately on the Wheatstone's bridge, using a secohmmeter to reverse the battery and galvanometer connexions simultaneously. Even at the highest speed at which the secohmmeter could be rotated, the resistance as measured altered from 50 to 57 ohms when the cell AB was reversed on the bridge. This is due to the fact that the cell polarizes much more rapidly when the current passes through it in the direction from B to A than when from A to B (the electrode B being larger than A). As an accurate determination of the resistance was impracticable, in the following tables the values of the total potential-differences are simply tabulated as observed in the columns headed $(V + CR_{AB})_{obs.}$, and their agreement with the theory tested as follows :—During the period in which a constant polarizing current is being applied, the excesses of the values of $(V + CR_{AB})_{obs.}$ over the calculated values of V should represent the quantity CR_{AB} and remain constant. From the mean value for each experiment values of R_{AB} may be subsequently calculated and compared with each other.

During the recovery of the cell, when no current is being applied, the two columns should of course be identical.

The most interesting case to which formula (24) applies is that in which a constant polarizing current C is applied for a time t_1 minutes, and then removed, the cell being left on open circuit to recover. In this case

$$\delta_0 C = C \text{ at time } t = 0,$$
$$\delta_1 C = -C \text{ at time } t = t_1,$$

and the formula for the polarization becomes

$$V = \cdot 0578 \log_{10} \left\{ 1 + \frac{23 \cdot 80}{A c_1} C \sqrt{t} \right\}$$

for all times up to $t = t_1$, and

$$V = \cdot 0578 \log_{10} \left\{ 1 + \frac{23 \cdot 80}{A c_1} (C \sqrt{t} - C \sqrt{t - t_1}) \right\}$$

for all times after $t = t_1$, *i. e.* during the recovery of the cell. Since the term $C \sqrt{t - t_1}$ becomes imaginary when $t < t_1$, the second form may be taken to express the polarization at all times if we exclude imaginary values. The experiments numbered I. to V. were designed to exhibit the dependence of the time-curves of the polarization on the quantities C, t_1, and c_1. A comparison of I. and II. shows the greater polarization produced, and in particular the much slower rate at which it dies away when the polarizing current is applied for a long time than when it is only applied for a short one.

I.

Solution : $\cdot 002$ n. $AgNO_3$ in $\cdot 2$ n. HNO_3 (i. e. $c_1 = \cdot 002$).
Anode (1) $A = 1 \cdot 713$ sq. cm. Temp. $21^\circ \cdot 1$ C.
Current of $1 \cdot 736 \times 10^{-3}$ ampere applied for 1 minute.

$$V_{\text{calc.}} = \cdot 0584 \log_{10} \left\{ 1 + 11 \cdot 63 (\sqrt{t} - \sqrt{t - 1}) \right\} \text{ volt.}$$

Time from start (mins.).	$V + CR_{AB}$ (obs.).	V (calc.).	Diff.
0·42	0·1554	0·0544	+0·1010
0·78	·1594	·0615	·0979
1·00	Current stopped.	·0753	
1+ 0·92	0·0464	0·0453	+ ·0011
„ 1·72	·0410	·0404	+ 6
„ 3·25	·0355	·0352	+ 3
„ 8·67	·0275	·0272	+ 3
„ 13·67	·0231	·0236	− 5
„ 21·0	·0190	·0206	− 16
„ 28·0	·0164	·0187	− 23
„ 42·5	·0127	·0161	− 24

II.

Solution : $\cdot002$ n. $AgNO_3$ in $\cdot2$ n. HNO_3.
Anode (2) $A = 1\cdot504$ sq. cm.   Temp. $22°\cdot5$ C.
Current of $1\cdot950 \times 10^{-3}$ ampere applied for 10 minutes.

$$V_{calc.} = \cdot0587 \log_{10}\{1 + 14\cdot61(\sqrt{t} - \sqrt{t-10})\} \text{ volt.}$$

Time from start (mins.).	$V + CR_{AB}$ (obs.).	V (calc.).	Diff.
1·03	0·1890	0·0704	0·1186
2·23	·1956	·0797	·1159
4·83	·2046	·0892	·1154
9·67	·2134	·0977	·1157
10·00	Current stopped.	·0984	
10+ 0·57	0·0951	0·0923	+ ·0028
,, 1·25	·0916	·0895	+ 24
,, 2·57	·0878	·0860	+ 18
,, 4·40	·0840	·0829	+ 11
,, 9·60	·0772	·0768	+ 4
,, 17·00	·0714	·0718	− 4
,, 32·0	·0640	·0665	− 15
,, 65·0	·0558	·0580	− 22

There is an agreement to within $\cdot003$ volt between the observed and calculated values of the polarization during the whole period of recovery observed. This is sufficiently good. It is evident, however, from the regularity of the variation of the differences in the fourth column that the observed polarization dies away somewhat more rapidly than that calculated from the formula. But this effect (which is observable in the experiments throughout) is only to be expected—every error, such as would be produced by convection currents, or by a slightly defective insulation of the electrodes A, B, which would allow a slight current to pass between them, would act in the same direction and cause the polarization to die away more rapidly than is indicated by the theory. The variation in the differences is probably not greater than would be accounted for by these sources of error.

In the two following experiments approximately the same initial polarization is produced, in the one case by a certain current acting for a given time, and in the second by half the current acting for four times the time. The slower rate of decay, both in theory and practice, of the polarization in the second case is noticeable. Of course, although the same concentration of the silver ions at the surface of the electrode is produced, in the one case twice as many silver ions have been dissolved as in the other, and the concentration gradients, on which the rate of decay depends, are different.

III.

Solution : ·002 n. $AgNO_3$ in ·2 n. HNO_3.
Anode (2) A = 1·504 sq. cm.   Temp. 22° C.
Current of $1·960 \times 10^{-3}$ ampere applied for 5 minutes.

$$V_{\text{calc.}} = ·0586 \log_{10} \{1 + 14·78(\sqrt{t} - \sqrt{t-5})\} \text{ volt.}$$

Time from start (mins.).	$V + CR_{AB}$ (obs.).	V (calc.).	Diff.
0·17	0·1694	0·0498	+0·1196
0 77	·1868	·0671	·1197
1·92	·1960	·0781	·1179
4·97	·2078	·0896	·1182
5·00	Current stopped.	·0898	
5+ 0 50	0·0854	0·0821	+ ·0033
,, 1·50	·0790	·0769	+ 21
,, 2·93	·0746	·0725	+ 19
,, 4·95	·0700	·0683	+ 17
,, 11·0	·0619	·0612	+ 7
,, 17·2	·0569	·0567	+ 2
,, 32·0	·0503	·0505	− 2
,, 53·0	·0448	·0453	− 8

IV.

Solution : 0·002 n. $AgNO_3$ in 0·2 n. HNO_3.
Anode (1) A = 1·713 sq. cm.   Temp. 23° C.
Current of $9·18 \times 10^{-4}$ ampere applied for 20 minutes.

$$V_{\text{calc}} = 0·0587 \log_{10} \{1 + 6·00(\sqrt{t} - \sqrt{t-20})\} \text{ volt.}$$

Time from start (mins.).	$V + CR_{AB}$ (obs.).	V (calc.).	Diff.
1·00	0·1030	0·0495	+·0535
3·25	·1126	·0630	·0496
7·5	·1240	·0712	·0528
19·95	·1350	·0847	·0503
20·00	Current stopped.	·0847	
20+0·60	0·0804	0·0805	− ·0001
1·70	·0780	·0777	+ 3
3·20	·0750	·0752	− 2
7·25	·0700	·0708	− 8
14 9	·0647	·0660	− 13
31·5	·0579	·0596	− 17
75·5	·0491	·0513	− 22
126	·0439	·0463	− 24
928	·0254	·0276	− 22

In experiment V. the concentration of the silver nitrate was reduced to one tenth of its value in the previous four. A considerable increase in the polarization both in theory and practice is the result, as may be seen by comparing the results with those of III., the current and its time of application being roughly the same in both experiments.

V.

Solution : $\cdot 0002$ n. $AgNO_3$ in $\cdot 2$ n. HNO_3.
Anode (2) A $= 1\cdot 504$ sq. cm.   Temp. $19°\cdot 2$ C.
Current of $1\cdot 925 \times 10^{-3}$ ampere applied for 6 minutes.

$$V_{calc.} = \cdot 0580 \log_{10}\{1 + 150\cdot 2(\sqrt{t} - \sqrt{t-6})\} \text{ volt.}$$

Time from start (mins.).	V + CR$_{AB}$ (obs.).	V. (calc.).	Diff.
1·12	0·2486	0·1279	$+0\cdot 1207$
2·67	·2564	·1387	·1177
4·83	·2630	·1461	·1169
6·00	Current stopped	·1489	
6+ 0·67	0·1402	0·1407	$-$ ·0005
,, 1·92	·1346	·1351	$-$ 5
,, 3·50	·1301	·1313	$-$ 12
,, 6·50	·1254	·1259	$-$ 5
,, 12·0	·1182	·1200	$-$ 18
,, 23·45	·1101	.1129	$-$ 28
,, 46·8	·1011	·1049	$-$ 38
,, 84·0	·0934	·0982	$-$ 48

The following are the values of R_{AB}, the resistance between the two electrodes, calculated as described on p. 655 for each of the two anodes and tubes used.

Anode (1).   Anode (2).
Exp. I. ...... $57\cdot 3$ ω.   Exp. II. ...... $59\cdot 8$ ω.
,,   IV. ...... $56\cdot 1$,,   III. ...... $60\cdot 7$

The two numbers agree well with each other in each case, and are in accordance with the approximate measured value for anode (1) given on p. 655. The variations correspond to differences of only $\cdot 002$ volt between the observed and the calculated polarizations. In exp. V., with anode (2), $R_{AB} = 61\cdot 6$ ω; the slightly higher value of the resistance in this case is partly accounted for by the smaller concentration of silver nitrate used.

The results of two experiments in which the polarizing

current is subjected to more complicated variations are to save space given only in the form of curves, fig. 2. In the

Fig. 2.
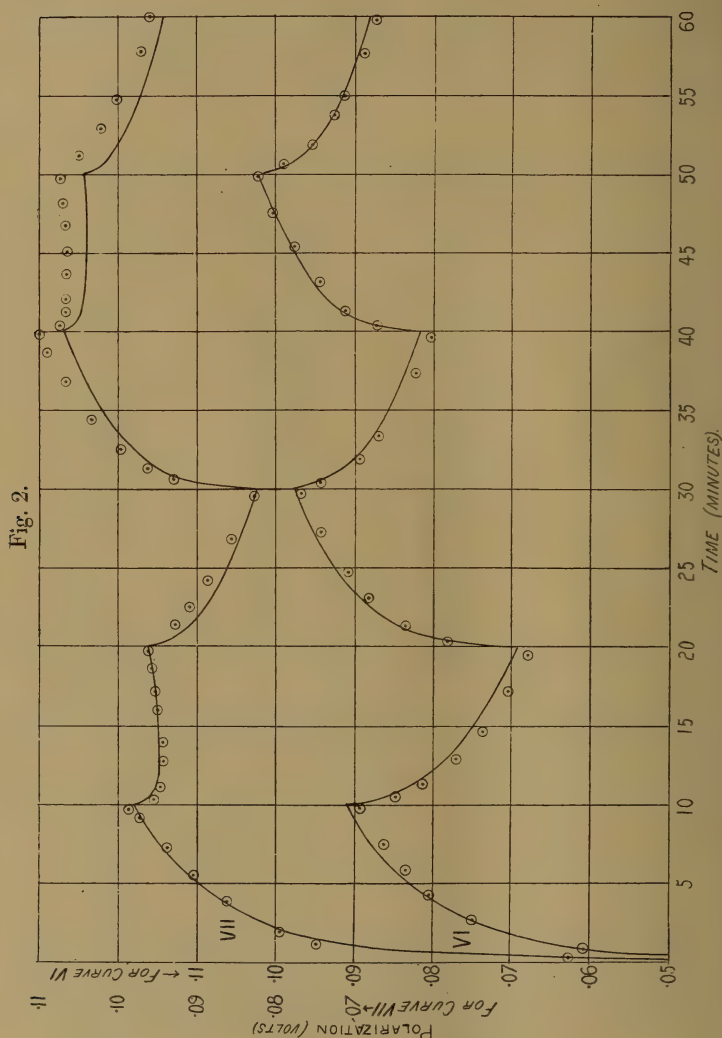
experiment represented by the lower curve (VI), a polarizing current of $1·695 \times 10^{-3}$ ampere was applied for 10 minutes, then cut off, leaving the cell on open circuit for 10 minutes, then again applied for 10 minutes, and so on. Thus in

equation (24)

$$\delta_0 C = 1.695 \times 10^{-3} \text{ at time } t = 0,$$
$$\delta_1 C = -1.695 \quad \text{,,} \qquad \text{,,} \quad t = 10,$$
$$\delta_2 C = 1.695 \quad \text{,,} \qquad \text{,,} \quad t = 20,$$

&c. The solution was as before 0·002 n. $AgNO_3$, in 0·2 n. HNO_3, area of anode and tube 1·713 sq. cm., temperature 20°·5 C., and the calculated polarization becomes

$$V_{\text{calc.}} = 0.0582 \log_{10}\{1 + 11.44(\sqrt{t} - \sqrt{t-10} + \sqrt{t-20} - \sqrt{t-30} + \&c.)\}$$

at any time t (real values only to be taken).

The lower curve (VI) in fig. 2 represents the values calculated from this equation plotted with the time. The $\odot$'s are the experimental values of the polarization; the observed potential-differences $V + CR_{AB}$ when the current is on have been reduced by the amount $CR_{AB} = ·0904$ volt, a value chosen arbitrarily so as to make the two portions of the experimental curve (current on, and current off) join well together.

In the case to which the upper curve (VII) refers, a current of $1·947 \times 10^{-3}$ ampere was applied for 10 minutes; it was then suddenly reduced to $·980 \times 10^{-3}$ ampere (approximately halved), at which value it was maintained constant for the next 10 minutes; this was then cut off and the cell left insulated for 10 minutes. The whole process was then repeated during the second half hour. The current variations to be introduced into (24) are thus as follows :—

$$\delta_0 C = 1.947 \times 10^{-3} \text{ at time } t = 0,$$
$$\delta_1 C = -0.967 \quad \text{,,} \qquad \text{,,} \quad t = 10,$$
$$\delta_2 C = -0.980 \quad \text{,,} \qquad \text{,,} \quad t = 20,$$
$$\delta_3 C = 1.947 \quad \text{,,} \qquad \text{,,} \quad t = 30,$$
$$\delta_4 C = -0.967 \quad \text{,,} \qquad \text{,,} \quad t = 40,$$
$$\delta_5 C = -0.980 \quad \text{,,} \qquad \text{,,} \quad t = 50.$$

The solution being ·002 n. $AgNO_3$ in ·2 n. HNO_3, A (anode (2)) 1·504 sq. cm., temperature 21°·5 C., the equation for the polarization becomes

$$V_{\text{calc.}} = ·0585 \log_{10}\{1 + 14·78(\sqrt{t} - ·497\sqrt{t-10} - ·503\sqrt{t-20}$$
$$+ \sqrt{t-30} - ·497\sqrt{t-40} - ·503\sqrt{t-50})\} \quad . \quad (25)$$

This is represented by the curve (VII). The observed polarizations, represented as before by $\odot$'s, have been deduced where necessary from the observed potential-differences

$V + CR_{AB}$ by assuming $R_{AB} = 59\ \omega$, which makes the different portions of the curve join on well to each other[*]. The second and fifth portions of the curve are very interesting. The sudden reduction in the current which has taken place causes the polarization to tend to diminish, but at a continually decreasing rate. Meanwhile the current still passing is all the time tending to increase the polarization, and the combination of the two effects gives rise to a minimum value of the polarization which is well marked, both theoretically and experimentally. It is striking that the extremely complicated state of distribution of the silver nitrate near the surface of the anode produced by so complex a cycle of changes should reduce at the surface itself to a state expressible in such simple terms as those of (25).

Mercurous-Nitrate Cells and Continuously Varying Current.

When the polarizing current undergoes a continuous variation instead of a series of sudden ones, formula (23) becomes applicable instead of (22). The simplest case to observe experimentally is that in which the polarizing current is produced by a small applied E.M.F. instead of by a high one acting through a high resistance. The resulting polarization of the cell will then appreciably reduce the effective E.M.F., and cause a gradual decrease in the polarizing current. This type of variation in the current was investigated in some experiments made on mercurous-nitrate cells, the results of two of which may be quoted. When the cathode was made unpolarizable, so that the anodic polarization was the whole polarization developed, I found in every case that over the greatest part of its course the current fell off at a rate very approximately proportional to the logarithm of the time. To obtain an analytical expression for the polarization we might thus represent the current by the formula

$$C = B - b \log t \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where B and b are constants, but it would be unsafe to introduce this empirical expression into the integral of (23) without

---

[*] The slight want of agreement in the latter half of the curve is possibly due to the fact that some lamps in another part of the building worked by the secondary cells used to supply the polarizing current may have been switched off temporarily, increasing the E.M.F. of the secondary cells and thus the polarizing current. Readings of the current were taken at 31·5 and 37·5 minutes, so that, if the lamps were turned on again before the latter reading, the effect would be missed in the current measurements, although a permanent increase in the polarization would be produced. This had happened in previous experiments which were rejected in consequence, but I have been unable to verify it in this instance.

further consideration. Equation (26) can obviously not hold at very small times, when it would give values for the current verging on infinity. The actual current must, however, start from the value $C_0 = E/R_0$, where E is the applied E.M.F., and R_0 the resistance of the circuit, and since the integration to be carried out in (23) extends from the instant $t = 0$, the substitution in it of (26) would be inadmissible. We may easily modify the empirical equation so as to make the current start from the correct initial value by writing it

$$C = B - b \log (t + \tau) \quad . \quad . \quad . \quad . \quad (27)$$

where τ is a small and suitably chosen time such that

$$B - b \log \tau = C_0. \quad . \quad . \quad . \quad . \quad . \quad (28)$$

If τ is sufficiently small this equation does not differ materially from (26), except when t is very small.

To obtain the polarization we substitute in equation (23) the value for C or $f(\theta)$,

$C = f(\theta) = 0$ before and at the instant $\theta = 0$.

$$\left. \begin{aligned} &= B - b \log_{10} (\theta + \tau) \\ &= C_0 - b \log_{10} \frac{\theta + \tau}{\tau} \text{ by (28)} \end{aligned} \right\} \text{ after the instant } \theta = 0 ;$$

and we have

$$\int_0^t \sqrt{t - \theta} f'(\theta) d\theta = C_0 \sqrt{t} - b \log_{10} e \int_0^t \frac{\sqrt{t - \theta}}{\theta + \tau} d\theta,$$

where the first term represents the effect of the sudden increase of $f(\theta)$ from 0 to C_0 at $t = 0$, and the second that of its gradual decay.

Assuming that τ is negligible with respect to t, the right-hand side may be evaluated as

$$\sqrt{t} \left\{ C_0 - b \log_{10} \frac{t + \tau}{\tau} + 2b \log_{10} e (1 - \log_e 2) \right\}$$
$$= \sqrt{t} \{ C + \cdot 267b \},$$

C being the value of the current at the time t substituted from (28). Consequently by (23)

$$V = \frac{RT}{\epsilon} \log_{10} \left\{ 1 + \frac{2}{Ac_1} \frac{C + \cdot 267b}{\sqrt{\pi \epsilon RTU}} \sqrt{t} \right\} \quad . \quad (29)$$

represents the polarization at any time.

The cell used for these experiments was composed of a mixture of mercurous nitrate and nitric acid with mercury electrodes contained in four beakers, WXYZ (fig. 3), con-

Fig. 3.
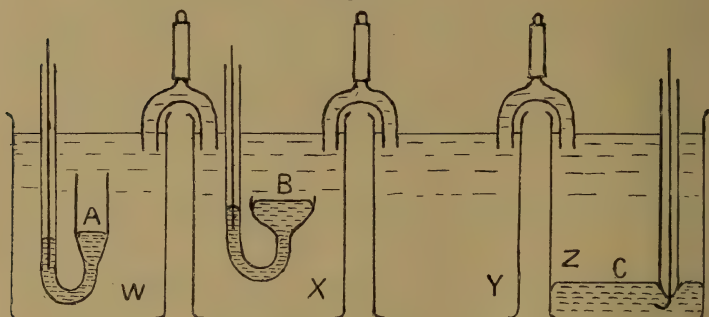
nected in series by siphons. In W was the mercury anode A surrounded by a glass tube, and in X a free electrode of mercury B, both under a solution of ·0001 normal $Hg_2(NO_3)_2$ in ·1 normal HNO_3; in Z the cathode, C, a large mercury surface under a half saturated solution of mercurous nitrate in ·5 normal HNO_3 (this rendered the cathode practically unpolarizable); and in Y a solution of ·1 normal HNO_3, the purpose of which was to prevent the diffusion of any appreciable amount of mercurous nitrate from Z to X.

The measured E.M.F. of this cell was ·0545 volt; and in the first experiment it was simply short-circuited through a galvanometer of 2900 ohms resistance and of known constant. The resistance of the cell itself between A and C was 4400 ohms, measured with the telephone and bridge *, so that the initial current produced should be

$$\frac{·0545}{4400 + 2900} = 7·46 \times 10^{-6} \text{ ampere.}$$

The actual current at any time can be accurately represented by the equation

$$C = \{6·33 - 1·48 \log_{10}(t + ·172)\} \times 10^{-6} \text{ ampere, . (30)}$$

t being the time in minutes. This may be seen from the following table of the observed values of the current and those calculated from (30).

* The experiments on the mercurous-nitrate cells were made in 1899, but not then published; this accounts for the difference between the methods of measurement used here and those already described used for the silver-nitrate cells.

Time from start (mins.).	Current (obs.).	Current (calc.).
0	$7 \cdot 46 \times 10^{-6}$	$7 \cdot 46 \times 10^{-6}$
3·0	5·43	5·59
5·5	5·20	5·21
14·0	4·67	4·62
27·5	4·23	4·19
50·0	3·83	3·81
82·0	3·45	3·50
179	2·95	3·00
250	2·76	2·78
755	2·05	2·08

In calculating the numerical values of the polarization from (29), it must be remembered that mercury exists in solutions of mercurous nitrate as a double ion $Hg-Hg$ with twice the usual ionic charge. This has been conclusively demonstrated by Ogg[*]. We must therefore put for ϵ the value

$$\epsilon = 2 \times 9654 \text{ E.M. units per gramme-ion of } Hg_2.$$

The ionic velocity U is not known with accuracy for mercury, but judging by the conductivity it is slightly less than that of silver. I have consequently taken it to be

$$U = 5 \times 10^{-12} \text{ cm. per sec. per E.M. unit per cm.}$$

Since it only occurs after a logarithm sign, a fairly large error in this value will only affect the result to a small extent. The sectional area of the tube round A was 0·789 sq. cm., temperature 16° C., and the concentration of the mercurous nitrate 0·0001 normal. Introducing these values into equation (29), we obtain

$$V_{\text{calc.}} = \cdot 0286 \log_{10} \{ 1 + 2 \cdot 34 \times 10^5 (C + \cdot 395 \times 10^{-6}) \sqrt{t} \} \text{ volt,}$$

where C is expressed in amperes, and t in minutes.

A comparison of the polarization calculated from this equation with that obtained in the experiment is given in the following table. The numbers in the column headed $(V + CR_{AB})_{\text{obs.}}$ are the observed differences of potential between the electrodes A and B at the times given. Since the resistance between A and B as measured by the telephone and bridge was 1480 ohms, the true polarizations, given under $V_{\text{obs.}}$, are obtained by subtracting 1480 C from the numbers in the previous column.

[*] *Zeit. Phys. Chem.* xxvii. p. 285 (1898).

Time from start (mins.).	Current (amp.).	V+CR$_{AB}$ (obs.).	V. (obs.).	V. (calc.).	Diff.	
11	$4·85 \times 10^{-6}$	·0274	·0204	·0202	+	·0002
20	4·45	·0297	·0233	·0225	+	8
40	4·00	·0310	·0252	·0252		0
69	3·57	·0328	·0276	·0273	+	3
130	3·15	·0346	·0301	·0296	+	5
195	2·87	·0361	·0319	·0301	+	18
251	2·75	·0370	·0330	·0318	+	12
756	2·05	·0412	·0382	·0353	+	29

In a second experiment in the same apparatus a solution made up to the same concentration as before ($c_1 = ·0001$ n. $Hg_2(NO_3)_2$) was employed in the cell, but instead of the cell being directly short-circuited through the galvanometer, a small reverse E.M.F. was applied in the external portion of the circuit, so that the total current-producing E.M.F. was reduced from the ·0545 volt of the previous case to ·0372 volt. The results are given below.

Empirical equation for current

$$C = \{4·44 - 1·20 \log_{10}(t + ·28)\} \times 10^{-6} \text{ ampere.}$$

Time from start (mins.).	Current (obs.).	Current (calc.).
0	$5·10 \times 10^{-6}$	$5·10 \times 10^{-6}$
5	3·54	3·57
11	3·20	3·17
19	2·92	2·90
39	2·54	2·53
90	2·07	2·10
180	1·75	1·73
278	1·55	1·50
825	0·95	0·94

Theoretical equation for polarization

$$V_{calc.} = ·0286 \log_{10} \{1 + 2·34 \times 10^5 (C + ·320 \times 10^{-6}) \sqrt{t}\} \text{ volt.}$$

Time from start (mins.).	Current (amp.).	V+CR$_{AB}$ (obs.).	V. (obs.).	V. (calc.).	Diff.	
43	$2·47 \times 10^{-6}$	·0237	0200	·0206	−	·0006
95	2·06	·0257	·0227	·0231	−	4
150	1·83	·0268	·0241	·0244	−	3
288	1·49	·0283	·0261	·0260	+	1
830	0·94	·0304	·0290	·0280	+	10

The agreement between the observed and calculated polarizations is as close as in the case of the silver-nitrate cells.

Discussion of Results.

The foregoing results show that the formulæ obtained represent the polarizations throughout with considerable accuracy, both as regards the absolute magnitude of the polarization and its rate of variation under very varied conditions of polarizing current. This may be taken as establishing the general correctness of the theory, at any rate as applied to the anodic polarization of the class of cell considered. But although a formula of the type of equation (22) is undoubtedly correct in its main features, I think it is possible that the numerical agreement found is to a certain extent a spurious one. For one thing, it turns out to be a closer one than we have any right to anticipate. The value $\cdot 0578$ volt of the factor $\dfrac{\mathrm{RT}}{\cdot 4343\epsilon} \times 10^{-8}$, on which the calculation of the polarization in the silver-nitrate cells is based, is invariably taken in all theoretical work on concentration-cells as accurately representing the E.M.F. of a concentration-cell the concentration of the metallic ions of which at one electrode is ten times that at the other; yet in actual practice the number obtained is always appreciably smaller. Thus a direct measurement of the E.M.F. of the cell

$$\mathrm{Ag} \; \left| \begin{array}{c|c} \cdot 002\,\mathrm{n.\ AgNO_3} & \cdot 05\,\mathrm{n.\ AgNO_3} \\ \hline \cdot 2\,\mathrm{n.\ HNO_3} & \cdot 2\,\mathrm{n.\ HNO_3} \end{array} \right| \; \mathrm{Ag}$$
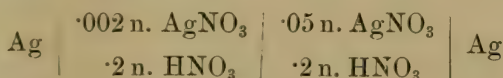
gave the value $\cdot 074$ volt at 19° C., which corresponds to a value of the factor of $\cdot 052$. There is no reason to suppose that this diminution in the factor as actually measured below its theoretical value does not hold in the polarization experiments, so that one would be justified in expecting that the observed polarizations would be some 10 per cent. lower than those calculated on the basis of the theoretical value for the factor. From this point of view, the agreement found means really a discrepancy of about 10 per cent., an amount which considerably exceeds the experimental error. I made an estimate of the experimental error by repeating two of the experiments under somewhat altered conditions, and found results consistent with the first to under $\cdot 002$ volt, which was equivalent to an error of about 3 per cent. This method of estimating the error, it is true, does not necessarily include

the effect of convection currents and imperfect insulation mentioned on p. 657; but the effect of these sources of error would be to reduce rather than to increase the observed polarization, and so could not account for the discrepancy. For the same reason the discrepancy cannot be ascribed to the neglect, which was made in the calculation of the polarization, of the effect of the slope of potential in the anode tube on the motion of the silver ions (see p. 648). An approximate calculation of the effect of this shows that the error is practically inappreciable, and in any case it would tend to reduce the observed polarization.

On the other hand, it is possible that the value assumed for the coefficient of diffusion of the silver ions in the presence of the nitric acid is in excess of its true value. The presence of nitric-acid ions undoubtedly reduces the equivalent conductivity of silver nitrate below its value at infinite dilution; this may be due to a diminution either in the dissociation, or possibly in the ionic mobility of the silver ions. Either of these effects would cause a decrease in the effective ionic mobility which determines the rate at which the silver diffuses away from the electrode. The assumption of too high a value for the coefficient of diffusion RTU/ϵ of the silver ions will diminish the second constant 23·80 of equation (24), and will so tend to counterbalance the error produced by the too high value of the constant ·0578. I think it is probable that both constants are really in error, and that the agreement found between the observed and calculated values of the polarization is due to the two errors being of such amounts as to approximately cancel each other. An examination of the equation shows that this might occur without the shape of the curve which expresses the variation of the polarization with the time being appreciably affected. To produce agreement between theory and experiment when the observed value ·052 is substituted for the theoretical ·0578, a diminution of some 20 per cent. must be assumed to be produced in the value of the coefficient of diffusion of the silver ions by the nitric acid. Since the second constant occurs under the sign of a logarithm in equation (24), the method of the experiments forms a very inaccurate means of measuring the coefficient of diffusion of the ions, so that only a very rough estimate of the value required is possible. The amount of the diminution suggested by these considerations seems perhaps larger than one would expect on the ordinary theory of dissociation. There are, however, no other experimental determinations with which to compare it, so that there is no reason to suppose it may not actually exist.

LXI. *On the Theory oj Experiments to detect Aberrations of the Second Degree.* By Edward W. Morley, *Ph.D., LL.D., Professor in Western Reserve University, and* Dayton C. Miller, *Ph.D., Professor in Case School of Applied Science, Cleveland, Ohio* [*].

[Plate IX.]

IN 1887 Michelson and one of the present writers made an experiment " On the Relative Motion of the Earth and the Luminiferous Æther " [†]. We found that, if there were any effect, it was not sensibly larger than one-fortieth of the amount expected. To explain this result, FitzGerald and Lorentz suggested that the motion of translation of a solid through the æther produces a contraction in the direction of the drift, with extension transversely, the amount of which is proportional to the square of the ratio of velocities of translation and of light.

Such a contraction can be imagined in two ways. It may be thought to be independent of the physical properties of the solid and governed only by geometric conditions; so that sandstone and pine, if of the same form, should be affected in the same ratio. On the other hand, the contraction may depend upon the physical properties of the solid : so that pine-timber would doubtless suffer a greater compression than sandstone. If the compression annul the expected effect in one apparatus, it may in another apparatus give place to an effect other than zero, perhaps with the contrary sign.

We have now completed an experiment in which two different pine-structures have been used, and in which the optical parts have been so enlarged as to produce an effect 2·3 times as great as the apparatus of 1887. The object was to determine whether there is any difference between the behaviour of sandstone and of pine.

When Michelson and Morley got a null result in 1887, it was thought sufficient to give the theory for merely the maximum and the minimum expected in the four principal azimuths, without mention of the phenomena at intermediate azimuths. The theory also neglected powers higher than the second of the ratio of the velocities. Recently, Dr. Hicks [‡] has published a profound and elaborate discussion of the theory, obtained by methods which are not approximate.

He develops expressions for angles of reflexion, for wave-length after reflexion, and for the conditions which determine the network of parallelograms formed by the two systems of wave-fronts. The diagonals of these parallelograms are alternately lines of maximum and minimum disturbance in the æther, so that they define the interference phenomena. These expressions are not only rigorous, but also general, applying to any adjustment whatever of the optical parts of the apparatus, and form a welcome contribution to the thorough understanding of the theory of the Michelson and Morley experiment.

In one passage he says that a term added by him " may entirely modify the nature of the changes produced as the direction of the drift alters " ; and some appear to think that the inference from the earlier experiment is involved in doubt by this discussion. It is therefore well to examine again the theory.

Let D, Pl. IX. fig. 1, be a plane-parallel glass plate, silvered so thinly that equal quantities of light are transmitted and reflected. S being a source of light, part of the light passes through D, moves on to the plane mirror II, where it is reflected back to D ; here, part is transmitted and lost, part is reflected to the observer at T. The other part of the entering light is reflected at the first surface of D, reflected again by the mirror I, is in part reflected by D, and lost, in part transmitted through D, and proceeds towards T. If distances and angles are suitable, the reunited rays between D and T will produce interference phenomena. If distances are equal, we may obtain interference phenomena in white light. In one of the usual adjustments of distances and angles, parallel fringes are seen when the eye or the telescope is made to give distinct vision of one of the mirrors I or II. The fringes apparently coincide with these surfaces. A central fringe is black ; on either side are coloured fringes, less and less distinct till they fade away into uniform illumination. If the path of either ray is shortened, the fringes move rapidly to one side. If we engrave a scale on I or II, we can, after any alteration of one of the paths, restore with great accuracy and ease the former relations by bringing the central dark fringe to its original place on this scale. If the motion of the earth through the æther were the cause of this change of path, we could measure the amount of change by measuring the displacement.

Suppose, fig. 2, that the apparatus moves in the direction of the arrow through the stationary æther. While the ray of light moves from D to I and returns to D, the mirror D

moves to the new position D_2. The angle of reflexion from D is no longer equal to the angle of incidence. The ray moving from D towards II finds the latter in the position II_2. returns to D_2, and is reflected from D_2 nearly in the same direction with the ray from I. In four azimuths of the apparatus the coincidence is exact ; for all others, the ray I and the ray II are inclined at a small angle which, at its maximum, is numerically equal to v^2/V^2, v and V being the velocities of the apparatus and of light. Since the angle δ, the total aberration, cannot be observed, being annulled by the motion of the observing telescope at T, we can hope to detect merely this aberration of the second degree, namely, the small angle between the emergent rays I and II.

With the adjustments just supposed, there are four methods of measuring interference phenomena which in turn measure the angle sought. We may use a micrometer in the telescope, or a scale engraved on I or on II ; we may use mechanical compensation to return a displaced fringe to its marked position, or we may use optical compensation.

In another adjustment the fringes are made infinitely broad. We are then limited to the last pair of methods. This pair, especially the last method, is capable of very great precision. When Michelson and Morley set up the first apparatus in which they utilized this method, the mean error of a setting, in which the observer did not himself see the reading, was less than the two-hundredth part of a wavelength. Since the theory of the apparatus in this special case is simpler, the discussion will assume this adjustment.

Accordingly, let the angles I B D, II B D, fig. 3 (Pl. IX.), be equal to each other and to $45°$. Let the three planes intersect in a common point B. For brevity, imagine that the mirrors themselves are produced so as to intersect in this point. Assume that the system is moving through the æther in a direction making an angle of $67\frac{1}{2}°$ with the direction of the light entering the telescope, as indicated in fig. 2. The velocities of the apparatus and of light being denoted by v, V, assume that $v/V = 1/5$.

A certain wave-front enters the apparatus, making with II an angle which is to be specified. If some given ray enters the apparatus so as to pass axially through the telescope, rays making an angle of 5 minutes on either side of it will pass through our actual apparatus. Almost any ray, wisely selected, may be used to determine what we desire to know about the whole pencil. For instance, we might select the ray which, after reflexion from II, shall

return to the point in the mirror D at which it first passed through this semitransparent mirror ; or the ray which, after reflexion from I, returns to D with the same exactness. The simplest treatment is possible when we select that system of wave-fronts which make with the mirror II an angle $\sin^{-1}\dfrac{v}{V}\cos\alpha$, where α is the azimuth of the apparatus measured from the position in which its motion through the æther is parallel to the axis of the observing telescope. The azimuth of the motion assumed in figs. 2 and 3 is $67^{\circ}\ 30'$.

We will now examine the condition of the wave-fronts in the apparatus, fig. 3, at two specified instants, using two diagrams to avoid confusing the numerous lines. In fig. 3 are shown nine wave-fronts making the specified angle with II. The wave-fronts of the transmitted fraction are denoted by accented letters. Seven have not yet reached the mirror II ; b' intersects II in the common point B ; a' has been reflected from II, and its upper part begins to be reflected from D. At the same instant are shown the wave-fronts of the other system by unaccented letters. All have been in part reflected from D ; c begins to be reflected from I ; b intersects I in the common point ; a is quite cleared from the mirror I. In fig. 4 we follow the same nine wave-fronts. Of the transmitted waves, a' has quite cleared the system of mirrors ; b' is just clearing D ; five are but partly reflected from D ; h' passes through the common point B, and is beginning to be reflected by D ; i' has not yet reached II. Of the other system of waves, a and b have cleared the system of mirrors ; five are passing through the semitransparent mirror D ; h passes through the common point B, having just finished its reflexion from I ; i is just beginning this reflexion. For all azimuths except four the general conditions are those of the diagram, but the amount and direction of the various inclinations alter with the azimuth.

The wave-front h is established in its whole length when it passes through the common point. The wave-front corresponding to it in the other system is, this instant, infinitely short, and a' is the first to be established in its entirety. But the position of a fictitious wave of this system, h'', is determined by two conditions—first, that it be parallel to a', and, secondly, that it coincide with the infinitely short wave-front h' at the common point B. Except at four certain azimuths, these two wave-fronts, in the same phase, and intersecting in a common point B, will be inclined to each other at a small angle. To measure or to detect this inclination is to measure or to detect the secondary aberration

which interests us. If we could measure the perpendicular distance between these wave-fronts at a sufficient distance from B, we should know the angle between them. But h'' is only a fictitious line. What we cannot measure between h and h'' we can measure between a and a', provided we can determine the point of intersection between a and a', and provided this be found in a convenient position. We have therefore to determine the point of intersection of a and a', knowing that of h and h''.

The observing telescope is shown at T, fig. 5. Its axis is parallel to B II. We will show that the phase-difference of a and a' is constant at all points on any line parallel to the line B II, or to the axis of the telescope.

If we write λ, λ', *not for wave-lengths*, but for the perpendicular distance between consecutive wave-fronts of the same phase, and δ, δ' for the total aberration of the wave-fronts of the two systems, we have to show that $\dfrac{\lambda'}{\cos \delta'} - \dfrac{\lambda}{\cos \delta}$ is identically zero for eight specified equidistant azimuths, and is not greater than $0 \cdot 3 \dfrac{v^4}{V^4}$ for other azimuths. Each of these quantities is determined by a complicated expression; and the equality specified can be most readily determined by trigonometric computation.

To prove the proposition, therefore, we will take that azimuth where, according to Dr. Hicks, the shifting of the intersection is a maximum, and we will assume the extreme case where the velocity of the apparatus is half that of light.

In fig. 6, the mirrors D, I, and II are accordingly supposed to move in the direction of the arrow. Let τ be the period of the waves of light incident on D; according to the previous specification, the angle between these wave-fronts and the plane of II is $\sin^{-1} \dfrac{v}{V} \cos \alpha$; that is, they are parallel to II. Lay off on $c\,d$, the line of motion of a certain point of the mirror D, the positions of this point at the times 0, τ, 2τ, &c. Positions of D and of I and II at certain times are also noted in the same way : all numerical subscripts denote times. The source moves with the apparatus, and therefore, with the assumed ratio of velocities, the apparent wave-length of the light incident at D is half the wave-length in the case of rest, and is half the distance described by a wave-front in the unit of time. Let the initial position be one in which a wave-front passes through the given point in the mirror and through the point O in the line of motion. At the time $t = \tau$ the mirror is at J, and the wave-front in

question cuts the line of motion in 2, and intersects the mirror in *e*. The wave-front reflected from D at *t*=0 will have reached the point f_1, and the tangent $e f_1$ establishes the reflected wave-front. At the times 2τ, 3τ, &c., this particular disturbance will be found at f_2, f_3, f_4, &c.

When D is at a position 2, a new disturbance will have been established at *g*, which, at the time *t*=5τ, will be found at g_5. In the same way, h_5, i_5, k_5 will have arranged themselves in the line $f_5 k_5$. At the time $t=10\tau$ the six wave-fronts will have been reflected from I, and will be placed along the line $o_{10} f_{10}$. The angle f_5 O I o is equal to the aberration of the wave-fronts after reflexion from D. As, at this azimuth, the angles of incidence and of reflexion at I are equal, this angle is also the aberration of the emergent rays.

Part of the wave-front *f*, indicated by *f'*, will be transmitted through the mirror D. It will overtake the mirror II at the time $t=8\tau$, when II will have position marked II_8, fig. 7. Returning after reflexion, it will take the position noted for the times $t=9\tau$, $t=10\tau$; and meeting the mirror D at $t=10\frac{2}{3}\tau$, it will be reflected as shown to f'_{11}, at an angle whose tangent is given by the formula below. The following wave-front g' will be reflected one period later, at II_9, and it is shown in several positions. The wave-front *f*, belonging to the other system, having passed through the mirror, and having reached the line, S*d*, at $t=10\tau$, is shown at f_{11}.

In fig. 8 is shown the position of the wave-fronts below the mirror D for the time $t=15\tau$. f_{15} and f'_{15} have moved along the paths indicated, while the other wave-fronts have moved in a corresponding manner, their position at the time $t=15\tau$ being as shown in the figure. The wave-fronts of the unaccented system are placed on the line *op* ; the aberration of the system is equal to the angle δ. The wave-fronts of the accented system are placed on the line *qr* ; the corresponding aberration is the angle δ' ; the line $T_{15}q$ being the position of the axis of the observing telescope at the time $t=15\tau$. Produce the planes of the wave-fronts, draw lines *l*, *l'*, parallel to qT_{15}, the axis of the telescope, each terminated by the planes of consecutive wave-fronts. Their lengths are $l=\dfrac{\lambda}{\cos\delta}$, $l'=\dfrac{\lambda'}{\cos\delta'}$. It is to be proved that $l=l'$.

Putting ϕ for the angle of incidence and ϕ' for the angle of reflexion, we have for both reflexions at D, $\phi=45°$. For this azimuth there is no change of angles at I and II. The

angles of reflexion are given by the equation

$$\tan \frac{\phi'}{2} = \tan \frac{\phi}{2} \cdot \frac{V-u}{V+u},$$

where u is the velocity of the surface perpendicular to its own plane ; negative, if moving away from the approaching light. *With the assumed adjustments*, we therefore have for the reflexion of the unaccented system at D:

$$\frac{\sin 45^\circ + \sin \delta}{\cos 45^\circ + \cos \delta} = \frac{\sin 45^\circ + \sin 0^\circ}{\cos 45^\circ + \cos 0^\circ} \cdot \frac{2 + \sqrt{0\cdot5}}{2 - \sqrt{0\cdot5}}$$

$$= \frac{\sqrt{0\cdot5} + 0\cdot6}{\sqrt{0\cdot5} + 0\cdot8} ; \therefore \cos \delta = 0\cdot8.$$

For the reflexion of the accented system we have :

$$\frac{\sin 45^\circ - \sin \delta'}{\cos 45^\circ + \cos \delta'} = \frac{\sin 45^\circ - \sin 0^\circ}{\cos 45^\circ + \cos 0^\circ} \cdot \frac{2 - \sqrt{0\cdot5}}{2 + \sqrt{0\cdot5}}$$

$$= \frac{\sqrt{0\cdot5} - \frac{5}{13}}{\sqrt{0\cdot5} + \frac{12}{13}} ; \therefore \cos \delta' = \frac{12}{13}.$$

By equation (3) of Dr. Hicks's[*] paper, putting L_1 for the perpendicular distance between wave-fronts of the light incident on D from the moving source,

$$\frac{\lambda}{L_1} = \frac{4 - 0\cdot5}{4 + 0\cdot5 - 2} = 1\cdot4; \text{ and } \frac{\lambda'}{L_1} = \frac{4 - 1}{4 + 1 - 4} \cdot \frac{4 - 0\cdot5}{4 + 0\cdot5 + 2} = \frac{21}{13}.$$

Therefore

$$\frac{\lambda}{\cos \delta} = L_1 \, 1\cdot4 \div 0\cdot8 = 1\cdot75 \, L_1,$$

and

$$\frac{\lambda'}{\cos \delta'} = L_1 \frac{21}{13} \div \frac{12}{13} = 1\cdot75 \, L_1 ;$$

accordingly,

$$\frac{\lambda'}{\cos \delta'} = \frac{\lambda}{\cos \delta} = l' = l.$$

Therefore, if the intersection of k_{15} and i'_{15} is on the line xy parallel to qT_{15}, the intersection of i_{15} and h'_{15} is also on the same line ; that is, the phase-difference of the two sets of waves is constant along any line parallel to the axis of the observing telescope. The same thing may easily be proved for any one of eight equidistant points of the circumference commencing from the point where the motion of the apparatus is parallel to the axis of the telescope.

[*] Phil. Mag. [6] iii. p. 17 (1902).

The equality is exact for the central ray.   For rays inclined as much as five minutes of arc on either side of the central ray,

$$\frac{\cos \delta'}{\lambda'} - \frac{\cos \delta}{\lambda} < 0 \cdot 2 \, \frac{v^3}{V^3}.$$

At azimuths other than those specified, the quantity $\frac{\cos \delta'}{\lambda'} - \frac{\cos \delta}{\lambda}$ is not greater than $0 \cdot 3 \, \frac{v^4}{V^4}$ for central rays, and not greater than $0 \cdot 5 \, \frac{v^3}{V^3}$ for extreme rays.   We may set side by side the magnitudes of this disturbing effect for central rays at several azimuths according to rigorous computation and according to Dr. Hicks's approximate formula.

Azimuths.	Disturbing Effects Compared.	
	Dr. Hicks's formula.	Rigorous formula.
90°	$1 \cdot 2 \, \dfrac{v^3}{V^3}$	$0 \cdot 0 \, \dfrac{v^4}{V^4}$
90° $\pm 11°$	$0 \cdot 9$,,	$0 \cdot 3$,,
90° $\pm 22°$	$0 \cdot 65$,,	$0 \cdot 3$,,
90° $\pm 45°$	$0 \cdot 0$,,	$0 \cdot 0$,,
90° $\pm 90°$	$0 \cdot 0$,,	$0 \cdot 0$,,
$-90°$ $\pm 45°$	$0 \cdot 0$,,	$0 \cdot 0$,,
$-90°$...............	$1 \cdot 2$,,	$0 \cdot 0$,,

It will be seen that the effect detected by Dr. Hicks proves, by rigorous computation, to be entirely negligible for the central rays.   Its extreme value for marginal rays is not greater than $0 \cdot 5 \, \frac{v^3}{V^3}$, which is entirely too small to influence the observations.   This result is very satisfactory. It is proved for the specified adjustment of the angles, but it is easy to see that the rotation of mirror I about a perpendicular line in its surface, by a quantity like ten seconds of arc, will not change all relations of residual aberrations by important amounts.   It is therefore established, at least for the adjustment specified, that the wave-fronts a and a' of fig. 4 intersect in the line B II, if the wave-fronts h and h'' do, rigorously for eight principal positions, very approximately for all other positions.   If, then, we can measure the linear distance between a and a' at some convenient position T, we may determine the angle between the wave-fronts a and a', which is the same as the angle between h and h'', the

angle of aberration of the second degree, which it is the object to detect.

We have shown that the wave-lengths of the two rays, when resolved *in the direction in which alone they are used,* are equal. One other point as to wave-lengths must be considered. We use wave-lengths to measure a length of less than 0·0002 mm., to determine the angle $h \, \mathrm{B} \, h''$. Is the scale of variable value? The light from a source moving with the apparatus has its wave-length modified by the motion. Dr. Hicks gives the formula for this effect in equation (4), page 17. If with this and the equation (2) we compute the wave-length resolved in the axis, at the azimuths where the effect is a maximum, and for the velocity ratio 100, the two minima are 0·9899995 L and 0·9899505 L, while the two maxima are both 0·9999500 L, where L is the wave-length in the case of no motion. For the ratio 10,000 these quantities differ from unity by about a hundredth part as much, and the inequality is negligible, even if we had to multiply this unit by a large number. But we have to do with only a fraction of the unit.

We next inquire as to the amount and the laws of aberration produced by reflexions from the mirrors of the apparatus. These can be developed in a series of powers of the velocity ratio, and of sines and cosines of the azimuth and of its multiples. But numerical estimates seem desirable, and the formulæ are such that these can more easily be obtained from trigonometrical computation. For the *actual* velocity ratio the computation is not easy, because trigonometrical tables of fifteen decimal places are not available. Imagine, then, three different apparatus, each of the dimensions proper to the special value of the velocity ratio for which it is specially designed. One apparatus, for the ratio 10, may have the length B II, fig. 3, equal to 10^2 L; another, for the ratio 100, may have the length 100^2 L; and the third, for the ratio 1000, may have the length 1000^2 L. What we can readily learn for the ratio 10, with seven place logarithms, will apply to the ratio 100, except for the circumstance that angles are not so small that sines and arcs are identical in value. What we compute for the ratio 100 with ten place logarithms tells us everything we desire to know for the ratio 1000 and for the actual ratio.

We have computed the aberrations of the two rays I and II, for certain azimuths with the velocity-ratios 10 and 1000, and for 18 azimuths of the apparatus with the velocity-ratio 100. From these aberrations we subtract that part of the aberration which is annulled by the motion of the telescope,

and then decompose these residual aberrations into terms depending on the squares and on the cubes of the ratio of velocities. To a thousandth part of the *residual* aberrations, their difference is represented by the equation

$$\delta' - \delta = \frac{v^2}{V^2} \cos 2\alpha + \frac{v^3}{V^3} (\sqrt{0\cdot5} \sin 2\alpha + \tfrac{1}{6} \sin 4\alpha + \cos \alpha).$$

For the velocity-ratio 10,000 the agreement would be much closer. Fig. 9 (Pl. IX.) A shows the laws of the variation in the residual aberrations of the two rays I and II, coming from the mirrors I and II. The unit for A and C is $\sin^{-1} \frac{v^2}{V^2}$, and for B and D, $\sin^{-1} \frac{v^3}{V^3}$. The curves A are nearly represented by $\frac{v^2}{V^2} (\sin 2\alpha \pm \tfrac{1}{2} \cos 2\alpha)$. Subtracting the ordinates given by this expression from the actual ordinates, we get the residuals shown (after multiplication by the reciprocal of the velocity-ratio) at B. C shows the difference of the curves I. and II. of A, and thus gives directly the angle of divergence of the emergent wave-fronts which is the object of our study. D gives the difference between this curve and the sine curve $\delta' - \delta = \frac{v^2}{V^2} \cos 2\alpha$. The latter curve shows that the difference of the aberrations of I and II is at an undisturbed maximum at 90° and at 270°; at 0°, it is less than the undisturbed maximum by the quantity $\frac{v^3}{V^3} \cos \alpha$, or $\frac{v^3}{V^3}$; at 180° it is greater by the same quantity.

It may be thought that the adjustment of the angles between the mirrors which has been assumed will limit too narrowly the use of the apparatus. We may simply say that experience with mirrors as nearly plane as those used by us has shown us that the method of observation supposed would suffice for angles of aberration at least twenty-five times that expected if the velocity-ratio is 10,000.

Since the experiment gives a null result, it is not worth the space to prove that what is true of this adjustment is true with sufficient approximation for an adjustment which differs from the assumed adjustment only by the rotation of mirror I by an angle of ten seconds around a perpendicular axis passing through its surface. Instead, we may compare the results here obtained with those of Dr. Hicks.

In the first place, he declares that the position of the fringes is displaced by aberration. This point is eliminated from our discussion by the fact that the fringes are infinitely

wide. We simply remark that, if we understand rightly his statement, this aberration is annulled by the motion of the telescope. Also, his discussion contains a term expressing the fact that the waves of one system gain upon those of the other while passing towards the observer. We have shown that, *in the conditions assumed* (and realized), this effect is *nil* for central rays in the eight principal azimuths, and is small in all others. At its maximum, for central rays, it is $0\cdot3\,\dfrac{v^4}{V^4}$. With our present large apparatus, whose length is $54\times10^6\,\lambda$, the gain of one wave-front over the other in the whole length is much less than $10^{-6}\,\lambda$.

In the theory of 1887, powers of the velocity-ratio higher than the second were expressly regarded as negligible. Dr. Hicks virtually supplies one such term. He writes, displacement of fringes $=\dfrac{\frac12\xi^2\mathrm{L}\cos2\alpha}{\sin\,(\mathrm{B}-\mathrm{A})-\frac12\xi^2\cos2\alpha}$, where ξ is the velocity-ratio, L is the length of path in the apparatus, from D to I, fig. 5, and $\mathrm{B}-\mathrm{A}$ is the difference between the angles DB I and DB II. Without the small term in the denominator, this gives precisely the same value as the expression in the paper of 1887, as a simple numerical computation shows. The effect of the small term is the following :—the value of the denominator is decreased or increased by $\frac12\,\dfrac{v^2}{V^2}$ at alternate quadrants, and the value of the fraction is therefore increased or decreased at alternate quadrants. But, according to the present solution, the expression should have a mean value at $90°$ and $270°$, and have, further, a maximum at $180°$ and a minimum at $0°$. At three quadrants we agree, but at the fourth we differ by twice the term in question. The difference is easily explained and is negligible, especially in view of the null result of experiment.

It should be noted that, when there is aberration of the wave-front, there are four closely related magnitudes. One is the distance travelled by the wave-front in the period ; a second is the perpendicular distance between consecutive wave-fronts, called λ in Dr. Hicks's paper ; a third is the distance between wave-fronts, resolved parallel to some line dictated by the geometric conditions of the case ; and the fourth is the distance between wave-fronts in the line of sight, which is the true wave-length. The perpendicular distance between wave-fronts is used rightly, as we conceive, in establishing the conditions of the network of intersecting wave-fronts in Dr. Hicks's admirable paper. But in one

paragraph, which is entirely distinct from the rest of the discussion, he uses an expression which is not sufficiently approximate; *e. g.*, if the expression be taken to mean the wave-length as stated above, and accordingly used to compute the number of waves in a given length in the line of vision, it differs from the truth by $\frac{v^2}{V^2}$, precisely doubling the result found otherwise.

We assert, then, that the theory of 1887 is correct to terms of the order retained, which were sufficient; that Dr. Hicks's theory agrees with it precisely as to numerical amount and sign[*] of the effect, and that a third examination of the theory gives results differing from those of the two others only by negligible terms of the third order.

LXII. *Report of an Experiment to detect the FitzGerald-Lorentz Effect. By* Edward W. Morley, *Ph.D., LL.D., Professor in Western Reserve University, and* Dayton C. Miller, *Ph.D., Professor in Case School of Applied Science*[†].

[Plate X.]

A NULL result was obtained in 1887[‡] in an experiment to detect, if possible, a difference of velocity of light in different directions, owing to the motion of the apparatus towards or away from waves of light in the stationary æther. FitzGerald and Lorentz then suggested that the dimensions of the apparatus might be modified by its motion through the æther. If this modification depend on the resilience or other physical properties of the materials, it may perhaps be detected by experiment.

We have constructed two apparatus with which to examine this question. In the first, we replaced the sandstone used in 1887 by a structure of white pine. A strong cross was built up of planks, 14 inches wide and 2 inches thick, and 14 feet long. One was laid east and west, then one across it north and south, and so on. They were slightly notched where they crossed. On their intersection was secured a cast-iron bedplate for certain optical parts of the apparatus.

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[*] Taking into account a note in 'Nature,' vol. lxv. p. 343 (1902).

[†] Communicated by the Authors: read at the New York Meeting of the National Academy of Sciences.

[‡] " On the Relative Motion of the Earth and the Luminiferous Æther." A. A. Michelson and E. W. Morley, Am. Journ. Sci. vol. xxxiv. p. 333.

At the ends. after filling the spaces between the planks, were bolted iron supports for our mirrors. The whole was placed on a round float, which in turn rested in a basin of mercury.

Our sixteen mirrors were each 4 inches in diameter. The mirrors rested each on the points of three adjusting screws, against which they were held by springs. On the bedplate, at the intersection of the arms of the cross, were placed a plane half-silvered mirror and a compensating plate ; these had been. as is usual, cut from the same plane-parallel disk.

Fig. 1 is a diagram, not to scale, of the optical arrangements.

Fig. 1.
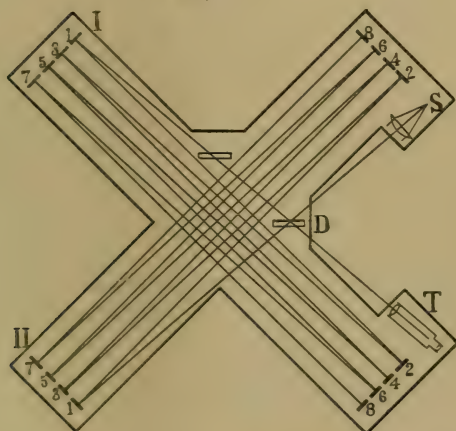
Fig. 2.
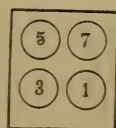
Light from a source S reaches the mirror D. Part is transmitted, reaching the mirror II. It is successively reflected to 2, 3, 4, 5, 6, 7, and 8. From 8 it returns by the same path to D. where part is reflected to the observer at T. Another part of the incident ray is reflected along the other arm of the cross. is similarly passed to and fro, returned, and at last transmitted to the observer. In the apparatus actually used, mirror 5 lay above 3, rather than to one side of it ; fig. 2 shows this arrangement. The whole path of the light along these mirrors was enclosed and covered, to lessen the effect of air-currents and other disturbances. An acetylene flame was carried as a source of light. A telescope magnifying thirty-five diameters gave distinct vision of mirror 8, at whose surface the interference-fringes are apparently localized.

The mirrors being silvered and polished were put in place, and the lengths of the two paths were measured with a split rod and then made nearly equal. Establishing interferences

in sodium light, we found the central part of a series of some
700 interferences which are brighter than the adjoining 300.
With no long search, we could see interferences in white light,
although we had provided no screw for moving a mirror with
its surface always parallel to a given surface. This we had
avoided, in order to have everything about the two arms as
symmetrical as possible.

We now computed the direction and velocity of the motion
of the centre of the apparatus by compounding the annual
motion in the orbit of the earth with the motion of the solar
system towards a certain point in the heavens. During part
of August, the whole of September, and nearly all of October,
this motion never coincides with the plane of our apparatus.
For other dates, there are two hours in each day when the
motion is in the desired plane, except for two days when
the two hours coalesce into one. At the beginning of June,
the two hours are about $11^{\text{h}}\ 20^{\text{m}}$ A.M., mean solar time, and
$9^{\text{h}}\ 50^{\text{m}}$ P.M. At the time of our last set of observations,
July 5th to July 9th, the hours were $11^{\text{h}}\ 40^{\text{m}}$ A.M. and
$8^{\text{h}}\ 20^{\text{m}}$ P.M., local mean time.

After many trials, with filar micrometer, and with a
scale on mirror 8, we found it advisable to accumulate a great
number of observations made as rapidly as might be. What
we had to do, in presence of all the local disturbances of
density of the air which sometimes made observation im-
possible and always made it difficult, was as if we were
trying to measure the diurnal solar atmospheric tide. If
we could vary the period of this tide at will by controlling
the revolutions of the earth, we should doubtless get a result
sooner by accelerating the latter and making a great number
of observations in a given time, rather than by retarding the
period in order to measure with very great precision the
hourly height of a barometer. We therefore proceeded as
follows :—One observer walked around with the moving
apparatus, his eye at the telescope, while he maintained the
rotation by an occasional gentle pull on a cord so fixed as not
to bring any strain to bear on the cross arms of the apparatus.
The room was darkened. The other observer also went around
with the apparatus; as an index showed the azimuth of the
apparatus to be that indicated by one of 16 equidistant marks,
he called out the number or some other signal. The first
observer replied with the reading for the given azimuth,
which the second observer recorded. The next azimuth was
called at the proper instant, the reading given, and so on.
Half the time, perhaps, the observations were interrupted before
they became numerous enough to be useful, being stopped

by excessive displacement of fringes owing to temperature changes and the like. But patience is a possession without which no one is likely to begin observations of this kind. Runs of twenty and thirty turns, involving 320 or 480 readings, were not uncommon. A run of thirty turns meant that the observer, who could sometimes make a turn of sixteen readings in 65 or 75 seconds, walked half a mile while making the severe effort involved in keeping his eye at the moving eyepiece without the least interruption for half an hour. The work is, of course, somewhat exhausting.

Observation with this apparatus could not begin till the month of August, and we had to stop without having accomplished as much as was desirable. During the busy season of the school year, observation is impossible. We had therefore expected to resume our work in June. But we then found that our pine apparatus had so much suffered from the dry air of the building, that we could not maintain the adjustment of our fringes. We could not, in the time, build another apparatus of timber which had not been dried all winter ; nor was it thought well to construct another apparatus closely resembling the first. While planning a new apparatus, we made a couple of experiments to show, what was well enough known, that difference of magnetic attraction on the iron parts of our apparatus could not disturb our observations. We suspended two massive pieces of iron at the ends of one arm, so that one should be in the lines of magnetic force of the earth's field, and the other transverse to them, these relations being reversed on reversing the position of the apparatus. But observations with this load of iron gave the same result as before. Next we placed an analytical balance on one arm, with which to weigh a bar of iron at the extremity of that arm. It was so placed that at one azimuth the bar was nearly in the lines of force, and at another was transverse to them. If there were a difference of half a milligramme in twelve hundred grammes, it would have been detected; but no such difference existed. We found by trial how much a weight of a hundred grammes displaced our fringes, and so learned, as was known before, that the influence of the earth's magnetism could not be a disturbing factor.

The Rumford Committee of the American Academy now came to our aid, and we carried out our original plan of making a steel structure which should permit easy and satisfactory observation. In this apparatus, all weights are carried by two steel girders which intersect in a cross. With steel, we could have perfect symmetry in the two arms, which is impossible with wood. On the steel framework, two sets of

four mirrors each are fastened by bolts through their sup-
porting frames.　Against these holders rest eight slender
pine rods, supported throughout their lengths by enclosing
them within the tubes of a brass truss.　Against the further
ends of these rods are held the freely suspended holders of the
other two sets of mirrors.　Springs apply a certain force to
hold the freely suspended holders against the rods, and so
against the first fixed mirror-holders, so that the distance
between the mirrors depends entirely on the pine rods.　In
fig. 3 (Pl. X.) the apparatus is shown in an incomplete state ;
the telescope is shown in its final position, but the lamp
and condensing-lens are not in place.　Fig. 4 (Pl. X.)
shows the apparatus ready for observations ; with the lamp
and lens placed as far as may be from the optical parts of the
apparatus, and shut off somewhat by screens.　The whole path
of the light in the apparatus is enclosed, and the observer is
protected by a dark cloth from the light which is necessary
for the recorder.

With this apparatus, we adjusted our fringes on a certain
Monday, and found that they remained in adjustment
throughout the week during which we were occupied in
observing them.

With this apparatus, observations were made precisely as
before.　We obtained 260 complete observations consisting
each of readings at sixteen azimuths around a circumference.
At the date of the observations, the annual motion of the earth
together with the motion of the solar system may be taken as
33·5 kilometres a second.　The velocity of light being 300,000
kilometres a second, the ratio of the squares of the velocities
is $0·72 . 10^8$.　The length of path of a ray in our apparatus
was 3224 centimetres, in which distance there are contained
$5·5 . 10^7$ wave-lengths of sodium light.　The expected effect
being doubled by rotation through 90°, the displacement of
fringes expected on the simple kinematic theory will be
$11 . 10^7 \div 0·72 . 10^8$.　This is 1·5 wave-length.

As was indicated, there were two times in the day when
observation was advisable.　The direction of the motion with
reference to a fixed line on the floor of the room being com-
puted for the two hours, we were able to superimpose those
observations which coincided with the line of drift for
the two hours of observation.　Doing this, and subtracting
a constant so as to make the algebraic sum of the observations
equal to zero, we get a certain result.　Then adding the
first term to the ninth, and so on, since the effect repeats
itself after a half revolution, we get our final result, as
follows :—

Result of observations at various azimuths.

Azimuths	8.	7.	6.	5.	4.	3.	2.	1.
Wave-lengths...	+0·0075	+0·0088	+0·0113	−0·0102	−0·0123	+0·0027	−0·0021	−0·0062

Azimuth mark 1 denotes that the telescope of the apparatus was directed N. 29° E., 3, N. 16° W., 5, N. 61° W., &c.

These numbers may be confidently pronounced to be due to errors of observation. We computed from them several curves of the theoretical form, having their origins at sixteen equidistant points in the half circumference; this was done by the method of least squares. The most probable of these curves had an amplitude of 0·0073 wave-length, and its zero was half way between the azimuths marked 4 and 5. The average of the given observations is 0·0076 wave-length: after subtracting the ordinates of the computed curve, the mean residual was 0·0066 wave-length. The sum of the squares of the residuals before was $565 . 10^{-4}$; afterwards it was $329 . 10^{-4}$.

We may therefore declare that the experiment shows that if there is any effect of the nature expected, it is less than the hundredth part of the computed value. If pine is affected at all as has been suggested, it is affected to the same amount as is sandstone. If the æther near the apparatus did not move with it, the difference in velocity was less than 3·5 kilometres a second, unless the effect on the materials annulled the effect sought.

Some have thought that the former experiment only proved that the æther *in a certain basement-room* was carried along with it. We desire to place the apparatus on a hill, covered only with a transparent covering, to see if an effect could be there detected. As the Rumford Committee have allowed us thus to utilize an unexpended balance, we hope to make the experiment in this form, should it be possible to make observations in trying conditions.

LXIII. Poiseuille's *Law at Very Low Rates of Shear.* By A. WILMER DUFF, M.A., D.Sc. (*Edin.*), *Professor of Physics in Worcester Polytechnic Institute, Worcester, Mass.*[*]

POISEUILLE[†] showed that the law of flow of a viscous liquid through a capillary tube, now usually called Poiseuille's law, holds true through a very wide range of velocity of flow. The rate at which the moving liquid is

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[*] Communicated by the Author.
[†] Poiseuille, *Compt. Rend.* vol. xv. p. 1167 (1842).

being sheared varies from 0 in the axis of the tube to a maximum of $pr/2l\eta$ at the surface of contact of tube and liquid, l and r being the length and radius respectively of the tube, p the difference of pressure at the ends of the tube, and η the coefficient of viscosity. In Poiseuille's experiments the maximum rate of shear varied with different tubes and different pressures between 5000 and 500,000 radians per second; and all subsequent measurements of viscosity by Poiseuille's method have, I believe, been made within these limits.

A number of considerations, relating to the discrepancies between different methods of measuring viscosity, the contradictory results obtained by myself [*] and others in testing the effect of electrostatic stress on viscosity, and Poisson and Maxwell's [†] view of a liquid as differing from a solid only in the magnitude of its elastic constants and rate of relaxation under stress, suggested the desirability of a test of Poiseuille's law at much lower rates of shear than those stated above.

Method and Apparatus.

The rate of outflow from a capillary tube varies, according to Poiseuille's law, as the product of the pressure and the fourth power of the radius, while the maximum rate of shear varies as the product of the pressure and the radius. Hence, to get a low rate of shear together with a measurable rate of outflow, it is necessary to use small pressures and a tube of large radius, the dimensions being such that the flow shall be rectilinear.

Two large beakers were connected by the tube which passed into the vessels through rubber stoppers. Differences of level of the liquid in the beakers were measured by means of optical levers. Two legs of each lever rested on an outside support, while the third leg was borne by a small vessel, or float, that floated on the surface of the liquid. When the liquid had come to the same level in the two vessels, as indicated by the readings of the scales becoming stationary, a slight initial difference of level was produced by opening for a moment a siphon that connected a third vessel to one of the other two. Or the initial difference of level was produced by lowering a body of known volume into one of the vessels: in this case one of the levers might be dispensed with, since the subsequent changes of level in one vessel could be deduced from that of the other and the known

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[*] *Physical Review,* vol. iv. no. 19 (1896).
[†] *Phil. Mag.* [4] vol. xxxv. pp. 133, 210 (1868).

cross-sections of the beakers. The tube and beakers were immersed in a large water-bath, and the work was performed in a suitable "constant temperature" room.

If h be the difference of level of the liquid in the two vessels at time t, the volume of flow in time dt from one vessel to the other is, according to Poiseuille's law,

$$v = \frac{\pi g \rho r^4 h}{8 l \eta}\, dt.$$

If A_1 and A_2 be the areas of the sections of the vessels at the level of the liquid, the change of the difference of level between the two vessels in time dt is

$$-dh = \frac{v}{A_1} + \frac{v}{A_2} = \frac{\pi g \rho r^4}{8 l \eta}\left(\frac{1}{A} + \frac{1}{A_2}\right)\frac{h\,dt}{\eta}\,;$$

whence it readily follows by integration that

$$\eta = \frac{\pi g \rho r^4}{8 l}\left(\frac{1}{A_1} + \frac{1}{A_2}\right)\frac{t_1 - t_0}{\log_e \dfrac{h_0}{h_1}},$$

h_0 being the difference of level at time t_0, and h_1 that at time t_1.

The flow must be slow enough for the motion to be rectilinear. This condition, according to O. Reynolds' criterion, is amply satisfied in the following experiments.

Sources of Error.

The method is simple, but there are several sources of possible error the effects of which may become very large when very small differences of level are to be measured.—(1) *Surface tension.* A sensible curvature of the surface of the liquid between the vessel and the float would be a source of pressure (positive or negative) on the liquid, and the magnitude of this could not be easily ascertained. Also the float would probably be drawn to one side and would not rise or fall vertically. Both sources of error were eliminated by choosing large vessels 10 cm. in radius and floats 4 cm. in radius, so that the surface-film between vessel and float was about 6 cm. wide. The vessels first tried were much smaller, and the error was considerable, the results showing no consistency.—(2) *Change of pressure of lever on float.* If the centre of gravity of the lever be not in the plane of the ends of the legs, any change of tilt of the lever will cause a redistribution of pressure of the lever on the

float and the fixed support. Hence the lever must be very light, or its centre of gravity must be in the plane of support. Both precautions were taken. Each lever, including mirror, weighed about 0·9 gm., and it was made of such a shape that the centre of gravity was in the plane of support. The latter adjustment was made by hanging the lever up by two legs on an inclined piece of paper, and adjusting the mirror until the end of the third leg was in the vertical plane through the other two, as tested by the vertical cross-hair of a reading-telescope. The length of each lever was about 1 cm., and a calculation showed that the residual error in the determination of the difference of level in the two vessels could not exceed 1/50 per cent.—(3) *Surface rigidity of impure surfaces.* The quasi-rigidity of an impure surface (sometimes called sur-face viscosity) would prevent free movement of the float, and cause a tilt that might greatly affect the readings. With clean vessels and distilled water it was found sufficient to avoid taking readings when the water had been more than 24 hours at rest, or, better still, to remove the float and thoroughly stir the surface before beginning readings, flow between the vessels being meanwhile prevented by a cap placed on one end of the tube. (I expect to record some observations on surface rigidity in another article.) (4) *Inequalities of the bore of the tube.* The tube with which the measurements were made was carefully selected from a large number that were roughly calibrated, and was 139·5 cm. long and had a mean radius of 0·1675 cm. Calibrated by a column of mercury in 18 steps from end to end, the radius showed a maximum deviation from the mean of 1·4 per cent., and a mean deviation from the mean of 0·7 per cent. The difference between the greatest and least diameters of the end sections did not exceed 1 per cent. It is very improbable that these inequalities were sufficient to produce errors equal to those of observation.—(5) *Correction for the ends of a capillary tube.* If the usual (empirical) correction for the ends were applied, namely, three times the diameter to be added to the length of the tube *, it would diminish the values obtained for the viscosity by less than 1 per cent. ; but, since the velocity of flow in the following experiments was exceedingly small compared with those in which the correction has been found necessary, and therefore the continuation of the stream-lines in the direction of the tube much less marked, it would seem probable that the necessary correction is much smaller. The correction for the kinetic energy of the outflowing liquid * is wholly negligible.

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* Thorpe and Rodger, Phil. Trans. vol. clxxxv. (1894).

Results of First Experiments.

Two liquids were tried, a heavy kerosene and distilled water. As regards the earlier experiments, it will be sufficient to state the results without entering into the details of the measurements. The maximum rate of shear in these experiments was about 5 radians per second for water and 0·2 for kerosene. The following results were obtained for kerosene, the first figure being in each case the viscosity at the low rate of shear, while the figure following it in brackets is the viscosity at the same temperature as obtained by the ordinary capillary-tube method at a maximum rate of shear of about 20,000 radians per second :—0·0232 (0·0235), 0·0243 (0·0246), 0·0239 (0·0239), 0·0244 (0·0244). In these experiments the method of a siphon for raising the level in one of the vessels together with the use of two optical levers was employed. The constants of the levers were obtained by supporting the movable leg on a micrometer screw-gauge. Using exactly the same method and instrumental constants, the results for distilled water were * :—0·0114 (0·0105), 0·0116 (0·0105), 0·0115 (0·0104). In each case the water had been left at rest in both tube and vessels from 20 to 24 hours after it was introduced in order to insure the level being the same in both vessels.

As these results seemed to indicate that the viscosity of water is 9 or 10 per cent. greater at the lower rates of shear, it was considered advisable to vary the experiment somewhat. For this purpose, the method of initially raising the level of the liquid in one of the vessels by placing in it a body of known volume (a steel cylinder) was used, and the constant of the optical lever was determined by running a known volume of liquid from a pipette into the vessel on which the optical lever was mounted, the capillary tube being meanwhile closed by a rubber cap ; also the surface of the water in each vessel was stirred up before readings were begun in order to destroy any surface-rigidity. On each occasion several determinations of the viscosity were made, the first after the liquid had been at rest in the tube about 24 hours, and the succeeding ones in each set as quickly as the water could be brought back to the same level in the two vessels. The following are the percentages by which the viscosity of water at slow shear exceeded the ordinary value at high shear, the successive sets of determinations being separated by dashes :— (1) 12, 11, 10, 10, 10—(2) 10, 9, 5, 4—(3) 13, 9, 8, 7, 6. The same method was applied to kerosene, and as before its viscosity was found the same at high and at low rates of shear.

* The numbers in brackets were found by graphical interpolation from Thorpe and Rodger's results, Phil. Trans. vol. clxxvii. (1886).

These results seemed to show that the difference between the viscosity of water at low rates of shear and that at high rates was greater the longer the liquid had been at rest. In each determination, the total amount of flow was so small that only a small part of the water was expelled from the tube. The only possible explanation seemed to be that the water dissolved from the tube a very small amount of some constituent that had an extraordinarily large effect on the viscosity at low rates of shear. The following experiments were decisive.

Final Results.

The first of the following sets of experiments was made with a view to measuring the viscosity of water as soon as possible after the water had been allowed to enter the tube, and then at increasing intervals of time afterwards. Everything having been made ready for reading, a bubble of air was first blown through the tube to completely remove the water in it. As soon as the reading of the level became stationary, the body of known volume was quickly lowered beneath the surface of the water in one vessel, and readings of the level of the water in the other vessel were made every minute for five minutes. After the intervals noted in the first column of the Table the process was repeated without the water being blown out of the tube.

Section of first vessel at level of water = 317·6 cm.^{2} = A_1.
Section of second vessel at level of water = 321·7 cm.^{2} = A_2.
Volume of steel cylinder .... = 28·84 c.c.
Rise of level produced by steel cylinder = ·09081 cm. = h_0.
Length of tube .... = 139·5 cm. = l.
Radius of tube .... = ·1675 cm. = r.
Time of flow .... = 300 sec. = $t_1 - t_0$.
Volume of calibrating pipette .... = 8·547 c.c.

Time liquid was in tube.	Temp.	Calibration reading.	Change of scale in 5 min.	Value of scale-division.	h_1.	Viscosity (low shear).	Viscosity (high shear).	Per cent. difference.
22 m	6·47	15·84	6·51	·001677	·06881	·01469	·01450	1·3
37 m	6·49	,,	6·51	,,	·06881	·01469	·01449	1·4
2 h 48 m	6·75	,,	6·48	,,	·06892	·01477	·01437	2·8
3 h 3 m	6·76	,,	6·47	,,	·06895	·01480	·01437	3·0
3 h 21 m	6·82	,,	6·47	,,	·06895	·01480	·01435	3·1
20 h	6·70	15·72	6·38	·01691	·06909	·01491	·01440	3·5
,,	6·72	,,	6·37	,,	·06913	·01494	·01439	3·8
,,	6·76	,,	6·40	,,	·06902	·01486	·01437	3·4
40 h	6·42	,,	6·33	,,	·06926	·01504	01451	3·7
,,	6·45	,,	6·32	,,	·06930	·01508	·01450	4·0

The tube was then removed and very lightly silvered inside by means of a silvering solution. It was then replaced, and the apparatus filled with water from the same source as previously and allowed to stand 44 hours, at the end of which time the following results were obtained :—

Time liquid was in tube.	Temp.	Cali-bration reading.	Change of scale in 5 min.	Value of scale-division.	h_1.	Viscosity (low shear).	Viscosity (high shear).	Per cent. dif-ference.
44 h.	12·17°	15·83	7·62	·001679	·06506	·01222	·01226	− ·3
44½ h.	12·20	,,	7·60	,,	·06512	·01226	·01225	0
45 h.	12·32	,,	7·69	,,	·06486	·01211	·01221	− ·8
46 h.	12·53	,,	7·61	,,	·06509	·01224	·01220	+ ·3

Thus the difference disappears when a silvered tube is used. The silvering was too slight to affect the diameter of the tube in any appreciable proportion, and besides, an appreciable decrease of diameter would have caused an *increase* in the result for the coefficient of viscosity while the effect was a decrease. The results stated seem to show conclusively that small quantities of some constituent of glass, probably silicates of the alkali metals, dissolved out by water, have a very great effect on the viscosity of the water at very low rates of shear. The maximum rate of solution of glass (of glass vessels) in distilled water at 18° was found by Kohlrausch * to be 0·48 mg. per square dcm. per day, the rate for most forms of glass being much smaller than this. At this maximum rate the amount dissolved in the water in the tube used in these experiments would produce a concentration of 0·000007 gm. per c.c., and this (or probably a much smaller amount) increased the viscosity at low rates of shear by 3 per cent. The large effect produced may be due to the tendency of silicic acid to separate out from the solution in a gelatinous form. The result was of course not due (at least directly) to impurity existing in the water before it entered the tube : for this would leave the effect of the silvering wholly unexplained. The distilled water used was that prepared for ordinary laboratory use by distillation in a block-tin still. The glass tube was of soft German sodium-glass.

Another tube gave similar high and somewhat variable results for the viscosity of water and the ordinary value for that of kerosene. Silvering the tube was not tried as it was accidentally broken. The results obtained by these two tubes

* *Ber. der deutsch. Chem. Ges.* vol. xxiv p. 3561.

seemed so conclusive that it was not thought necessary to try a third.

A test of Poiseuille's law for liquids in a constant physical state being the chief object of the present inquiry, the effect of impurity at very low rates of shear may be left for closer study at a future date.

Conclusions.

1. Poiseuille's law holds true to within 1 per cent. for kerosene and pure water between rates of shear of 5 and 500,000 radians per second in the case of water, and between 0·2 and 20,000 radians per second in the case of kerosene ; or the coefficients of viscosity of those liquids are constant within these limits.

2. Very minute quantities of the constituents of glass dissolved out by water have very large effects on the viscosity of water at low rates of shear.

3. Differences of rates of shear (at least within the above limits) do not seem to account for the lack of perfect agreement between different methods of measuring viscosity and between different methods of testing the effect of electrostatic stress on viscosity; water and kerosene (within the above limits) do not afford any support for the view that at low rates of shear liquids may approximate in condition to plastic solids.

LXIV. *Short Spark-Discharges.* By CARL KINSLEY,
Assistant Professor of Physics, University of Chicago [*].

THE relation between spark-potential, distance, and the pressure of the gas used in the discharge-chamber was obtained by Paschen [†] from a long series of experiments. His conclusion has been reaffirmed by Carr [‡] from a most satisfactory series of observations which covered a much wider range. He found that in all the gases examined, at any particular potential-difference used, the product of the gas-pressure and the length of the spark-discharge is a constant. Thomson [§], in developing a theory of the spark-discharge, has found that the above relation can probably be extended to include all gases and "expressed by an equation of the

[†] Paschen, Wied. *Ann.* vol. xxxvii. p. 379 (1889).
[‡] Carr, Proc. Roy. Soc. vol. lxxi. p. 374 (1903).
[§] J. J. Thomson, ' Conduction of Electricity through Gases,' pp. 374 *et seq.*

form $V = K + f\left(\dfrac{d}{\lambda}\right)$, where d is the spark-length and λ is the mean free path, and f a function which is the same for all gases," and " K a constant which may vary from one gas to another."

While the distance is comparatively long, *i. e.* greater than 3×10^{-4} cm., it seems probable that the carrier of the electricity remains the same even though the spark distance is considerably less than the mean free path of the molecules. It follows, however, from Earhart's [*] experiments, as well as from the results obtained by G. M. Hobbs [†], that when the distance is less than 3×10^{-4} cm. there is an abrupt divergence from the previous relation between spark-potential and length. This probably results from a change in the carrier of the electricity. Experiments with coherers [‡] and other loose contacts have shown that when the break in the metallic circuit is small, only a small difference of potential is needed to cause a discharge, followed usually by a coherence between the terminals. The difference of potential necessary to cause coherence was found to be unaffected by a change in the pressure of the gas. If between 0 and 3×10^{-4} cm. there is no other abrupt change in the relation between spark-potential and distance, it is probable that there is no other change in the carrier of the electrical discharge.

Some preliminary experiments made by me during the winter of 1904 to determine the relation between potential-difference and spark-lengths less than 3×10^{-4} cm. were not decisive, and so a special instrument was designed and its construction begun in April. Shaw [§] soon after published a "preliminary report," to which further reference will be made, upon the same subject; but the question seemed of sufficient importance to warrant a prosecution of the work upon which I was engaged.

The apparatus (a horizontal section of which is shown in fig. 1) was so designed as to have the greatest possible rigidity, since measurements of distances as short as $1\mu\mu$, *i. e.* 10^{-7} centim., were desired. The discharge surfaces were iridio-platinum faces soldered on to the brass rods P and N. These were truncated cones carefully ground to fit their

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[*] Earhart, Phil. Mag. [6] vol. i. p. 147 (1901).

[†] An elaborate investigation carried on in the Ryerson Laboratory between 1902 and 1905, but as yet unpublished, in which Hobbs has extended the work of Earhart.

[‡] Kinsley, Phys. Rev. vol. xii. p. 177 (1901) : Eccles, Electrician, vol. xlvii. p. 682 (1901).

[§] Shaw, Proc. Roy. Soc. May 28, 1904.

sleeves—in the one case the insulating glass sleeve G, which was also of a conical form and ground to fit A′, and in the other case the truncated cone D which was ground to

Fig. 1.
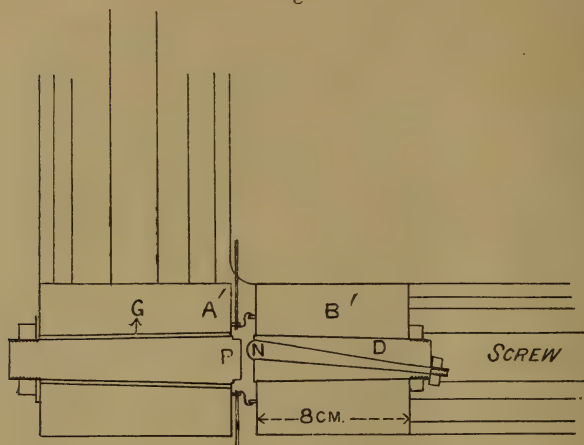
fit B′. It will be noted that without opening the discharge-chamber, the points of contact of both discharge-surfaces could be changed by revolving D and N through small angles in the opposite direction. The thin flexible rubber of the discharge-chamber allowed B′ to be slipped away from A′ while the change was being made.

The method of measurement, as shown by fig. 2, involved the use of a Michelson interferometer with a special arrangement of mirrors to allow the measurement of the excessively short distances. To Chamberlain * is due, however, the first application of multiple reflexions to a Michelson interferometer.

The two mirrors A and B were rigidly fastened to A′ and B′ respectively. The block B′ slides on straight ways and is controlled by a differential-screw of ratio 50 threads per inch to 20 threads per centim. The screw was driven by a worm-gear having a ratio of 100 to 1. By means of the interference-fringes observed in the compound microscope

* During the summer of 1903 Professor C. W. Chamberlain of Dennison University, while working in the Ryerson Laboratory, first devised and used two mirrors making a slight angle with each other, as a part of an interferometer. The investigation, upon a subject totally different from the present one, has not yet been published. This arrangement promises to be a very useful method of attacking many difficult problems.

at C, the relative movement of the discharge-surfaces P and N could at all times be followed.

Fig. 2.
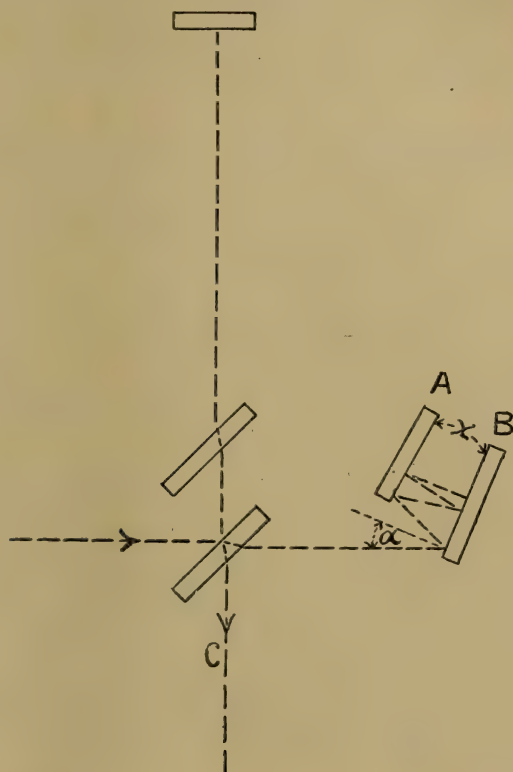
The relation between the actual movement of the carriage in which B is mounted and the increase in the length of the path of the ray is given by the following equation.

Let a = angle of incidence.

x = angle between mirrors A and B.

D_n = actual distance passed over by the ray between the mirrors due to a movement of the mirror B.

n = number of the order, i. e. n gives the number of times the ray is reflected by the mirror B.

d = actual movement of the mirror B along the original path of the ray.

f_n = number of fringes that passed the cross-hair of the microscope when the nth order of reflexions was used.

λ = wave-length of the light used.

a = $d \cos a$.

The average distance between the mirrors A and B was 2·5 mm.

$$D_5 = \frac{4a}{\cos \alpha} + \frac{4a}{\cos (\alpha - 2x)} + \frac{2a}{\cos (\alpha - 4x)}.$$

$$D_{10} = 4a \left\{ \frac{1}{\cos \alpha} + \frac{1}{\cos (\alpha - 2x)} + \frac{1}{\cos (\alpha - 4x)} + \frac{1}{\cos (\alpha - 6x)} + \frac{1}{\cos (\alpha - 8x)} \right\}.$$

The tenth order was used in every case given in this paper except on October 28th, when the fifth order was used.

$$\alpha = 22\tfrac{1}{2}°. \quad \text{For the tenth order } x = 2\tfrac{1}{2}°.$$

Substituting in the above equations

$$d = \frac{D_{10}}{19·09}, \quad D_{10} = f_{10}\lambda.$$

If the green light of mercury is used, $\lambda = 546\mu\mu$.　When sodium light is used $\lambda = 589\mu\mu$.

∴　$d = 28·6 f_{10}$, for mercury light and the tenth order.

$d = 61·4 f_5$, for sodium light and the fifth order.

The spark was obtained by discharging a condenser across the gap separating the surfaces.　The condenser was charged by a set of small storage-cells, and the charge was renewed each four seconds to ensure the constancy of the potential. The leakage, however, was small, as the insulation resistance of the glass sleeve was 16,500 megohms, of the switches and conductors 170,000 megohms, which gave a total insulation resistance of 15,000 megohms.　The diagram of connexions is shown in fig. 3.

The capacity, C, was 1 microfarad.　The voltage used with the high-resistance galvanometer for finding the zero was about $\frac{1}{5000}$ of a volt.　Non-inductive resistances r_1 and r_2 were bridged across the inductive parts of the circuit.　The difference of potential that produced the spark-discharge was measured by the Kelvin electrostatic voltmeter, or by the ballistic throw of the galvanometer upon discharging the condenser.　The use of the single galvanometer was desirable, as it was necessary to note the position of the fringes with reference to the cross-hair of the microscope at both the instant that the circuit was broken in obtaining the zero, and also at the instant the spark-discharge occurred, as shown by the ballistic throw.　This was accomplished by mounting the galvano-

meter-telescope beside the microscope in such a position that the two could be used as a binocular.

In making a test the following method was used :—

(*a*) Before each test unused portions of both discharge-surfaces were brought together. The surfaces were polished on a clean cloth buffing-wheel with soft dry rouge just before closing the discharge-chamber.

Fig. 3.
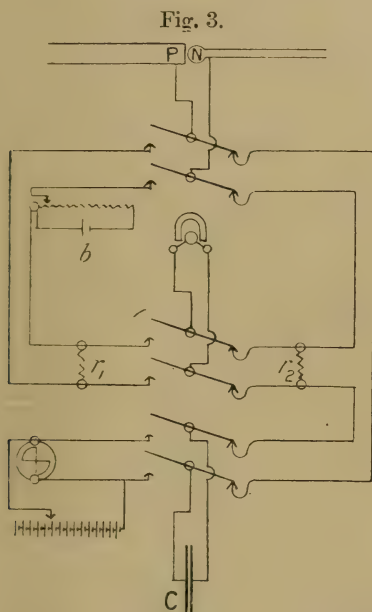
(*b*) The chamber was blown out with dust-free air from the drying-tubes.

(*c*) The zero was obtained by noting the position of the fringes in the microscope when upon separating P and N the galvanometer showed that the testing circuit, shown on the left of fig. 3, was broken. Throughout the test the zero was re-determined each five or ten minutes. If the zero was inconstant, or if a regular change large enough to necessitate a correction was obtained, the testing was discontinued until the zero was found to be sufficiently constant.

(*d*) A considerable time was allowed to elapse between changes of potential, and during this time the condenser was recharged each four seconds. Thus during nine-tenths of the time it stood connected to the discharge-surfaces.

Fig. 4.
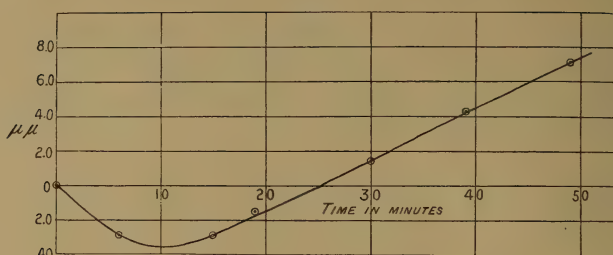
TABLE for fig. 4.

Drift of zero obtained just after resetting the discharge-surfaces.

Time in Minutes.	Zero Position.	
	In Fringes.	In $\mu\mu$.
0	0	0
6	−0·1	−2·86
15	−0·1	−2·86
19	−0·05	−1·43
30	+0·05	+1·43
39	+0·15	+4·29
49	+0·25	+7·15

The accompanying curve (fig. 4) shows a characteristic variation of the zero, *i. e.*, the point at which the surfaces break contact. This may be explained as follows :—

The handling of the instrument in readjusting the discharge-surfaces causes a rise of temperature. The subsequent fall of temperature was shown by a shift of the zero in such a direction as to open the gap. As the effect of handling gradually disappeared, the instrument returned to its normal condition, when the steadily rising curve indicated a rising temperature. It often required six or eight hours of careful testing and adjustment of room-temperatures to find the correct temperature at which the apparatus must be used. As an hour was required to complete each observation, it was found advisable to do most of the work at night, when the temperature could most easily be kept constant during the long period required for a complete series.

The results given below are chosen from a great number in such a way as to illustrate the total range covered by this investigation. The notation employed is as follows :—

f_n = distance in number of fringes of the nth order between the discharge-surfaces when the spark passed.

d = distance in $\mu\mu$ between the discharge-surfaces.

V = difference of potential at which the spark occurred.

Vi = interval by which the potential was changed during the test.

a = time in seconds, after increasing the potential to its final value, before the spark occurred.

wt = empirically assigned weight which depended on the sharpness and drift of the zero as well as on Vi and a.

Fig. 5.
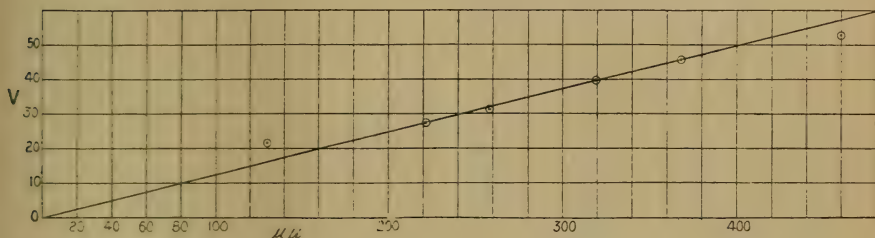
TABLE for fig. 5.

Data taken October 28, 1904.

This series was taken between 24 and 27 hours after closing the discharge-chamber. Thirty seconds was the time-interval used between changes of potential.

During this series the drift of the zero was not obtained.

f_5.	d.	V.	Vi.	a.	wt.
2·1	129	21·6	2	2	6
3·6	221	27·6	2	20	6
4·2	258	31·3	2	15	6
5·2	319	39·6	2	30	6
6·0	368	45·8	2	26	6
7·5	460	52·5	2	30	6

Fig. 6.
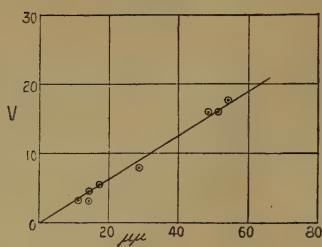
TABLE for fig. 6.

Data taken November 23, 1904.

This series was taken between 19 and 25 hours after closing the discharge-chamber. The time-interval between changes of potential was not constant.

f_{10}.	d.	V.	Vi.	a.	wt.
1·9	54·3	17·8	......	20	6
1·8	51·5	16·0	......	20	8
1·7	48·6	16·0	......	8	8
1·0	28·6	8·0	......	0	10
0·6	17·1	5·6	......	30	10
0·5	14·3	4·6	......	4	6
0·4	11·4	3·1	......	28	10
0·6	17·1	3·1	......	0	4

Fig. 7.
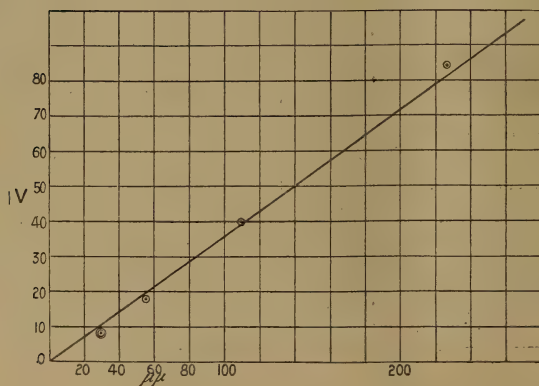
TABLE for fig. 7.—Data taken November 27, 1904.
This series was taken between 2 and 5 hours after closing the discharge-chamber. Forty seconds was used as the time-interval between changes of potential.

f_{10}.	d.	V.	Vi.	a.	wt.
1·9	54·3	17·8	1·0	12	10
1·0	28·6	8·0	0·5	30	10
3·8	109·0	39·9	2	14	10
7·9	226·0	84·0	2	34	10
1·9	54·3	17·8	...	50	8

Fig. 8.
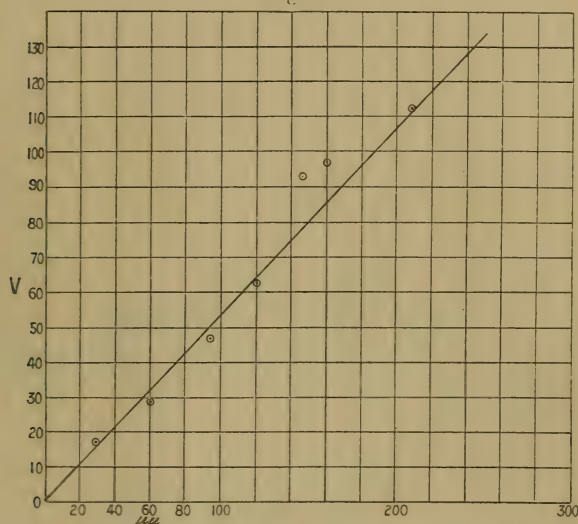
TABLE for fig. 8.—Data taken November 16, 1904.
This series was taken between 3 and 8 hours after closing the discharge-chamber. Sixty seconds was used as the interval between changes of potential*.

f_{10}.	d.	V.	Vi.	a.	wt.
1·0	28·6	17·0	1·0	30	10
2·1	60	28·7	4·0	30	8
3·3	94	46·9	2	35	10
4·2	120	62·5	2	10	10
5·1	146	93·0	2	30	10
5·6	160	97·0	6	40	6
7·3	208	112·5	2	50	10
8·0	228	97—	97	0	0
8·2	234	97—	24	0	0
8·0	228	50—	50	0	0

* The zero drift of the last three points had not been obtained.

Fig. 9.
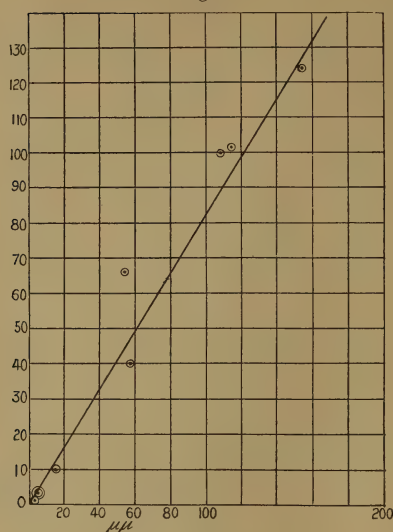
TABLE for fig. 9.

Data taken December 20, 1904.

This series was taken 6 and 17 hours after closing the discharge-chamber. Ten seconds was the interval between changes of potential.

f_{10}.	d.	V.	Vi.	a.	wt.
* 4·1	117	42·0	2·0	4	6
* 8·5	243	136·0	2·0	4	10
2·0	57	40·0	2·0	6	10
0·17	4·9	3·2	0·2	4	10
6·5	186	95·0	95·0	0	0
0·17	4·9	3·0	0·5	...	8
0·10	2·9	1·0	...	...	4
less than 0·1	...	0·2	...	...	...
0·5	14·3	10·0	1·0	4	10
4·1	117	101·8	2·0	6	10
4·0	114	99·8	2·0	0	10
2·0	57	66·0	2·0	8	10
5·4	154	124·0	2·0	4	10
6·0	171	120·0	24·0	0	0

* The first two points are not shown on the plot.

It is to be expected that erratic values would sometimes be obtained. These values would usually be too low for the reason that—

(1) The minute irregularities of the surface might oppose sharp points in spite of the final polishing with soft rouge, and the pressing of the surfaces together before taking the zero reading.

(2) A particle of dust of too high a resistance to appreciably affect the resistance of the galvanometer circuit might carry the high potential discharge.

(3) A particle of dust not at first in contact with both surfaces might be made to bridge the gap by the electrostatic attraction while testing.

(4) Any vibration of the building or even sound-vibrations might decrease the distance momentarily.

It was therefore thought best to occasionally disregard certain values which were much too low to be consistent with the rest of the series. All of the observations are recorded in the data sheets, and except for two values noted in the table for fig. 9, all of the observations not given a weight of zero are shown on the plots. The weight of zero was generally given when the discharge was so premature that the zero drift had not been obtained.

The individual curves show plainly that there is no sudden change in the potential gradient within the range covered by them. In every case, the potential seems to be a linear function of the distance. It is to be noted that the shortest distance actually determined was $3 \mu\mu$, which corresponded to an apparent movement of only one-tenth of a fringe. It was found also that a shorter distance—unmeasured—required but 0.2 of a volt to produce a discharge which, at the rate found by the previous observations of this series, would indicate a spark-distance of only $0.2 \mu\mu$. Such short distances could very readily have been accurately measured by using a larger number of reflexions had it been thought worth while to do so.

After each discharge it was found that the surface had cohered. An example of the coherence will be discussed in an appendix. At each spark-distance, however, it is possible to prevent coherence by using a sufficiently small capacity.

It will be noted that there is a striking difference between the different series of observations, as shown by fig. 10 (p. 707).

The potential gradient was found to vary widely from day to day, although on each day a linear relation between the potential and distance of spark-discharge was sufficiently evident.

It was finally found that the potential gradient could be always raised by thoroughly washing the surfaces with alcohol before buffing them on a soft cotton wheel with dry

soft rouge. The potential gradient, which was thus made high, would gradually fall.

Fig. 10.
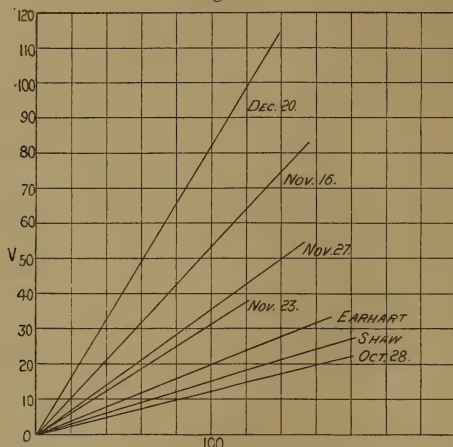
In general it was also found that the potential gradient shown after rubbing with dry rouge on the buffing-wheel would fall whether the initial treatment was such as to give at first a high value to the potential gradient or not. A particular case is given as an illustration in Table A.

TABLE A.

Data taken December 1, 1904. —Forty seconds was in most cases the time-interval between changes of potential.

$f_{10}.$	V.	Vi.	$a.$	V. $d=100\,\mu\mu.$	Hours after closing chamber.	$wt.$
4·1	107·0	2	22	91·5	2	10
3·9	74·0	2	4	66·5	3	5
3·9	87·5	2	20	78·5	4	10
3·9	95·5	2	22	85·5	5·5	10
4·0	85·5	2	46	75·0	6	10
4·2	111·0	2	36	92·5	9	10
			Average...	84	5	
3·9	77·9	2	...	70·0	19·0	10
3·9	75·9	2	12	68·0	20·6	10
4·1	83·5	2	4	71·0	21·2	10
3·8	79·5	2	22	73·0	24·0	10
3·8	73·9	2	16	68·0	24·4	7
4·0	70	...	...	...	...	0
			Average...	70	22	

The rate at which the potential gradient falls depends somewhat on the drying substances used. Calcium chloride gave the slowest fall, and was much better used alone than when used with either phosphorus pentoxide or sulphuric acid, or with both together. In the results already given an elaborate system of drying-tubes and large reservoirs partly filled with fresh anhydrous calcium chloride was alone used.

The drying-tubes and discharge-chamber were at one time filled with illuminating-gas in order to vary the experimental conditions. Table B gives the results obtained. It is seen that the potential gradient fell, but no more than might have been expected had air alone been used.

TABLE B.

Data taken December 17, 1904.—Ten seconds was the time-interval used between changes of potential.

Air was used in the discharge-chamber.

$f_{10}.$	V.	Vi.	a.	V. $d = 100\,\mu\mu.$	Hours after closing chamber.	$wt.$
4·0	40·2	2	0	35·0	1·5	10
3·9	38·4	2	10	34·5	3·0	10
4·6	50·0	2	0	38·0	3·5	10
			Average...		3	

Illuminating-gas was used in the discharge-chamber.

$f_{10}.$	V.	Vi.	a.	V. $d = 100\,\mu\mu.$	Hours after closing chamber.	$wt.$
4·1	31·3	2·0	8	27·0	14·0	10
4·3	35·1	2·0	6	29·0	16·0	10
			Average...	28	15	

In order to vary the condition at the point of discharge, the surfaces which had been polished in the same way each time were now polished by a jeweller. They were thoroughly cleaned with caustic potash, nitric acid, water, and finally washed in alcohol, but not rubbed on the buffing-wheel again. The tests showed a medium value for the potential gradient. No important change in the action of the surfaces was observed.

Considering all of the observations taken, it cannot be said that the potential gradient approached any constant value. It could be brought to any value within the wide range of $12\frac{1}{2}$ volts per $100\,\mu\mu$ to 83 volts per $100\,\mu\mu$. This range

includes the values obtained by Earhart and by Shaw, as shown in fig. 9. There was no indication that values both higher and lower could not also have been obtained.

The constancy of the potential gradient throughout every independent series of observations is good evidence that the carrier of the electrical discharge is unchanged between $450\,\mu\mu$ and $3\,\mu\mu$.

Since the relation between the spark-potential and spark-distance gives a straight line passing through zero, it is evident that throughout the range examined the discharge occurs along the shortest path between the two surfaces.

The sudden change of potential gradient depending on the treatment of the surface, and the slow change while isolated in the discharge-chamber, seems to show that the carriers of the electrical discharge come neither from the gas in the chamber nor the metal of the electrodes.

The lag in the discharge, as shown in the tables, is the same phenomenon so well known in the case of long spark-discharges. It points to the same mechanism of discharge for short as well as for long sparks.

Probably a determination of the cause of the change in the potential gradient will settle the question of the real carrier of the electrical discharge in the case of short sparks.

I am pleased to acknowledge that I have been greatly indebted to Professor A. A. Michelson for his continued interest and many suggestions throughout the course of this investigation.

Ryerson Laboratory, University of Chicago,
December 23, 1904.

Appendix *to paper on* "*Short Spark-Discharges.*"

Note on the Coherer.

In the preceding paper it was found that the discharge caused the surfaces to cohere. Although it has already been convincingly shown that the coherence is caused by the formation of a metallic bridge* between the surfaces, the following evidence on that point may be of interest.

When two clean metallic surfaces are pressed together they will form a low-resistance metallic contact. If now they are slowly separated, the resistance of the contact will rapidly increase. Under normal conditions the resistance will increase

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* For a *résumé* see Gute, Proc. International Electrical Congress, St. Louis, 1904.

from 0·1 ohm to ∞ at complete rupture within the distance of 30 $\mu\mu$. The point of rupture can be determined within 1 $\mu\mu$ under favourable conditions.

If a discharge is passed between the surfaces which have been separated, to a distance of say 100 $\mu\mu$, they will cohere. If now the surfaces are slowly separated the resistance, which had become very low, will gradually increase. They can with care be separated to a great distance before rupture will again occur. The accompanying curve is quite characteristic.
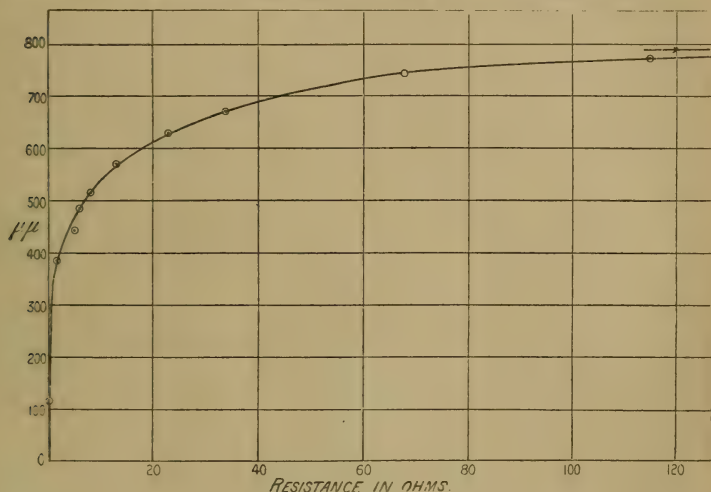
TABLE.

Data taken December 14, 1904.—After the cohering following a discharge, the surfaces were separated and the resistance of the coherence determined.

F_{10}.	Galvanometer reading.	d.	Resistance.
Open	33·5	...	∞
Short circuit	23·5	...	0
4·1	23·5	117	0·15
13·5	23·6	386	1·5
15·5	23·8	443	5·0
17·0	23·9	486	6·0
18·0	24·0	515	8·0
20·0	24·3	572	13·0
22·0	24·9	630	23·0
23·5	25·4	672	34·0
26·0	26·6	743	68·0
27·0	27·8	772	115·0
27·5	33·5	787	∞ Break

If the original separation of the surfaces is less than $117 \mu\mu$, as in the accompanying illustration, then the final distance obtained before rupture will be less than $800 \mu\mu$.

For values of d between 0 and $730 \mu\mu$ the curve given follows very exactly the equation

$$R = 4 \cdot 5 \times 10^{-16} \, d^6,$$

R = resistance in ohms,

$d =$ distance between surfaces in $\mu\mu$.

It will, therefore, hardly do to assume that the conducting bridge has a uniform cross-section throughout its length. If, however, we consider it uniform when d equals $400 \mu\mu$ and R equals 2 ohms, *i. e.* before the extension has greatly increased the resistance, then, assuming the material is pure platinum, the diameter of the bridge is found to be $4 \cdot 4 \times 10^{-5}$ cm. This distance is considerably shorter than the mercury-light waves used in this investigation, and consequently we need not be surprised at our failure to observe by microscopic examination any evidence of cohesion.

The behaviour of the coherence, however, can very readily be explained upon the assumption of the formation of a very small metallic bridge which ruptures at once upon receiving any sudden jar, causing a lateral movement, but which may be stretched until it rapidly forms a neck and breaks.

The process of the formation of the bridge is still unknown. It is probable that an explanation of the production of short spark-discharges will also make clear the coherer action.   C. K.

LXV. *The Properties of Radium in Minute Quantities. By* A. S. Eve, *McGill University, Montreal* *.

IN a recent communication to the *Physikalische Zeitschrift* (Dec. 1, 1904), A. Voller has published some results which appear to be diametrically opposed to the theory of radioactivity, a theory which has already received a remarkable amount of experimental confirmation.

Voller found that radium, distributed on a plate in minute quantities and exposed to the air, disappeared or decayed in a few days or weeks; whereas other investigations lead to the conclusion that radium, whether in larger or smaller quantities,

has a life of more than a thousand years. He also found that the activity of radium thinly distributed over a few square centimetres was not proportional to the mass present, but had an excess of activity very markedly greater, proportionally, in the case of small quantities. Other observers have found that in the case of the radioactive substances, and their products, the activity of a given mass is independent of the degree of concentration.

Some test experiments, made by the present writer, do not give results in agreement with those obtained by Voller, but they are in agreement with the generally accepted theory of radioactivity within the probable limits of experimental error.

The activity as measured by Voller of a small mass of radium, first dissolved in water, and then evaporated over a small area (1·2 cm.), are given in the following table :—

Amount of Radium in milligrams.	Observed Activity.
10^{-3}	38·4
10^{-4}	11·6
10^{-5}	6·2
10^{-6}	6·0
10^{-7}	3·6
10^{-8}	2·6
10^{-9}	1·6

It will be seen that on increasing the amount one million-fold the activity was increased, not one million times, but only twenty-four times.

So also for the duration of the effect, the quantities did not last for several hundred years, but disappeared in a few days, thus :—

Amount of Radium in milligrams.	Days to complete disappearance.
10^{-4}	126
10^{-5}	61
10^{-6}	26
10^{-7}	17
10^{-8}	16
10^{-9}	15

Whatever may be the true explanation of the phenomena observed by Voller, the experimental method adopted is very unsuitable as a test of the two main points under investigation.

A plate, covered with a small quantity of radium, was placed at a distance of 10 cms. from one extremity of a hollow metal cylinder, along the axis of which was an insulated rod connected with a gold-leaf system. There would thus be considerable absorption of the α rays by the air traversed, the capacity of the system would be large, and the ionization effect at the far end of the cylinder would be uncertain in character.

In the present experiments 1 c.c. of water, containing 10^{-6} milligrams of radium, was slowly evaporated over the bottom of a silvered glass flask so as to cover an area of 76 sq. cms. An ebonite cork was then carefully waxed to the mouth of the flask so that it was hermetically sealed. The ebonite supported a wire, sulphur bead, and gold-leaf system. This electroscope was charged to 300 volts when a magnet was brought near, so as to cause a small piece of magnetized watch-spring to bridge the sulphur bead insulation. The copper wire, as well as the silver coating of the electroscope, were then connected to earth. The natural leak of the electroscope had previously been determined with care for several days, and was not more than 5 per cent. of the effect under measurement.

Voller found that 10^{-6} milligrams, distributed over 1·2 sq. cm., completely disappeared in 26 days. In my experiment more than 40 days have elapsed since the maximum was reached, and I have found no trace of disappearance, in the case of 10^{-6} milligrams spread over 76 sq. cms.; and an effect of 2 or 3 per cent. could have been detected with certainty.

Experiments were also made to measure the activity produced by various small quantities of radium. Professor Rutherford kindly gave me some solutions of radium, and the required amounts were distributed by evaporation over small clean zinc or platinum dishes 4·9 sq. cms. in area. These were then placed in the bottom of a gold-leaf electroscope having a small natural leak.

Attempts to measure 10^{-3} and 10^{-8} milligrams were not successful, as the effects on the electroscope were respectively too rapid and too slow, a result which might be expected from theoretical considerations, if the activity is proportional to the mass. In all cases the minimum values were observed, when the emanation, if present, had been driven off by evaporation, and the excited activity, if any, had time to decay; that is, two or three hours after evaporation.

A comparison of the results obtained by Voller and by

the writer is given in the following table, in which the observed activity for 10^{-4} milligrams in each case is raised to 100.

Quantity of Radium in milligrams.	Activity (Voller).	Activity (Eve).
10^{-4}	100	100
10^{-5}	53·5	10·6
10^{-6}	51·7	1·18
10^{-7}	31	·125

It will be seen that when the quantity of radium is increased one-thousandfold, Voller finds the activity increased between three and four times, whilst in the present experiment the activity is increased 800 times.

It appears desirable to remove the impression that the activity of radium in minute quantities decays in a few days. The occurrence of radium in pitchblende refutes such a suggestion.

It is also desirable to point out that the activity is at least approximately, and is probably accurately, proportional to the mass present, and that the properties of a given quantity of radioactive matter are independent of the degree of concentration.

It must be borne in mind, however, that Voller's experiments were carried out on plates exposed to the air, and the writer's experiments were entirely made in a closed vessel.

No disappearance can be detected in a closed vessel, and therefore Voller's effects cannot be due to the decay of the activity of the radium; but they may indicate a dissipation of the radium itself from the plate.

Note by Professor E. RUTHERFORD.

The results given by Mr. Eve in this paper are supported by experiments which I have made on the radioactivity of weak radium solutions. A year ago I made up a standard solution of radium bromide containing 10^{-6} milligrams per c.c., which was kept in a stoppered bottle. The quantity of radium in a cubic centimetre of the solution, measured by the amount of emanation produced by it, has not sensibly altered in the course of a year.

A large number of experiments have been made by different observers on the rate of decay of the radioactive products, but in no case has the constant of decay of activity been

changed by altering the degree of concentration of the product. For example, the constant of decay of the radium emanation has been tested over a range of at least one million times, but no alteration of its value has been detected.

The experiments of Eve undoubtedly show that the activity of the radium bromide, *kept in a closed vessel*, after reaching its maximum, does not exhibit the decrease observed by Voller for a deposit of radium exposed to the open air. This shows that the decrease of activity, noticed by Voller, has nothing to do with the actual change in the rate of disintegration of radium, but is a result of the exposure to the open air. Not the slightest evidence has so far been obtained to indicate that the rate of disintegration of radium is in any way altered by the amount of radium present. In this respect its behaviour is analogous to the other numerous radioactive products.

On the other hand, there is some evidence that small quantities of radium, deposited on a plate by evaporation of a radium solution, apparently lose their activity fairly rapidly when left in the open air. I have observed that the activity of a plate 50 sq. cms. in area over which half a milligram of radium was deposited, lost more than half of its activity in the course of a year.

This disappearance of the activity has no connexion with the actual life of the radium itself, but is probably due to the escape of the radium from the plate into the surrounding gas. A quantity of 10^{-9} grams of radium bromide, spread over an area of 10 sq. cms., is far too small to form a layer of even molecular thickness. During the process of evaporation, the radium bromide would tend to collect together and form small crystals which are deposited irregularly over the surface of the plate. These particles would not be held firmly to the plate, and would gradually escape from it.

Such an effect would probably be also shown by inactive matter, if deposited in such minute amount. This action, however, can only be experimentally tested with a substance like radium, where the activity can be used as a means of detecting the change in an excessively minute quantity of deposited matter.

McGill University, Montreal.
    March 28, 1905.

LXVI. *On Direct Reading Resistance-Thermometers, with a Note on Composite Thermocouples. By* ALBERT CAMPBELL, *B.A.*[*]

THE measurement of high temperatures is a matter of growing importance both in scientific and technical work, and thanks mainly to Prof. Callendar's work on the subject, platinum resistance-thermometers at present hold the foremost place for accuracy in such measurements. In their most usual form, however, resistance-thermometers do not indicate the temperature directly, but it has to be deduced from the reading by means of a formula, table, or curve. In order to get rid of this trouble, I have recently invented two simple arrangements, by either of which the reading of a resistance-box, in connexion with a resistance-thermometer, gives directly the actual temperature to a good degree of accuracy.

I shall designate the two methods :—

> (A) The Rectifying Shunt Method,
>
> and (B) The Rectifying Loop Method.

I have named them thus, because the result of using the device is to reduce to a straight line the curve connecting the box-reading with the temperature. I shall first describe (A), which is the simpler but the less accurate of the two.

(A) *The Rectifying Shunt Method.*

In this method the arrangement is extremely simple, and consists merely of the addition of an appropriate shunt to one of the arms of the Wheatstone's bridge used to measure the resistance. Before I show how this shunt is applied, let us consider the connexion between resistance and temperature. For platinum, the formulæ ordinarily used are equivalent to the following

$$\frac{R_t}{R_0} = 1 + \alpha t - \beta t^2,$$

where R_0 and R_t are the resistances at $0°$ C. and t respectively, and α and β are the constants. As to actual values, α and β vary slightly in different specimens, α being of the order of $0\cdot003$, while β is near $0\cdot0000005$. This parabolic formula has been found to hold with considerable accuracy over a wide range of temperature (up to $1000°$ C.) [†]. For simplicity we may for the present make R_0 equal to 1 ohm.

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[*] Communicated by the Physical Society : read March 10, 1905.

[†] See, for example, J. A. Harker, Phil. Trans. vol. cciii. p. 343.

Then
$$R_t = 1 + \alpha t - \beta t^2.$$

Fig. 1.
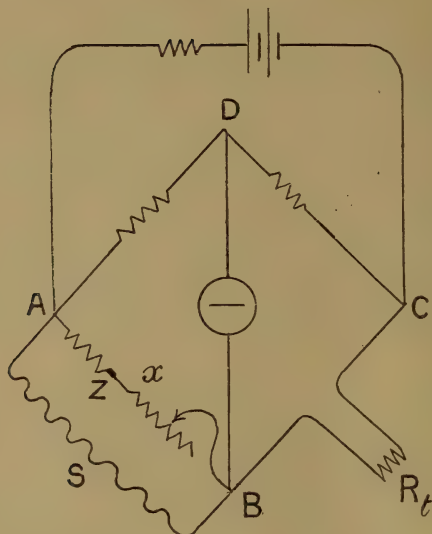
Now in fig. 1 let the Wheatstone's bridge shown have, at 0° C., four equal arms, of which BC is the platinum thermometer, and AB consists of a variable resistance x shunted by a fixed resistance s.

If the temperature of R_t is t, and x is altered until a balance is obtained, then

$$\frac{sx}{s+x} = R_t = 1 + \alpha t - \beta t^2.$$

Therefore

$$x = \frac{sR_t}{s - R_t} = \frac{s(1 + \alpha t - \beta t^2)}{s - 1 - \alpha t + \beta t^2}$$
$$= \frac{s}{s-1} \left[(1 + \alpha t - \beta t^2) \Big/ \left(1 - \frac{\alpha t}{s-1} + \frac{\beta t^2}{s-1} \right) \right].$$

Let $\dfrac{1}{s-1} = q.$

Then
$$x = sq \cdot \frac{1 + \alpha t - \beta t^2}{1 - \alpha q t + \beta q t^2}$$
$$= sq [1 + (1+q)\alpha t + (1+q)(q\alpha^2 - \beta)t^2$$
$$+ \alpha q (1+q)(\alpha^2 q - 2\beta)t^3 + \ldots] \quad . \quad (1)$$

Since α^2 and β are small let us neglect for the present the terms beyond t^2.

If we can now make the term in t^2 vanish, x will reduce to the form

$$x_0(1+mt).$$

The condition for this is

$$(1+q)\,(q\alpha^2-\beta)=0.$$

The factor $(1+q)$, being equal to $\dfrac{s}{s-1}$, cannot vanish since s is not zero, and thus the requisite condition is

$$q\alpha^2-\beta=0.$$

Hence

$$\frac{1}{s-1}=q=\frac{\beta}{\alpha^2};$$

therefore

$$s=1+\frac{\alpha^2}{\beta}. \qquad \cdots \cdots \quad (2)$$

Since α and β are both positive for platinum, $\left(1+\dfrac{\alpha^2}{\beta}\right)$ is positive and therefore s is a real resistance. When s is made to have this value, then

$$x=x_0(1+mt)$$

where x_0 is the reading when $t=0^\circ$ C.

We have

$$x_0=\frac{s}{s-1}, \qquad \cdots \cdots \quad (3)$$

$$m=\alpha(1+q)=\frac{\alpha s}{s-1}, \qquad \cdots \cdots \quad (4)$$

and

$$t=\frac{x-x_0}{m}.$$

Thus t is directly proportional to the increase of x above the zero-reading x_0.

Clearly this proportionality still holds when the bridge-arms are not all equal, so long as x_0 and s are kept in the proper ratio. The compensating leads, being in the measuring arm, must have resistance in proportion to it. In practice, it is desirable to be able to use for the variable part ZB an ordinary resistance-box with sets of 10-ohm, 1-ohm, and 0·1-ohm coils (with $\frac{1}{100}$ths also, if desired for the highest accuracy); and, in order to do this with a platinum thermometer of any resistance, the arm DC (fig. 1) is made to have the

same resistance (at 0° C. say) as the thermometer, whilst the arm AD is made adjustable within a small range so that it can be made equal to AB at the zero-reading. Then if s, m, and x_0 be the values found from α and β as shown above,

AZ is set to be $\dfrac{x_0}{10m}$, and the shunt is made $\dfrac{s}{10m}$. When

these adjustments are right to start with, then each $\frac{1}{10}$ ohm between Z and B represents 1° C.

It will be remembered that in the above investigation we neglected the powers of t above t^2 [equation (1)]. The result of this is that the values of s, x_0, and m calculated as above, give readings slightly in error, particularly at the higher temperatures. It is not difficult, however, by one or two trials to alter the value of s (and hence x_0 and m) so as to give readings whose accuracy is within two or three degrees, from 0° to 1000° C.

I have described this method mainly for two reasons: firstly, that it led up to the discovery of the more exact way (B), and secondly, because it is of some theoretical interest to observe that the resistance of a platinum wire at varying temperature can be very nearly represented by two resistances in parallel, one of them having a constant temperature coefficient and the other being invariable with temperature. Perhaps the first part represents pure metal, while the second (which is about $\frac{1}{30}$ of the first) may be an alloy produced by the impurities present.

If the resistance-thermometer be of iron (at moderate temperatures) or any other metal for which

$$R_t = R_0(1 + \alpha t + \beta t^2),$$

the temperature-reading curve can usually be brought very nearly to a straight line by shunting the thermometer-arm itself by a suitable invariable resistance. Neglecting the terms above t^2 as in the preceding investigation, the requisite value

of the shunt would be $R_0\left(\dfrac{\alpha^2}{\beta} - 1\right)$. This value is not

necessarily always positive, and thus it may happen that the result is not attainable (—it *is* possible for *iron*). As in the case of metals like platinum, a better result can be got by

trying various values of the shunt near to $R_t\left(\dfrac{\alpha^2}{\beta} - 1\right)$ in this case.

By introducing a little more complication, as in fig. 2, a nearer approximation to linear variation with temperature can be secured.

Here the main resistance R is shunted by $s_1 + s_2$, of which s_1 is invariable and kept cold, while a small part s_2 is of the

Fig. 2.
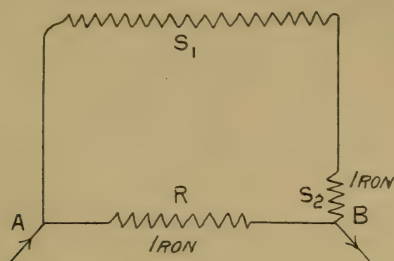
same metal as R and at the same temperature. By suitable choice of s_1 and s_2 the resistance from A to B can be made very nearly of the form $R_0(1 + mt)$.

To take an actual example of Method (A),—a certain platinum resistance-thermometer follows approximately the formula

$$\frac{R_t}{R_0} = 1 + 0.003946t - 0.0000005798t^2. \quad . \quad . \quad (5)$$

Fig. 3 shows the values of the resistances necessary to give direct readings for this instrument.

Fig. 3.

It will be seen that the shunt is here equal to $25x_0$, while the approximate formula gives $27.86x_0$.

I need not discuss the first method further, but will pass now to the second, which is much more exact.

(B) *The Rectifying Loop Method.*

The arrangement of the measuring arm in this method is shown in fig. 4, in which a contact F can be moved along a loop of fixed resistance (n). (The resistance of the contact is assumed small and constant.)

Fig. 4.
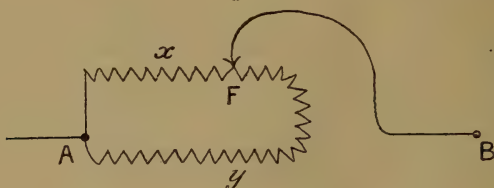
As before, let us assume the parabolic formula

$$R_t = R_0(1 + \alpha t - \beta t^2),$$

and let us choose $R_0 = 1$, making

$$R_t = 1 + \alpha t - \beta t^2.$$

In fig. 4 let $x + y = \text{constant} = n$, and let resistance AB be proportional to R_t when a balance (in the bridge) is obtained. Then

$$kR_t = \frac{xy}{x+y} = \frac{x(n-x)}{n}$$

$$= x - \frac{x^2}{n}.$$

Now suppose that x is altered in proportion to the rise of temperature of R,

i. e. $\qquad x = x_0(1 + mt).$

Let $x_0 = 1$, and therefore $x = 1 + mt$. Then

$$k(1 + \alpha t - \beta t^2) = kR_t = x - \frac{x^2}{n} = 1 + mt - \frac{(1 + mt)^2}{n};$$

or

$$k(1 + \alpha t - \beta t^2) = \left(1 - \frac{1}{n}\right) + m\left(1 - \frac{2}{n}\right)t - \frac{m^2}{n}t^2.$$

If this is true for all values of t, we have

$$k = 1 - \frac{1}{n} = \frac{n-1}{n}; \quad \cdots \quad (6)$$

$$\alpha = m\left(\frac{n-2}{n-1}\right); \cdots \quad (7)$$

$$\beta = \frac{m^2}{n-1}. \quad \cdots \quad (8)$$

Hence

$$\alpha = \frac{\beta(n-2)}{m}.$$

Therefore

$$n = \frac{\alpha m + 2\beta}{\beta} = \frac{\alpha m}{\beta} + 2.$$

Hence from (7) $m^2 - \alpha m - \beta = 0.$

Therefore

$$m = \frac{\alpha}{2}\left(1 \pm \sqrt{1 + \frac{4\beta}{\alpha^2}}\right). \quad . \quad . \quad . \quad (9)$$

Also

$$n = \frac{\alpha m}{\beta} + 2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$k = \frac{\alpha m + \beta}{\alpha m + 2\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Since α and β are both positive (for metals like platinum), real values for m, n, and k can thus always be found by equations (9), (10), and (11). As no terms in t^3 and higher powers have been neglected, *this arrangement gives an exact equivalent of the parabolic formula.* In other words, when m, n, and k have the values found from α and β by these equations, the alteration in x necessary to produce a balance for a given change of temperature will be proportional to that change of temperature.

From equation (9) it will be seen that two values of m satisfy the required conditions. The value got by taking the positive sign in (9) is the convenient one to use, as the other value gives very small change of reading for a given change of temperature.

In order to make the increase of x by 0·1 ohm correspond

Fig. 5.
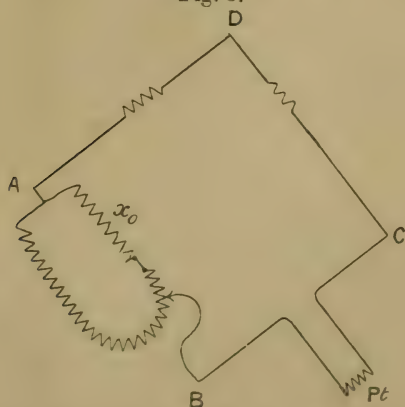
to a rise of temperature of 1° C., as in Method (A) the resistances (fig. 5) in the loop are raised in suitable pro-

portion *, and the arm AD is set to give a balance at some
known temperature (say 0° C.), the compensating leads,
increased in like ratio, being between B and the slider.

For example, for the thermometer already mentioned
[equation (5)] we could make the total resistance of the
loop 603·4 ohms, and the resistance x_0 equal to 24·46 ohms.
Then each 0·1 ohm moved by the slider will accurately
represent 1° C. Without altering these resistances, the
combination could be used with fair accuracy with any other
thermometer of nearly similar platinum. For the best work,
however, the simplest way is to have two small resistance-
coils adjusted to correspond to each thermometer, one repre-
senting x_0, and the other $n - x_0 - 100$.

The arrangement shown in fig. 6 is convenient for the
Rectifying Loop, the two adjusted coils being at P and Q.

Fig. 6.
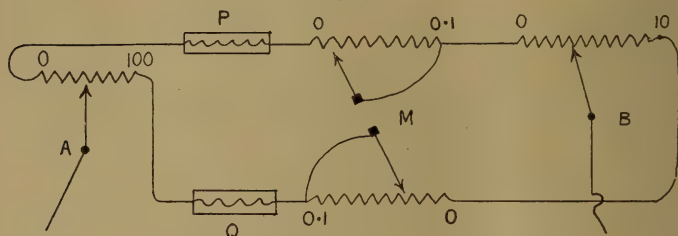
Here A and B, the ends of the bridge-arm, are connected to
turning-sliders on the ten 10-ohm and ten 1-ohm coils
respectively, while the central turning head is a double one,
altering the upper and lower resistances by equal and opposite
steps of 0·1 ohm at a time. For clearness I have put this
head (M) in the middle of the figure, but in the actual
construction the dials should be in their proper order of
magnitude, so as to give the temperature at a glance.

APPENDIX.

On Composite Thermocouples.

For the measurement of moderate or small differences of
temperature, up to 150° C., sensitive thermo-junctions of
several kinds have been successfully used. For example, in
the measurement of the small temperature-differences due
to the Peltier Effect, I found Iron-German silver thermopiles
very convenient (Proc. Roy. Soc. Edin. 1882). The great

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* A similar procedure is required to give equal bridge-arms.

advantage of this combination lies in the fact that the thermo-electric power is large and almost constant over a good range of temperature, the lines of the iron and German-silver in the thermoelectric diagram being very nearly parallel: thus the voltages given are very nearly proportional to the temperature-differences. Fortunately the Nickel line is also very nearly parallel to that for Iron, and is lower than that for German-silver, making the combination Iron-Nickel considerably more powerful than that of Iron-German silver. Since the introduction of Constantan (or Eureka, a copper-nickel alloy), it has been largely used, particularly in Germany, in conjunction with iron, as a sensitive thermojunction. The thermoelectric power is likewise very nearly constant and it is considerably greater than either of the others already mentioned, being about 50 microvolts per degree C. The Iron-Nickel combination, however, has for some purposes the advantage over the other two, in that its resistance is much lower for the same size of wires. In all these combinations, the voltage obtained is only approximately proportional to the temperature difference. To make the proportionality much closer I would suggest the use of composite junctions, first employed by Tait in mapping out the thermoelectric diagram

Fig. 7.
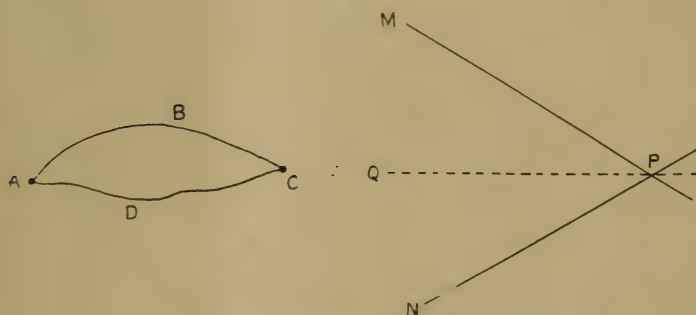
In fig. 7 let ABC and ADC be wires of two different metals joined in parallel, and let MP and NP be their respective lines on the thermoelectric diagram. Then the line for the composite wire AC will pass through P (the Neutral Point of ABC and ADC) in some direction such as QP, and, by altering the ratio of the resistances of ABC and ADC, the direction of QP can be altered continuously so as to lie anywhere desired between MP and NP.

Thus by combining with the iron or the constantan (say) a "modifying" wire of another metal such as copper, the line of the composite wire can be turned into more exact parallelism to the line of the other metal used with it by merely altering the resistance of one of the branches of the composite wire. In the cases already mentioned the "modifying" wire (copper) will have a circuit of relatively high resistance.

In measuring high temperatures with platinum-platinum-rhodium (or PtIr) junctions, a "modifying" wire of less pure platinum, for example, would enable the thermoelectric voltage-curve to be modified at will, and perhaps "normalized" or even rectified.

Teddington, Feb. 23, 1905.

LXVII. *Simple Diagram connecting the Various Motions in the so-called Bohnenberger's Machine.* By J. J. TAUDIN CHABOT *.

BOHNENBERGER'S machine † is but seldom noticed in the general run of text-books on physics. This may be partly due to the fact that an exhaustive numerical representation of its mode of action cannot be there given, but partly it is also due to the circumstance that in consequence of its remaining unnoticed it has gradually come to be regarded by many as a mere curiosity, instead of a necessary part of the equipment of every well arranged physical laboratory.

Nevertheless this piece of apparatus presents a stimulating means, whose value is not to be underrated, of demonstrating the composition of motions, such as we already come across in the parallelogram of velocities ; provided only that its mode of action may be rendered plain in a correspondingly simple manner. The following lines are an effort in this direction.

Let the massive disk be in the x–y plane, and its axis of rotation along the z-axis of an orthogonal three-dimensional system of co-ordinates $x\,y\,z$; and let the direction of rotation be R_z (fig. 1). Then we have to distinguish two main combinations : (1) an independent torque, acting in the direction R_y^+, imparts to each element of the rotating disk as it passes the x-axis a tendency to move along a resultant R_{yz} ; but

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* Communicated by the Author.
† "Beschreibung einer Maschine zur Erläuterung der Gesetze der Umdrehung der Erde um ihre Achse und Veränderung der Lage der letztern." Tuebingen, 1817.

since the various elements are rigidly connected together, it is only possible for the *entire* disk to rotate about the x-axis,
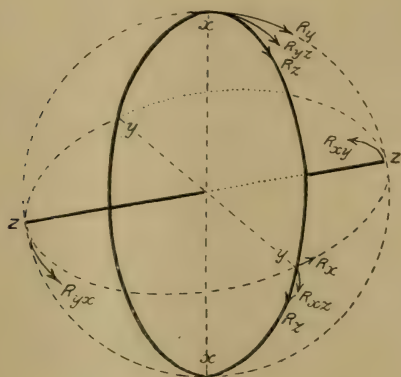
whereby a rotation of the axis originally along z takes place in the y–z° plane, here to be regarded as the limiting case of a conical surface whose apex is at the centre of the disk, while its generating line glides along a loxodromic or logarithmic spiral along the direction R_{xy} on the concentric spherical surface. (2) Similarly, an independent torque acting in the direction R_z imparts to each element of the rotating disk as it crosses the y-axis a tendency to move along the direction of a resultant R_{xz}, which with respect to the solidly built up disk now means a tendency towards rotation about the y-axis, so that the axis of rotation (originally along z) becomes displaced in the x–z plane, and in the general case follows the direction R_{yx} of a spiral on the spherical surface.

Each eccentrically situated point of the rotating disk itself obviously describes, during a very brief interval of time, a geodetic line on the corresponding spherical surface, so that during such an interval the resultant, the hypothenuse of a spherical right-angled triangle, is given by $\cos R_{yz} = \cos R_y \cos R_z$, and $\cos R_{xz} = \cos R_x \cos R_z$ respectively.

These relations hold for every position of the coordinate system in space.

With the stationary z-axis along the horizontal, the first main combination gives the phenomenon of precession, the second that of nutation. Now, as is immediately evident from the diagram, both phenonema mutually influence each other: if (first main combination) a nutation (in the direction R_y) be artificially produced, there necessarily follows a precession (in

the direction R_{xy}); and if (second main combination) a precession (in the direction R_x) be artificially produced, there always results a nutation (in the direction R_{yx}).

In accordance with the law of equality of action and reaction, a direct acceleration, for example, of precession (by the rotation of an axis parallel to the y-axis) calls forth a nutation (about an axis parallel to the y-axis) opposed to that which itself originally caused the precession—a relation which under certain conditions is very well illustrated by the Fessel-Plücker apparatus, although the construction of this latter was not, properly speaking, undertaken for the purpose of demonstrating this most complicated phenomenon. If the rotating disk be arranged with its axis at the end of a long arm, sufficiently flexible to vibrate readily, and if the apparatus be vibrated (*e. g.* by attaching it to an electromagnetically driven diaphragm), then the vibrations take place in succession in every possible direction about the arm so long as the disk is at rest. But if the disk be at the same time rotating, so that there occurs, according to the adjustments, a more or less rapid precession (first main combination), then the vibrations of the axis appear—to use a frequently criticised (and justly so) phrase of the optician—polarized, elliptically at first, until the horizontal component presently becomes reduced to zero, and only plane-polarized vibrations take place in the vertical plane : these denote an alternating nutation which becomes associated with the precession.

If the rotating disk forms part of a larger rotating system, with whose axis of rotation the axis of rotation z of the disk makes an angle $\alpha > 0$, the disk tends, by becoming inclined to an instantaneous diameter which crosses at right angles the axis of rotation of the rotating system, to bring its own axis of rotation z into parallelism with the axis of rotation of the system, so that, as is evident again from the diagram, both rotations get the same direction. This gives rise to a directive force, inherent f. i. to any rotating body at the earth's surface : Fixing the x- or the y-axis of a Bohnenberger's machine, viz. causing it to follow the earth's rotation, Foucault * saw the axis of rotation z of the disk turning about the remaining y- or x-axis and setting itself as near as possible—with widest possible angle α—parallel to the axis of rotation of the earth.

Rotterdam, Feb. 13, 1905.

* " Sur les phénomènes d'orientation des corps tournants entraînés par un axe fixe à la surface de la Terre." Paris, 1852.

LXVIII. *Anomalous Dispersion of the Magnetic Rotation of the Plane of Polarization.* By R. W. WOOD*.

THE question of the form of the curve representing the magnetic rotatory dispersion of absorbing media in the vicinity of the absorption-band is one of considerable importance. The formula for the rotation, developed from the hypothesis of molecular currents, calls for anomalous dispersion, *i. e.* a dispersion-curve with oppositely directed branches at the edges of the band, while the equation built up on the " Hall effect " hypothesis calls for rotations of similar sign on opposite sides of the band (see Drude's 'Optics'). The latter condition occurs in the case of sodium vapour as is well known, while the long series of investigations by Schmauss, published in the *Annalen der Physik* during the past three years, appeared to establish the fact that curves of the former type were exhibited by solutions of the aniline dyes and solutions of the salts of didymium. Having found that ordinary anomalous dispersion could be shown to much better advantage by means of prisms of fused cyanine than by means of alcoholic solutions, I made an attempt to verify the results of Schmauss by employing thick films of this substance formed by pressing the fused dye between plates of glass, which were then separated by the blow of a hammer. The film was then detached from the glass plate, to avoid the rotation produced by the glass, and mounted between the poles of a powerful electromagnet. Absolutely no trace of magnetic rotation could be detected, however, even when the films were so thick that only red light could be made to pass through them. Solutions were also tried, but so far as could be determined with the apparatus employed, the dye was without effect upon the rotatory power of the solvent. At about the time of the completion of these experiments, a paper by Bates appeared in the *Annalen (Ann. der Physik,* xii. pp. 1080, 1091), who pointed out that the method employed by Schmauss was open to objections, and that his anomalous curves could be explained as due to errors in the determination of the position of the dark band in the spectrum. Bates repeated the work by a method which was claimed to be many times more accurate than that employed by Schmauss, and free from the objections to which the latter was open. The results of his experiments were absolutely negative; that is, the dye was found to be without influence upon the magnetic rotatory power of the liquid in which it was dissolved. The character of the work

* Communicated by the Author.

done by Bates is well shown by the tables of readings which accompany the paper, and there seems to be no means of avoiding the conclusion that these substances are without rotatory power.   No reply to these criticisms has appeared to my knowledge, the two pieces of work standing without comment, each on its own merits.   Having already come to conclusions similar to those of Bates, I was the more ready to accept his experimental refutation of any anomalous dispersion effects in the magnetic rotation of these substances. As the method which he employed was much more sensitive than the one which I had made use of, publication of my own results was of course superfluous.

I have recently made a series of experiments with a saturated solution of praseodymium chloride, and have found what appears to me to be indisputable evidence of the effect. Curves were obtained for the substance by two different methods, and both showed anomalous effects at the edges of the absorption-band.   The solution was contained in a cell 1 cm. in thickness, furnished with windows of the thinnest cover-glass.   The light from the monochromatic illuminator (spectroscope with slit in place of eyepiece) was sent through the axis of the magnet and the rotations measured by setting the analysing nicol for extinction.   As the monochromatic light furnished by this apparatus is extremely brilliant, the setting could be made to within a tenth of a degree without difficulty.   A Savart plate was also employed, which increased the accuracy of the observations, and the curves accompanying this paper were plotted from readings obtained with this form of polarimeter.

Strong elliptical polarization was observed when working in the vicinity of the absorption-band, due to the unequal absorbing power of the magnetized medium for the two circular components.   This made accurate readings difficult, for it was impossible to completely extinguish the light when employing merely the two nicols, or to cause the complete disappearance of the Savart fringes.   It was found that the readings were prejudiced by a knowledge of what was expected.   The first curves were made by progressively changing the wave-length of the light emitted by the monochromatic illuminator.   The rotation, which amounted to about three degrees for the extreme red, increased nearly a whole degree as the absorption-band was approached, dropping back half a degree or so on crossing the band, and then rising again to a value of 4·5 in the blue-green region.   A curve obtained in this manner is shown in fig. 1 (Curve 1).   On taking a number of readings at random close to the edges of

the absorption-band it was found that they did not agree very well, the same values for the rotation occasionally being

Fig. 1.
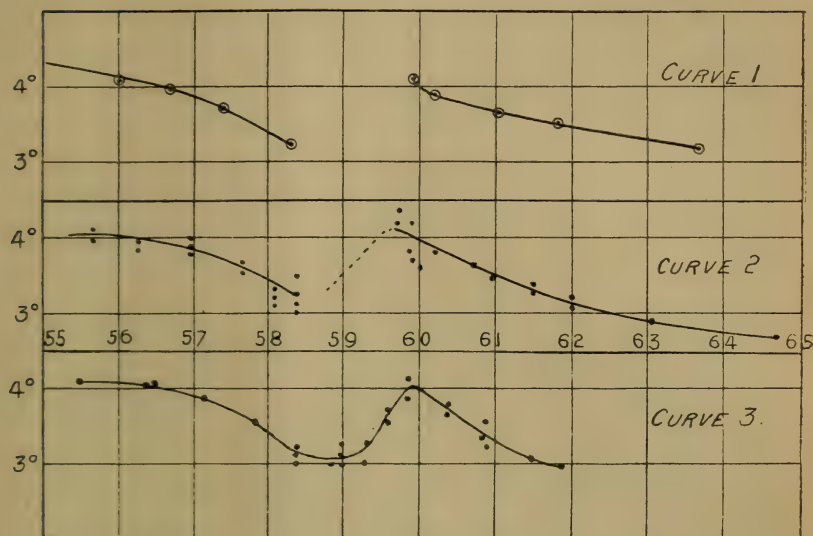
found on opposite sides of the band. On this account it seemed best to take a large number of readings for various wave-lengths, the monochromatic illuminator being adjusted by another observer. The readings obtained in this manner are recorded in curve 2, which, while showing the large errors which are liable to be made at the edges of the absorption-band, gives unmistakable evidence of anomalous rotatory dispersion. By opening the slit of the monochromatic illuminator, and working with more intense, though less homogeneous, light, it was possible to obtain readings through the region of absorption, though a good deal of trouble was experienced with the elliptical polarization in this case. The values found in this case are plotted in curve 3. I do not, however, attach much importance to them, as the light was far from homogeneous.

The observations recorded in this note were made merely for the purpose of convincing myself of the reality of the phenomenon of anomalous magnetic rotatory dispersion, and as the only other important piece of work on the subject, that of Schmauss, has been called into question, they may be worth putting on record.

LXIX. *On Clausius' Theorem for Irreversible Cycles, and on the Increase of Entropy.* By Prof. W. McF. ORR, *M.A.**

1. FROM the remarks with which Professor Planck, in the January number of this Magazine, and in private communications, has honoured my paper which appeared in that of October, I conclude that on the points to which he refers, although there has been a little mutual misconception, there is in reality not much difference between our views: this conclusion affords me much pleasure.

I quite agree with Prof. Planck that his definition of reversibility has a practical bearing; it appeared necessary, however, for me to point out that he uses the term in a different sense from the other writers whom I ventured to criticise, as of course the meaning of Clausius' Theorem for Irreversible Cycles and its mode of proof must to some extent depend on the meaning of the word " Irreversible."

My statement that Prof. Planck gives one definition of " reversibility " but uses another was based on a slight misconception which I regret. I understood him to state † that the change from one state (A) to another (B) is reversible if, and only if, it is possible starting from B to obtain A, and leave all the materials and machines used in the same condition exactly (at A) *as before their application (at* B). I have since learnt from him, however, that the words here italicized should be replaced by " *as when the system was in the state* A *in the first instance.*" This being so, my accusation must be withdrawn; and Prof. Planck's argument does not seem open to any serious objection on the ground of logic. I must confess, however, that I have found it just a little obscure : for instance, in § 124 and elsewhere it seems necessary to imply an appeal to the First Law in order to show that the change considered would really be completely reversed.

In contesting my statement that " Under ordinary circumstances, however, no body can expand without producing a change of density in some other body," Prof. Planck has not attached sufficient importance to the words " under ordinary circumstances," which I used designedly.

I note with gratification that Prof. Planck agrees that in some cases of violent motion of a gas the usual definition of Entropy is inapplicable : I had drawn the contrary conclusion from his book. I should like, however, to see him go so far as to admit that this definition applies only to equilibrium states.

* Communicated by the Author.
† *Vorlesungen über Thermodynamik*; or Ogg's translation, Arts. 109, 112.

Prof. Planck objected that a proof which I suggested of the Principle of the Increase of Entropy starts with the assumption expressed in Lord Kelvin's version of the Second Law. I understand that he withdraws this objection on learning that by Lord Kelvin's version of the Second Law I mean the principle that it is impossible for a substance to go through a cycle in which it does work and exchanges heat with bodies at one definite temperature only. This principle constitutes his own starting-point. I regret that in this matter also I unintentionally misrepresented him in my paper: I stated that he assumes as a result of experience that the expansion of a gas without doing external work is "irreversible" ("*irreversibel*"); but on reperusal I find that this is deduced from Lord Kelvin's axiom*.

2. With most of Prof. Buckingham's remarks † I find myself in substantial agreement, as was indeed to be anticipated from the fact that he also had, in his 'Theory of Thermodynamics,' expressed the view that Clausius' inequality cannot be deduced from the Two Laws.

If, however, I am to say with him that any logical physicist finds the so-called proofs of this theorem worthless, I fear I must add an expression of regret that so few logical physicists have ever given publicity to their views on the subject.

Whether the "new substance" on which this theorem is to be based should take the form of an emendation of the Second Law as I suggested, or should form, so to say, a Third Law, as he prefers, is a question chiefly of individual taste. It may indeed be considered presumptuous to suggest any modification of the famous and time-honoured Second Law, though here it may be urged that the words of Lord Kelvin and of Clausius do not, without comment, convey clearly what is generally considered to be their authors' meanings.

That the choice also among the various methods of Carnot's cycles, Entropy, Available Energy, and Thermodynamic Potential should be largely governed by considerations of expediency and individual liking I of course admit, provided all these methods are logically sound. My plea for the use of cycles is, however, based partly on the view that the conceptions of Entropy, of Thermodynamic Potential, and, as a measurable quantity, that of Available Energy also, are intelligible only for such states as can be reached in a reversible way: there seems therefore a difficulty in justifying the statement, for example, that when a system is in stable

* *Lec. cit.* Art. 118.
† Phil. Mag. Feb. 1905, p. 208.

equilibrium the Thermodynamic Potential is a minimum, and though Available Energy may be *defined* for all states, in order to apply the test of its being a minimum it is necessary to express it in terms of other functions, while such expressions are valid only for states which may be obtained in a reversible manner.

LXX. *Notices respecting New Books.*

A Magnetic Survey of Japan reduced to the Epoch 1895·0 *and the Sea Level, carried out by order of the Earthquake Investigation Committee. Reported by* A. TANAKADATE, *Professor of Physics, Imperial University, Tōkyō.* (Forming Vol. XIV. of the Journal of the College of Science . . . Tōkyō.) Text pp. xii + 180, with 1 plate ; Appendix pp. (iv) + (347), with 87 plates and 11 maps. Tōkyō, 1904.

IN carrying out this important survey Prof. Tanakadate had the assistance of fifteen Japanese men of science who devoted the summers of four years, 1893 to 1896, to the work. The instruments employed—shown in a plate at the beginning—present several novelties, notably the silk suspension of the collimator magnet described on p. 13.

In the absence presumably of data from fixed observatories, the diurnal variations were eliminated by multiplying observations throughout the day. The declination observations were so numerous at each station as to give curves of the diurnal variation occupying 59 plates in the Appendix. The other plates show the exact position and surroundings of each of the 320 stations. The method of reducing the observations to a common epoch is unusual. The declination, for instance, at a place whose latitude is ϕ and longitude λ is assumed to be given by a formula such as

$$\delta = \delta_0 + A(\phi - \phi_0) + B(\lambda - \lambda_0) + C(\lambda - \lambda_0)^2,$$

where the suffix $_0$ refers to a certain central latitude and longitude. The constants A &c. being determined from the present survey and a previous one by Profs. Knott and Nagaoka, and so for two different epochs, results are obtained for the secular change as depending on ϕ and λ. The values thus found for the mean annual change at different parts of Japan vary from $+3'\cdot 82$ to $-1'\cdot 85$ in Declination, from $+2'\cdot 10$ to $-4'\cdot 17$ in Inclination, and from $+29\cdot 9\gamma$ to $-7\cdot 9\gamma$ in Horizontal Force. Even allowing for the fact that a range of 16° of latitude and 18° of longitude is covered, the variety in these results is remarkable, and confirmation from fixed observatories seems highly desirable.

The question of the existence of a vertical earth-air electric current is minutely considered. Table xviii. gives the results obtained over Japan from formulæ of the type $4\pi w = d\mathrm{Y}/dx - d\mathrm{X}/dy$, and analogous results for Great Britain and Austria are deduced from the surveys of Rücker & Thorpe and of Liznar. In all three

cases a locus of no current is found running through the centre of the country, and the + and − currents deduced on either side of this are held to not improbably represent observational uncertainties.

A special feature is the reduction to sea-level which is actually applied to every observation. Assuming electric currents negligible, the vertical variation of, say, the northerly component X is deduced from the observed northerly variation of the vertical component Z by the formula

$$dX/dz = dZ/dx.$$

Vertical variation tables are thus calculated for all the elements and the results are compared with those deduced from formulæ of the type $\partial X/X = -3h/R$, where R is the Earth's radius, and ∂X the increment of X with height h above sea-level. The latter type of formula follows from the Gaussian potential, neglecting harmonics above the first. The comparison is made for Britain and Austria as well as Japan, and in most cases the accordance is remarkably close; there are differences, however, too systematic to be wholly due to chance. The corrections for reduction to sea-level in the case of the highest stations—fully $3\frac{1}{2}$ kilometres above sea-level— are of the order 45γ in Horizontal Force and $0'\cdot4$ in Inclination; at most stations they are trifling.

Considerable mathematical calculation is devoted to an attempt to arrive at the depth and position of magnetic foci to which the observed irregularities in the lines of magnetic force may be ascribed. For the foci affecting respectively Japan, Austria, and Britain, depths are found of the order 1000, 1300, and 2600 kilometres; but it is admitted that much uncertainty enters into the figures.

The isogonic and isoclinal lines and lines of equal force are given in charts at the end, smoothed curves on thick paper, and curves showing the local irregularities on thin transparent sheets, through which the smoothed curves are visible. A coloured geological map is added. The volume, which is handsomely printed in English, is a remarkable tribute to the intellectual activity of Japan, and merits the careful consideration of all interested in Magnetic Surveys.   C. CHREE.

Experimentelle Untersuchung von Gasen. Von Dr. MORRIS W. TRAVERS, *F.R.S. Mit einem Vorwort von* Sir WILLIAM RAMSAY, *K.C.B.* Deutsch von Dr. TADEUSZ ESTREICHER, Privatdozent an der K. K. Jagellonischen Universität in Krakau. Mit 1 Tafel und 144 in den Text eingedruckten Abbildungen. Braunschweig: F. Vieweg und Sohn. 1904. Pp. xii + 372.

SINCE the appearance of Bunsen's *Gasometrische Methoden* no comprehensive treatise on the subject of the experimental investigation of gases was published until Dr. Travers gave an account of recent work in this direction in his well-known book, which

appeared in 1902. Lest any reader of this review should be misled into supposing that the German version of this work is a mere translation, we hasten to assure him that such is by no means the case, as by reason of numerous additions which have brought it up to date, and important original contributions by the translator, it has become practically a new treatise on the subject.

The investigations connected with the recently discovered gases—argon, helium, neon, crypton, and xenon—have necessitated the development of much more refined and elaborate methods than any of those formerly in use. The details of many of these methods are not always to be found in the original papers which contain the results of experimental investigations, but they will be found clearly described in the book under review.

The opening chapters deal with the fundamental laws of gases, the construction of mercurial pumps, the handling of gases, their collection and storage, the preparation of gases in a state of purity, pressure and volume measurements, the calibration of vessels and tubes, and gas analysis. The next two chapters deal with the composition of the atmosphere and the gases of the helium group. Then follows an account of density determinations, of the relations connecting temperature, pressure, and volume over narrow and wide ranges of variation, the liquefaction of gases and the methods of handling them, vapour pressures and the critical constants, the solubility of gases in liquids, specific heats and heats of vaporization, effusion, transpiration and diffusion, refractive indices, and spectrum analysis. A useful appendix contains an account of methods for producing and maintaining constant known temperatures.

The author and translator are to be congratulated on having produced a work which is not only of very great value to the specialist in this subject, but which will prove of interest to a much wider circle of readers—for every chemist and physicist will derive solid benefit from a careful study of this unique work.

Vier- und Fünfstellige Logarithmentafeln nebst einigen Physikalischen Konstanten. Vieweg und Sohn, Braunschweig.

THIS clearly printed book of tables contains the usual four-figure logarithms on two pages, followed by 18 pages of five-figure logarithms with proportional parts alongside. Two brief tables of natural sines and cosines, tangents and cotangents for complete degrees only, a few analytical expansions, and condensed tables of various physical constants fill three other pages. The book appeals especially to the needs of workers in physical laboratories.

Fünfzehnter Bericht der Naturwissenschaftlichen Gesellschaft zu Chemnitz, umfassend die Zeit vom 22. Oktober 1899 bis zum 30. September 1903. Mit 3 Tafeln und 7 Figuren im Texte. Chemnitz : Carl Brunner. 1904. Pp. cxxiv+186.

IN addition to the List of Members and other official information, this volume contains a number of papers, mainly on botanical subjects, and an interesting article on " Our Calendar " by Professor O. Müller.

Diagram 5.
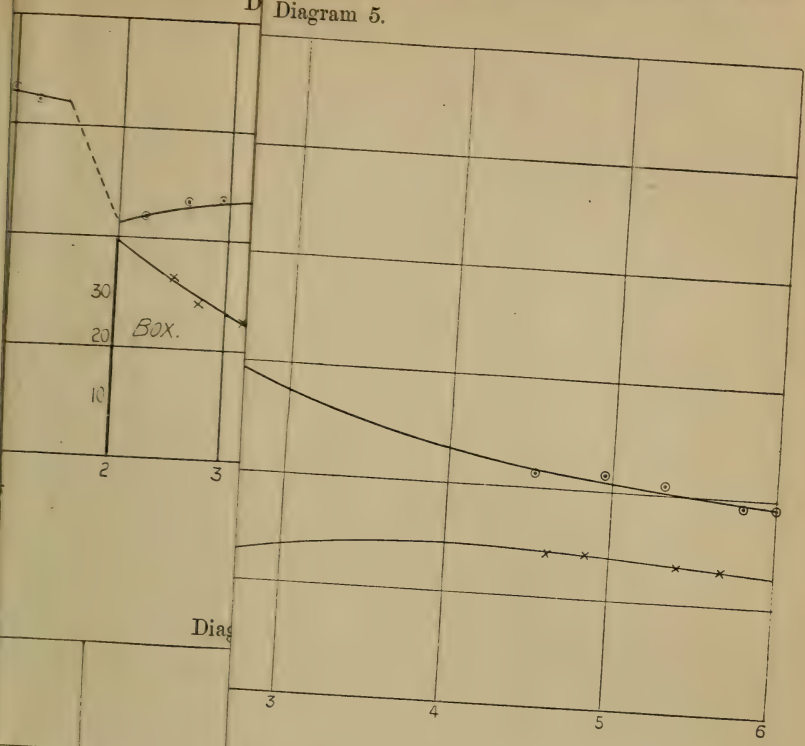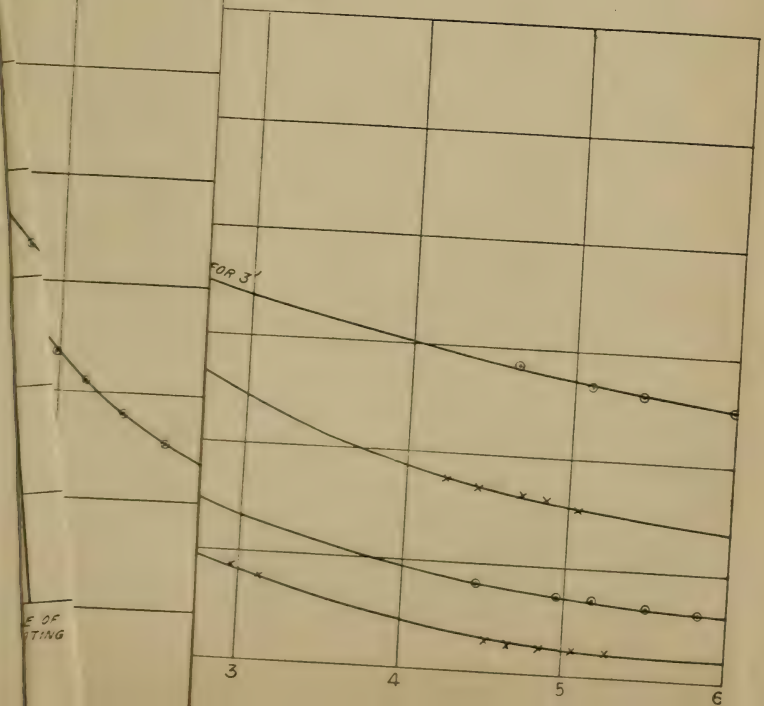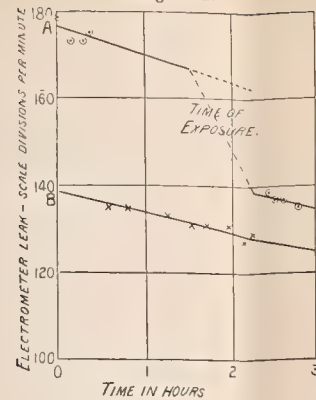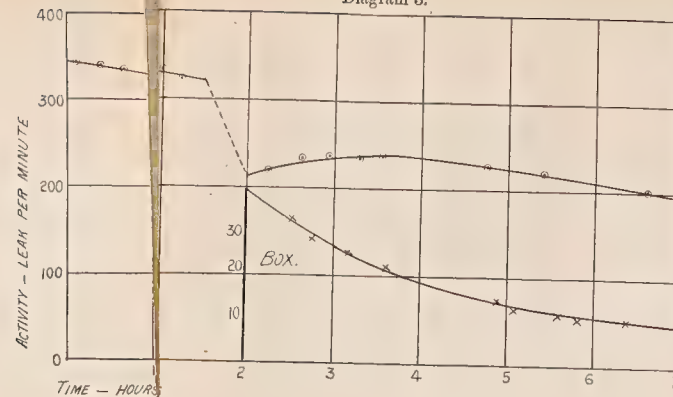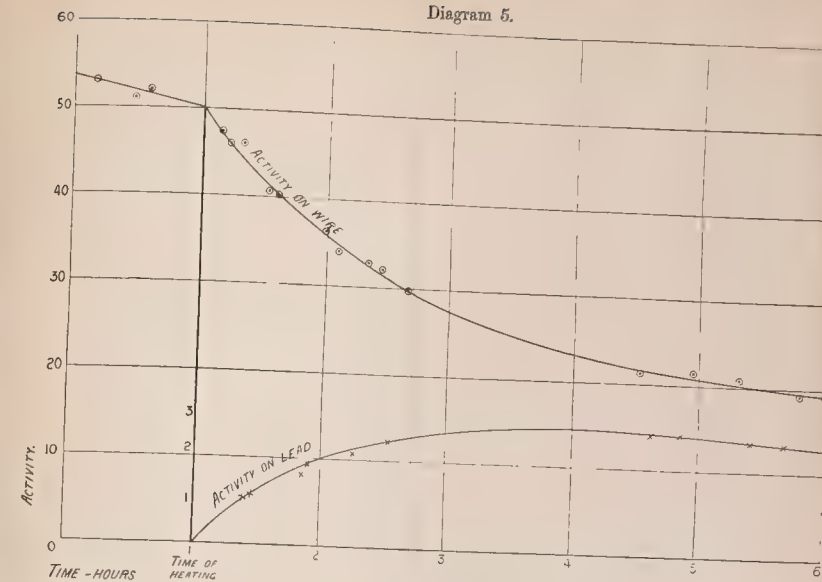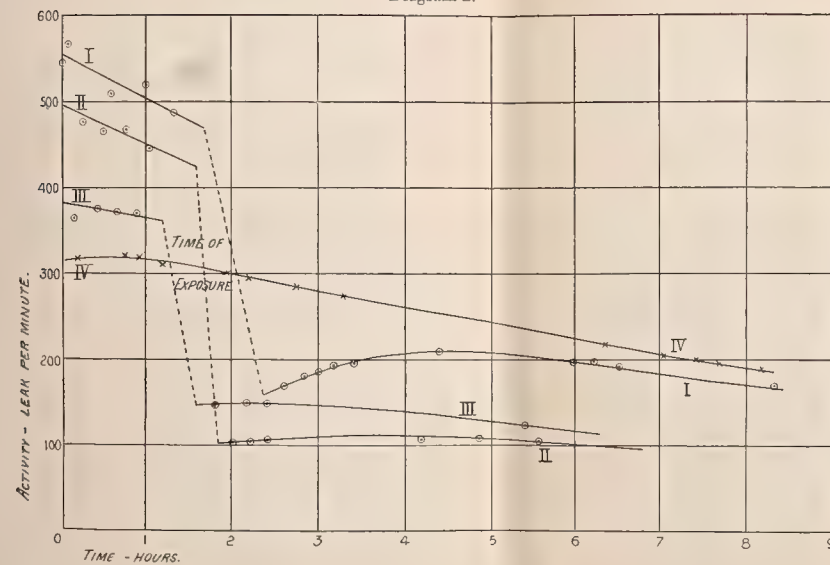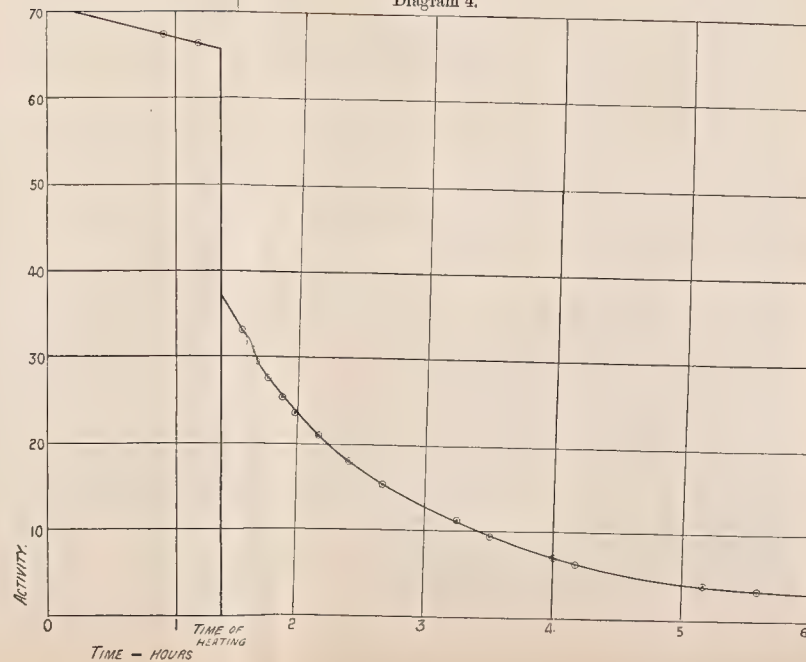
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Diagram 2.

Diagram 3.

Diagram 4.

Diagram 5.

Diagram 6.

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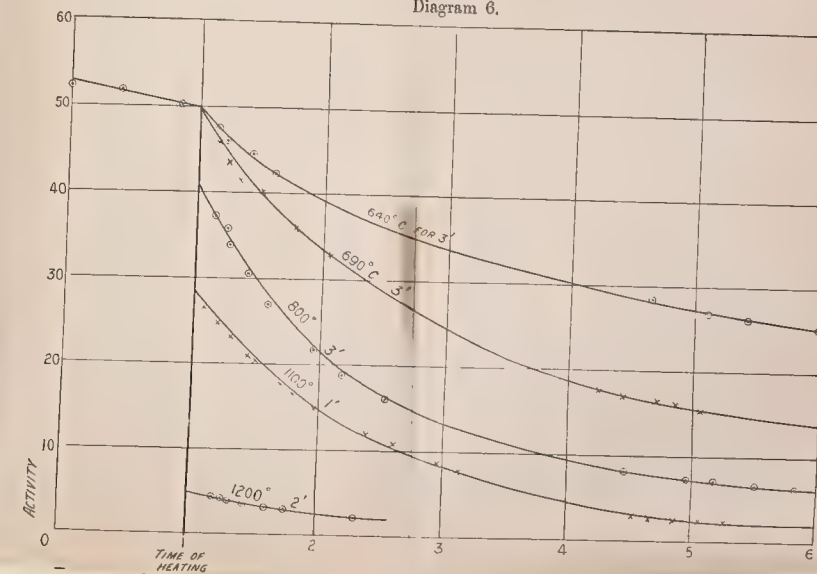
FIG. 1.
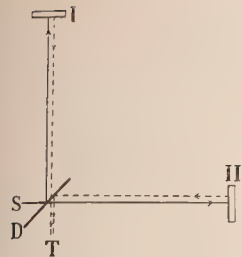
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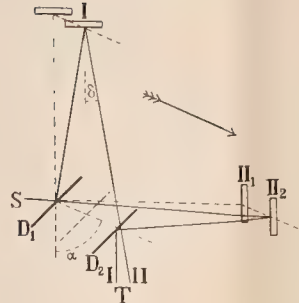
FIG. 5.
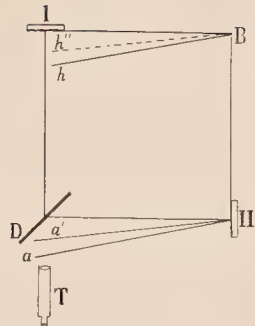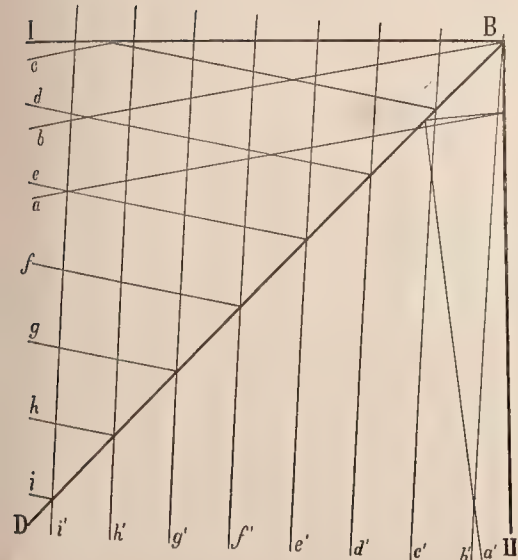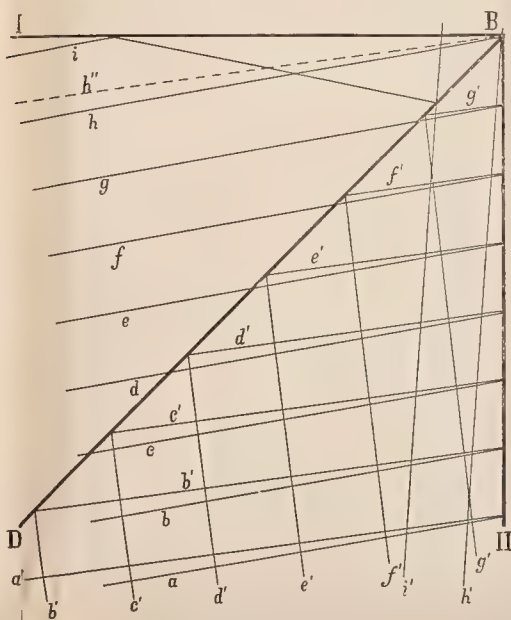
FIG. 6.

FIG. 3.

FIG. 4.

FIG. 7.

FIG. 9.
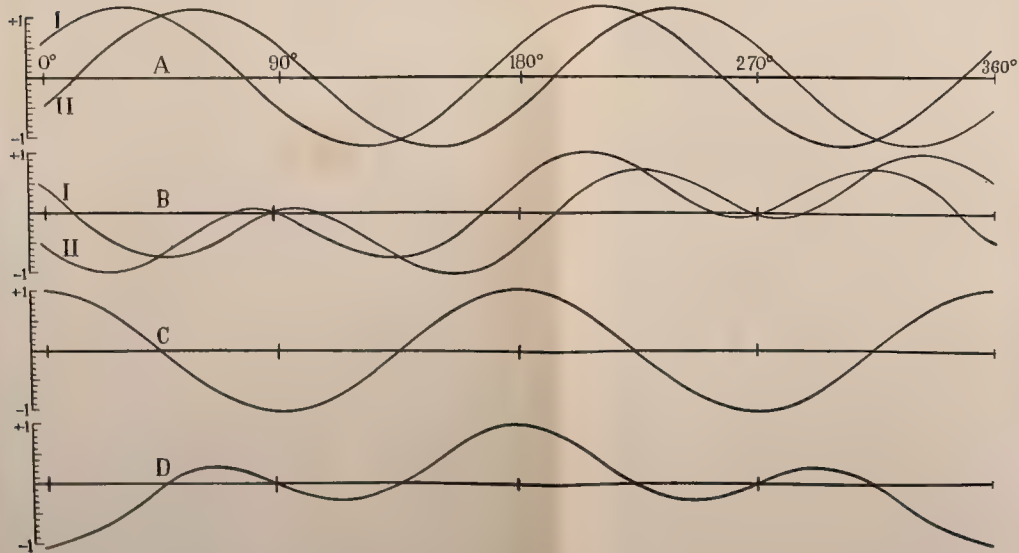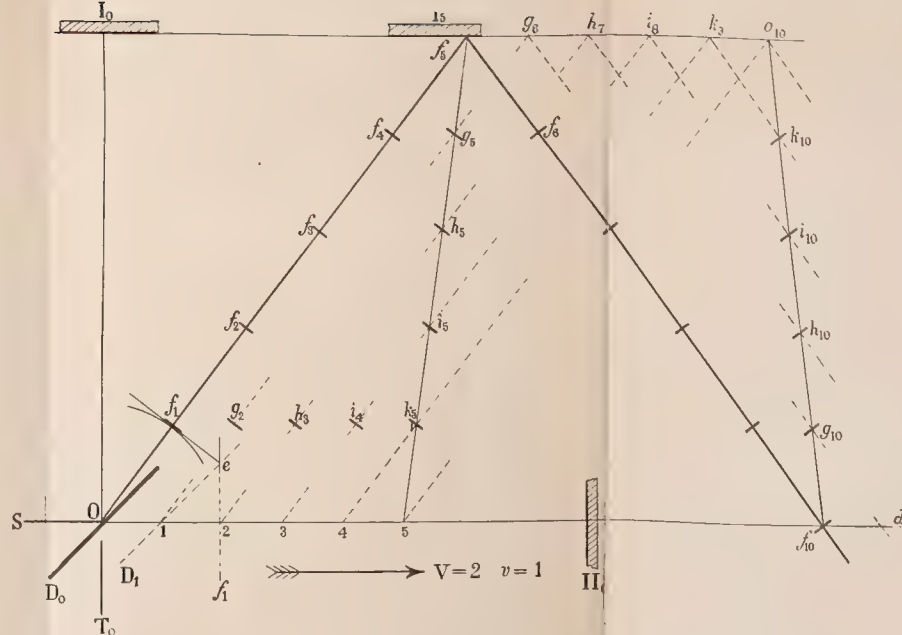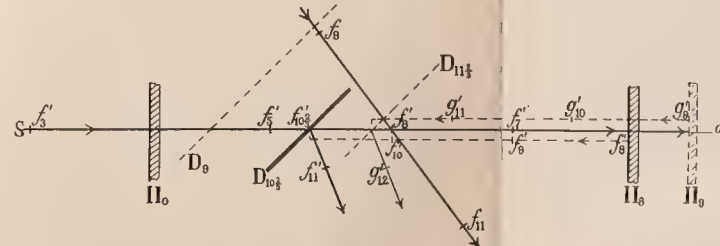
FIG. 8.
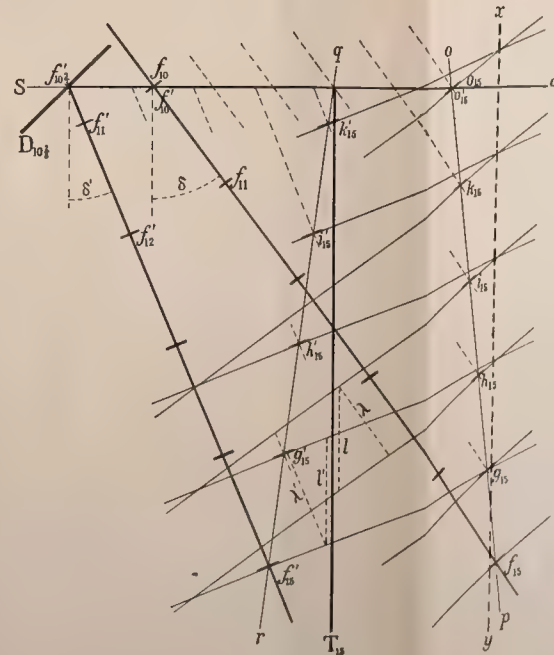
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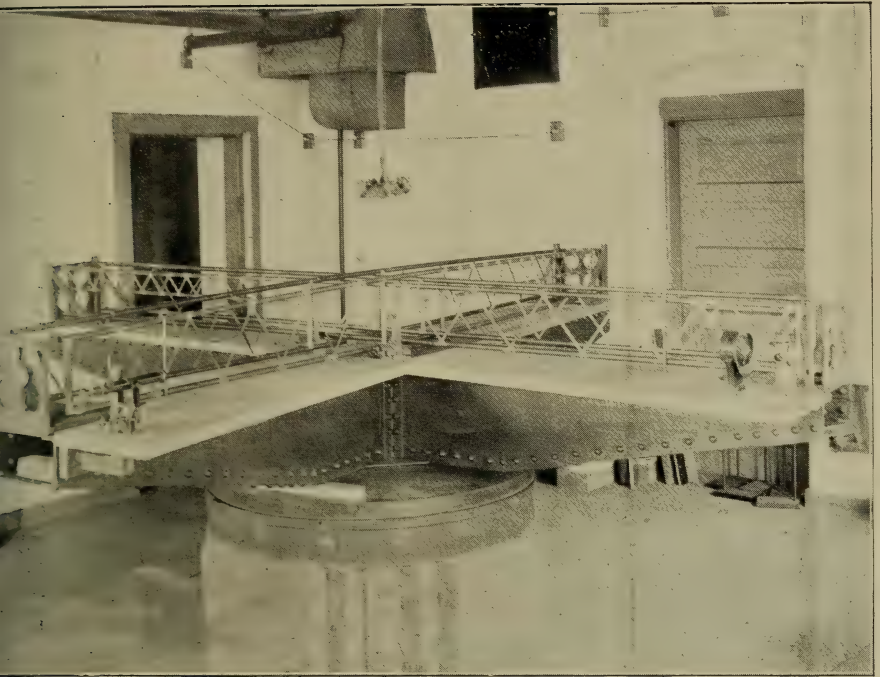
Fig. 4.
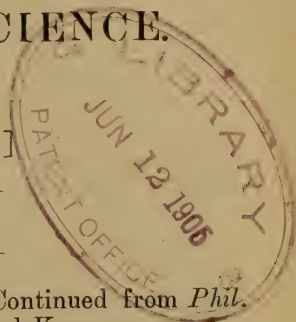
LXXI. *Deep Water Ship-Waves* [*]. (Continued from *Phil. Mag., October* 1904.) *By* Lord KELVIN.

§§ 32–64. Canal Ship-Waves.

§ 32. To avoid the somewhat cumbrous title "Two-dimensional," I now use the designation "Canal[†] Waves" to denote waves in a canal with horizontal bottom and vertical sides, which, if not two-dimensional in their source, become more and more approximately two-dimensional at greater and greater distances from the source. In the present communication the source is such as to render the motion two-dimensional throughout; the two dimensions being respectively perpendicular to the bottom, and parallel to the length of the canal: the canal being straight.

§ 33. The word " deep " in the present communication and its two predecessors (§§ 1–31) is used for brevity to mean infinitely deep; or so deep that the motion does not differ

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[†] This designation does not include an interesting class of canal waves of which the dynamical theory was first given by Kelland in the Trans. Roy. Soc. Edin. for 1839 ; the case in which the wave-length is very long in comparison with the depth and breadth of the canal, and the transverse section is of any shape other than rectangular with horizontal bottom and vertical sides.

sensibly from what it would be if the water, being incompressible, were infinitely deep. This condition is practically fulfilled in water of finite depth if the distance between every crest (point of maximum elevation), and neighbouring crest on either side, is less than two or three times its distance from the bottom.

§ 34. By "ship-waves" I mean any waves produced in open sea or in a canal by a moving generator; and for simplicity I suppose the motion of the generator to be rectilineal and uniform. The generator may be a ship floating on the water, or a submarine ship or a fish moving at uniform speed below the surface; or, as suggested by Rayleigh, an electrified body moving above the surface. For canal ship-waves, if the motion of the water close to the source is to be two-dimensional, the ship or submarine must be a pontoon having its sides (or a submerged bar having its ends) plane and fitting to the sides of the canal, with freedom to move horizontally. The submerged surface must be cylindric with generating lines perpendicular to the sides.

§ 35. The case of a circular cylindric bar of diameter small compared with its depth below the surface, moving horizontally at a constant speed, is a mathematical problem which presents interesting difficulties, worthy of serious work for anyone who may care to undertake it. The case of a floating pontoon is much more difficult, because of the discontinuity between free surface of water and water-surface pressed by a rigid body of given shape, displacing the water.

§ 36. Choosing a much easier problem than either of those, I take as wave generator a forcive* consisting of a given continuous distribution of pressure at the surface, travelling over the surface at a given speed. To understand the relation of this to the pontoon problem, imagine the rigid surface of the pontoon to become flexible; and imagine applied to it, a given distribution Π of pressure, everywhere perpendicular to it. Take O, any point at a distance h above the undisturbed water-level, draw O X parallel to the length of the canal and O Z vertically downwards. Let ξ, ζ be the displacement-components of any particle of the water whose undisturbed position is (x, z). We suppose the disturbance infinitesimal; by which we mean that the change of distance between any two particles of water is infinitely small in comparison with their undisturbed distance; and that the line

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* "Forcive" is a very useful word introduced, after careful consultation with literary authorities, by my brother the late Prof. James Thomson, to denote *any system of force*.

joining them experiences changes of direction which are infinitely small in comparison with the radian. For liberal interpretation of this condition see § 61 below. Water being assumed frictionless, its motion, started primarily from rest by pressure applied to the free surface, is essentially irrotational. But we need not assume this at present: we see immediately that it is proved by our equations of motion, when in them we suppose the motion to be infinitesimal. The equations of motion, when the density of the liquid is taken as unity, are :—

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + \dot\xi\frac{d\dot\xi}{dx} + \dot\zeta\frac{d\dot\xi}{dz} &= -\frac{dp}{dx} \\ \frac{d^2\zeta}{dt^2} + \dot\xi\frac{d\dot\zeta}{dx} + \dot\zeta\frac{d\dot\zeta}{dz} &= g - \frac{dp}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad (59),$$

where g denotes the force of gravity and p the pressure at (x, z, t). Assuming now the liquid to be incompressible, we have

$$\frac{d\dot\xi}{dx} + \frac{d\dot\zeta}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad (60).$$

§ 37. The motion being assumed to be infinitesimal, the second and third terms of the first members of (59) are negligible, and the equations of motion become :—

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= -\frac{dp}{dx} \\ \frac{d^2\zeta}{dt^2} &= g - \frac{dp}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (61).$$

This, by taking the difference of two differentiations, gives:—

$$\frac{d}{dt}\left(\frac{d\dot\xi}{dz} - \frac{d\dot\zeta}{dx}\right) = 0 \quad . \quad . \quad . \quad . \quad (62),$$

which shows that if at any time the motion is zero or irrotational, it remains irrotational for ever.

§ 38. If at any time there is rotational motion in any part of the liquid, it is interesting to know what becomes of it. Leaving for a moment our present restriction to canal waves, imagine ourselves on a very smooth sea in a ship, kept moving uniformly at a good speed by a tow-rope above the water. Looking over the ship's side we see a layer of disturbed motion, showing by dimples in the surface innumerable little whirlpools. The thickness of this layer increases from nothing perceptible near the bow to perhaps 10 or 20 cms. near the

stern; more or less according to the length and speed of the ship. If now the water suddenly loses viscosity and becomes a perfect fluid, the dynamics of vortex motion tells us that the rotationally moving water gets left behind by the ship, and spreads out in the more and more distant wake and becomes lost*; without, however, losing its kinetic energy, which becomes reduced to infinitely small velocities in an infinitely large portion of liquid. The ship now goes on through the calm sea without producing any more eddies along its sides and stern, but leaving within an acute angle on each side of its wake, smooth ship-waves with no eddies or turbulence of any kind. The ideal annulment of the water's viscosity diminishes considerably the tension of the tow-rope, but by no means annuls it; it has still work to do on an ever increasing assemblage of regular waves extending farther and

farther right astern, and over an area of $19° 28' \left(\tan^{-1} \sqrt{\dfrac{1}{8}} \right)$

on each side of mid-wake, as we shall see in about § 80 below. Returning now to two-dimensional motion and canal waves: we, in virtue of (62), put

$$\dot{\xi} = \frac{d\phi}{dx}, \quad \dot{\zeta} = \frac{d\phi}{dz} \quad . \quad . \quad . \quad . \quad (63),$$

where ϕ denotes what is commonly called the "velocity-potential"; which, when convenient, we shall write in full $\phi(x, z, t)$. With this notation (61) gives by integration with respect to x and z,

$$\frac{d\phi}{dt} = -p + g(z + \mathrm{C}) \quad . \quad . \quad . \quad . \quad (64).$$

And (60) gives

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dz^2} = 0 \quad . \quad . \quad . \quad . \quad (65).$$

Following Fourier's method, take now

$$\phi(x, z, t) = -k\epsilon^{-mz} \sin m(x - vt) \quad . \quad . \quad . \quad (66),$$

* It now seems to me certain that if any motion be given within a finite portion of an infinite incompressible liquid originally at rest, its fate is necessarily dissipation to infinite distances with infinitely small velocities everywhere; while the total kinetic energy remains constant. After many years of failure to prove that the motion in the ordinary Helmholtz circular ring is stable, I came to the conclusion that it is essentially unstable, and that its fate must be to become dissipated as now described. I came to this conclusion by extensions not hitherto published of the considerations described in a short paper entitled: "On the Stability of Steady and Periodic Fluid Motion," in the Phil. Mag. for May 1887.

which satisfies (65) and expresses a sinusoidal wave-disturbance, of wave-length $2\pi/m$, travelling x-wards with velocity v.

§ 39. To find the boundary-pressure Π, which must act on the water-surface to get the motion represented by (66), when m, v, k are given, we must apply (64) to the boundary. Let $z=0$ be the undisturbed surface; and let $\mathbf{d}$ denote its depression, at (x, o, t) below undisturbed level; that is to say,

$$\dot{\mathbf{d}}=\dot{\zeta}(x,o,t)=\frac{d}{dz}\phi(x,z,t)_{z=0}=mk\sin m(x-vt) \quad (67),$$

whence by integration with respect to t,

$$\mathbf{d}=\frac{k}{v}\cos m(x-vt) \quad . \quad . \quad . \quad . \quad . \quad (68).$$

To apply (64) to the surface, we must, in gz, put $z=\mathbf{d}$; and in $d\phi/dt$ we may put $z=0$, because $\mathbf{d}$, k, are infinitely small quantities of the first order, and their product is neglected in our problem of infinitesimal displacements. Hence with (66) and (68), and with Π taken to denote surface-pressure, (64) becomes

$$kmv\cos m(x-vt)=\frac{g}{v}k\cos m(x-vt)-\Pi+g\mathrm{C} \quad . \quad (69);$$

whence, with the arbitrary constant C taken $=0$,

$$\Pi=kr\left(\frac{g}{v^2}-m\right)\cos m(x-vt) \quad . \quad . \quad . \quad . \quad (70);$$

and, eliminating k by (68), we have finally,

$$\Pi=(g-mv^2)\mathbf{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (71).$$

Thus we see that if $v=\sqrt{g/m}$, we have $\Pi=0$, and therefore we have a train of free sinusoidal waves having wave-length equal to $2\pi/m$. This is the well-known law of relation between velocity and length of free deep-sea waves. But if v is not equal to $\sqrt{g/m}$, we have forced waves with a surface-pressure $(g-mv^2)\mathbf{d}$ which is directed with or against the displacement according as $v<$ or $> \sqrt{g/m}$.

§ 40. Let now our problem be :—given Π, a sum of sinusoidal functions, instead of a single one, as in (70) ;—required $\mathbf{d}$ the resulting displacement of the water-surface. We have by (71) and (70), with properly altered notation,

$$\Pi=\Sigma\mathrm{B}\cos m(x-vt+\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72),$$

$$\mathbf{d} = \Sigma \frac{\mathrm{B}}{g - mv^2} \cos m(x - vt + \beta) + \mathrm{A} \cos \frac{g}{v^2}(x - vt + \gamma) \quad (73),$$

where B, m, β are given constants having different values in the different terms of the sums; and v is a given constant velocity. The last term of (73) expresses, with two arbitrary constants (A, γ), a train of free waves which we may superimpose on any solution of our problem.

§ 41. It is very interesting and instructive in respect to the dynamics of water-waves, to apply (72) to a particular case of Fourier's expansion of periodic arbitrary functions such as a distribution of alternate constant pressures, and zeros, on equal successive spaces, travelling with velocity v. But this must be left undone for the present, to let us get on with ship-waves; and for this purpose we may take as a case of (72), (73),

$$\Pi = gc(\tfrac{1}{2} + e \cos \theta + e^2 \cos 2\theta + \&\mathrm{c.}) = gc \frac{\tfrac{1}{2}(1 - e^2)}{1 - 2e \cos \theta + e^2} \quad (74),$$

$$\mathbf{d} = \mathrm{J}c \left\{ \frac{1}{2\mathrm{J}} + \frac{e}{\mathrm{J} - 1} \cos \theta + \frac{e^2}{\mathrm{J} - 2} \cos 2\theta + \&\mathrm{c.} \right\} \quad . \quad . \quad (75);$$

where

$$\theta = \frac{2\pi}{a}(x - vt + \beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76);$$

$$v^2 = \frac{g\lambda}{2\pi}; \ \ \mathrm{J} = \frac{a}{\lambda} = \frac{ga}{2\pi v^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77);$$

and e may be any numeric < 1. Remark that when $v = 0$, $\mathrm{J} = \infty$, and we have by (75) and (74), $\mathbf{d} = \Pi/g$, which explains our unit of pressure.

§ 42. To understand the dynamical conditions thus prescribed, and the resulting motion :—remark first that (74), with (76), represents a space-periodic distribution of pressure on the surface, travelling with velocity v; and (75) represents the displacement of the water-surface in the resulting motion, when space-periodic of the same space-period as the surface-pressure. Any motion whatever; consequent on any initial disturbance and no subsequent application of surface-pressure; may be superimposed on the solution represented by (75), to constitute the complete solution of the problem of finding the motion in which the surface-pressure is that given in (74).

§ 43. To understand thoroughly the constitution of the forcive-datum (74) for Π, it is helpful to know that, n denoting any positive or negative integer, we have

$$2\pi(\tfrac{1}{2} + e \cos \theta + e^2 \cos 2\theta + \&\mathrm{c.}) = \sum_{n = -\infty}^{n = \infty} \frac{ba}{b^2 + (x - na)^2} \quad (78),$$

if

$$
\left.\begin{array}{l}
b = \dfrac{a}{2\pi} \log\left(1/e\right) \\[2mm]
x = \dfrac{a}{2\pi}\,\theta
\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (79).
$$

This we find by applying § 15 above to the periodic function represented by the second member of (78).

The equality of the two members of (78) is illustrated by
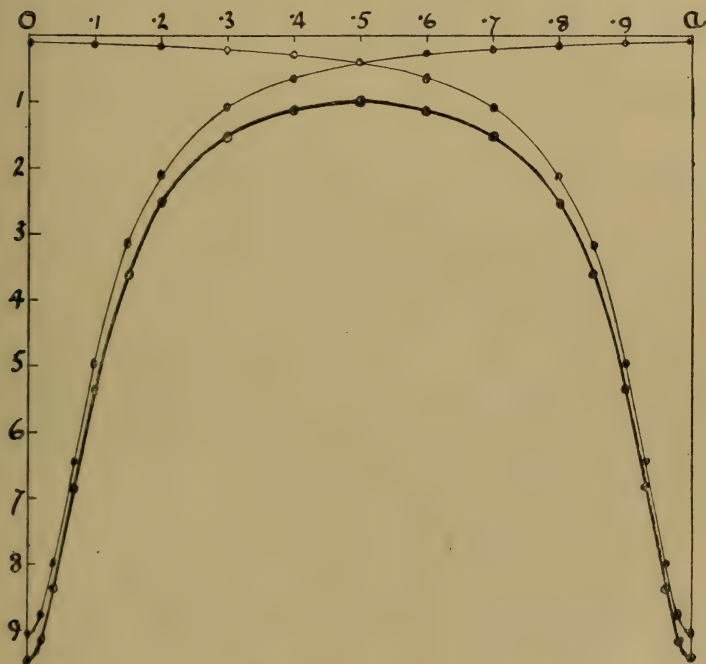
Fig. 11; $e = \cdot 5$.

fig. 11; in which; for the case $e = \cdot 5$ and consequently, by (79), $b/a = \cdot 1103$; the heavy curve represents the first member, and the two light curves represent two terms of the second member; which are as many as the scale of the diagram allows to be seen on it. There is a somewhat close agreement between each of the light curves, and the part of the heavy curve between a maximum and the minimum on each side of it. Thus we see that even with e so small as $\cdot 5$, we have a *not very rough* approximation to equality between successive

half periods of the first member of (78) and a single term of its second member. If e is < 1 by an infinitely small difference this approximation is infinitely nearly perfect. It is so nearly perfect for $e = \cdot9$ that fig. 12 cannot show any
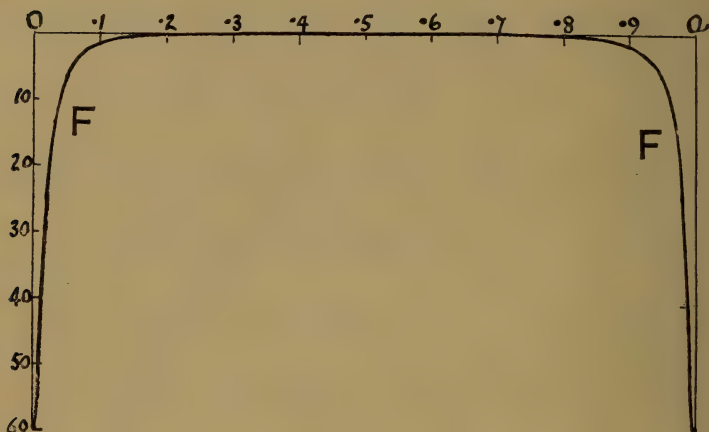
Fig. 12; $e = \cdot9$.

deviation from it, on a scale of ordinates 1/10 of that of fig. 11. The tendency to agreement between the first member of (78) and a single term of its second member with values of e approaching to 1, is well shown by the following modification of the last member of (74) :—

$$\Pi = gc \frac{\frac{1}{2}(1 - e^2)}{1 - 2e\cos\theta + e^2} = gc \frac{\frac{1}{2}(1 - e^2)}{(1 - e)^2 + 4e\sin^2\frac{1}{2}\theta} \quad (80).$$

Thus we see that if $e \doteq 1$, Π is very great when θ is very small; and Π is very small *unless* θ is very small (or very nearly $= 2i\pi$). Thus when $e \doteq 1$, we have

$$\frac{1}{gc} \Pi \doteq \frac{\frac{1}{2}(1 - e^2)}{(1 - e)^2 + e\theta^2} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (81);$$

which means expressing Π approximately by a single term of the second member of (78).

§ 44. Return to our dynamical solution (75); and remark that if J is an integer, one term of (75) is infinite, of which the dynamical meaning is clear in (70). Hence to have every term of (75) finite we must have $J = j + \delta$, where j is an integer and δ is < 1; and we may conveniently write (75) as follows :—

$$\mathbf{d} = c(\delta + j)\left\{ \tfrac{1}{2}\frac{1}{\delta + j} + \frac{e\cos\theta}{\delta + j - 1} + \frac{e^2\cos 2\theta}{\delta + j - 2} + \ldots + \frac{e^j\cos j\theta}{\delta} \right.$$

$$\left. - \frac{e^{j+1}\cos (j+1)\theta}{1 - \delta} - \frac{e^{j+2}\cos (j+2)\theta}{2 - \delta} - \text{ad inf.} \right\} \quad (82);$$

or

$$\mathbf{d} = \mathscr{F} + \mathscr{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83),$$

where $\mathscr{F}$ and $\mathscr{I}$ denote finite and infinite series shown in (82).

§ 45. We are going to make $\delta = \tfrac{1}{2}$; and in this case $\mathscr{I}$ can be summed, in finite terms, as follows. First multiply each term by $e^{j+\delta}\, e^{-j-\delta}$; and we find

$$\mathscr{I} = -c(\delta + j)e^{j+\delta}\left[\frac{e^{1-\delta}}{1 - \delta}\cos (j+1)\theta + \frac{e^{2-\delta}}{2 - \delta}\cos (j+2)\theta + \&\text{c.} \right]$$

$$= -c(\delta + j)e^{j+\delta}\int de\left[e^{-\delta}\cos (j+1)\theta + e^{1-\delta}\cos (j+2)\theta + \&\text{c.} \right]$$

$$= -c(\delta + j)e^{j+\delta}\int de\; e^{-\delta}\{\text{RS}\}q^{j+1}(1 + eq + e^2q^2 + \&\text{c.});$$

where q denotes $\epsilon^{\iota\theta}$; and, as in § 3 above, $\{\text{RS}\}$ denotes realisation by taking half sum for $\pm\iota$. Summing the infinite series, and performing $\int de$, for the case $\delta = \tfrac{1}{2}$, we find

$$\mathscr{I} = -c(j+\tfrac{1}{2})e^{j+\frac{1}{2}}\{\text{RS}\}q^{j+\frac{1}{2}}\log\frac{1 + \sqrt{qe}}{1 - \sqrt{qe}} \quad . \quad . \quad (84),$$

$$= -c(j+\tfrac{1}{2})e^{j+\frac{1}{2}}\{\text{RS}\}q^{j+\frac{1}{2}}\log\frac{1 + \sqrt{e}\cos\tfrac{1}{2}\theta + \iota\sqrt{e}\sin\tfrac{1}{2}\theta}{1 - \sqrt{e}\cos\tfrac{1}{2}\theta - \iota\sqrt{e}\sin\tfrac{1}{2}\theta}$$

$$= -c(j+\tfrac{1}{2})e^{j+\frac{1}{2}}\{\text{RS}\}q^{j+\frac{1}{2}}\left[\log\sqrt{\frac{1 + 2\sqrt{e}\cos\tfrac{1}{2}\theta + e}{1 - 2\sqrt{e}\cos\tfrac{1}{2}\theta + e}} + \iota(\psi - \psi') \right]$$

where

$$\psi = \tan^{-1}\frac{\sqrt{e}\sin\tfrac{1}{2}\theta}{1 + \sqrt{e}\cos\tfrac{1}{2}\theta}, \quad \psi' = \tan^{-1}\frac{-\sqrt{e}\sin\tfrac{1}{2}\theta}{1 - \sqrt{e}\cos\tfrac{1}{2}\theta} \quad (85),$$

and therefore

$$\psi - \psi' = \tan^{-1}\frac{2\sqrt{e}\sin\tfrac{1}{2}\theta}{1 - e}\;.$$

Hence finally

$$\mathscr{I} = c(j+\tfrac{1}{2})e^{j+\frac{1}{2}}\left\{ -\cos (j+\tfrac{1}{2})\theta \log\sqrt{\frac{1 + 2\sqrt{e}\cos\tfrac{1}{2}\theta + e}{1 - 2\sqrt{e}\cos\tfrac{1}{2}\theta + e}} \right.$$

$$\left. + \sin (j+\tfrac{1}{2})\theta \tan^{-1}\frac{2\sqrt{e}\sin\tfrac{1}{2}\theta}{1 - e} \right\}\;. \quad (86).$$

For our present case, of $\delta = \frac{1}{2}$, (82) gives

$$\mathscr{G} = c(j+\tfrac{1}{2})\left\{\frac{1}{2}\frac{1}{j+\frac{1}{2}} + \frac{e\cos\theta}{j-\frac{1}{2}} + \frac{e^2\cos 2\theta}{j-\frac{3}{2}} + \ldots + \frac{e^j\cos j\theta}{\frac{1}{2}}\right\}\ (87).$$

With $\mathscr{I}$ and $\mathscr{G}$ thus expressed, (83) gives the solution of our problem.

§ 46. In all the calculations of §§ 46–61 I have taken $e = \cdot 9$, as suggested for hydrokinetic illustrations in Lecture X. of my Baltimore Lectures, pp. 113, 114, from which fig. 12, and part of fig. 11 above, are taken. Results calculated from (83), (86), (87), are represented in figs. 13–16, all for the same forcive, (74) with $e = \cdot 9$, and for the four different velocities of its travel, which correspond to the values 20, 9, 4, 0, of j. The wave-lengths of free waves having these velocities are [(77) above] $2a/41$, $2a/19$, $2a/9$, and $2a$. The velocities are inversely proportional to $\sqrt{41}$, $\sqrt{19}$, $\sqrt{9}$, $\sqrt{2}$. Each diagram shows the forcive by one curve, a repetition of fig. 12; and shows by another curve the depression, d, of the water-surface produced by it, when travelling at one or other of the four speeds.

§ 47. Taking first the last, being the highest, of those speeds, we see by fig. 16 that the forcive travelling at that speed produces maximum displacement *upwards* where the *downward* pressure is greatest; and maximum *downward* displacement where the pressure (everywhere downward) is least. Judging dynamically it is easy to see that greater and greater speeds of the forcive would still give displacements above the mean level where the downward pressure of the forcive is greatest, and below the mean level where it is least; but with diminishing magnitudes down to zero for infinite speed.

And in (75) we have, for all positive values of $J < 1$, a series always convergent, (though sluggishly when $e \doteq 1$,) by which the displacement can be exactly calculated for every value of θ.

§ 48. Take next fig. 15, for which $J = 4\frac{1}{2}$, and therefore, by (77), $v = \sqrt{ga/9\pi}$, and $\lambda = a/4\cdot 5$. Remark that the scale of ordinates is, in fig. 15, only $1/2\cdot 5$ of the scale in fig. 16; and see how enormously great is the water-disturbance now in comparison with what we had with the same forcive, but three times greater speed and nine times greater wave-length $(v = \sqrt{ga/\pi}, \lambda = 2a)$. Within the space-period of fig. 15 we see four complete waves, very approximately sinusoidal, between M, M, two maximums of depression which are *almost exactly* (but very slightly less than) quarter wave-lengths

Fig. 13.

Fig. 14.

j=4.
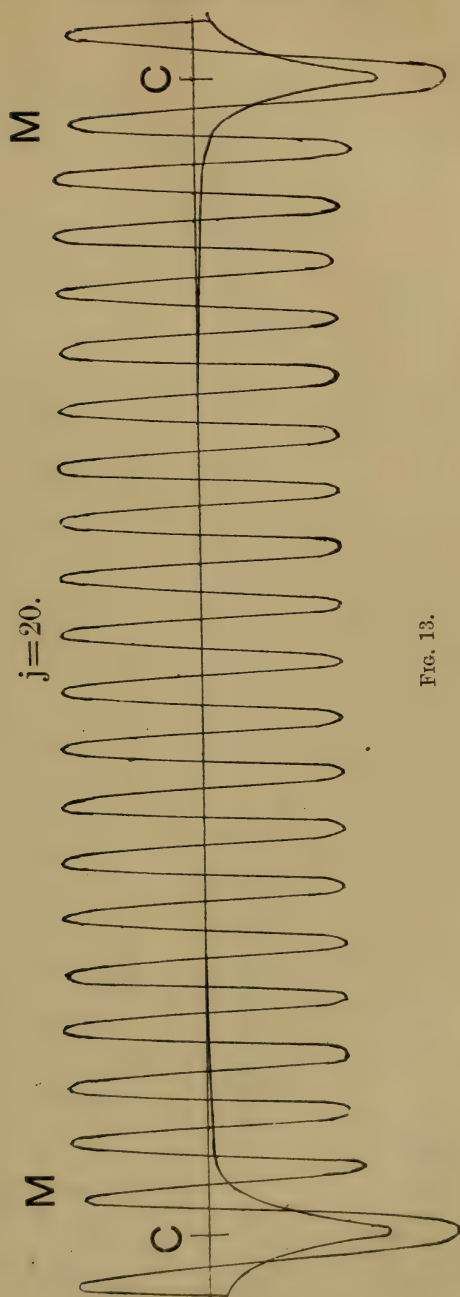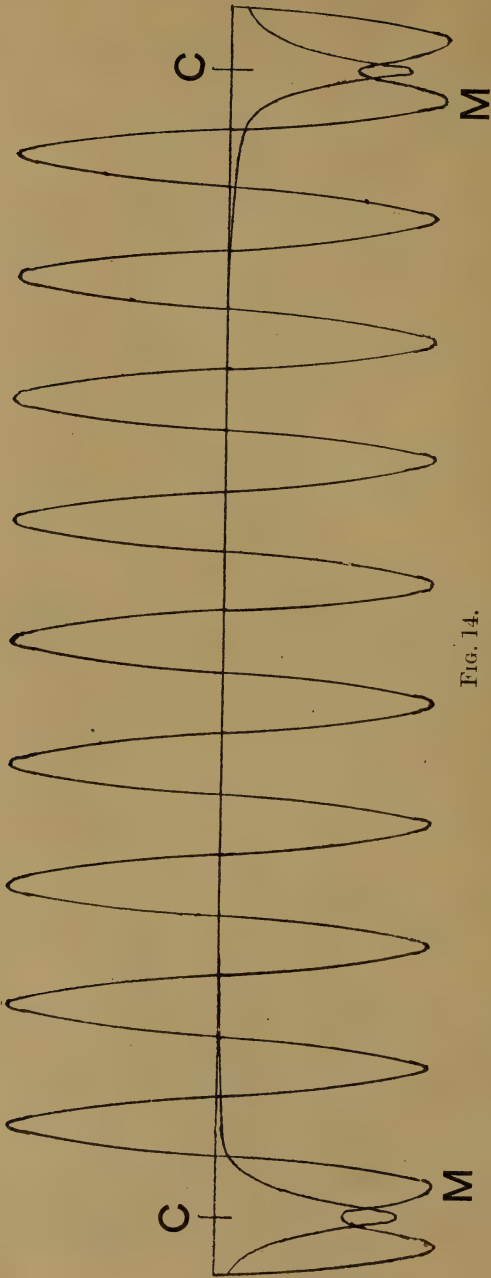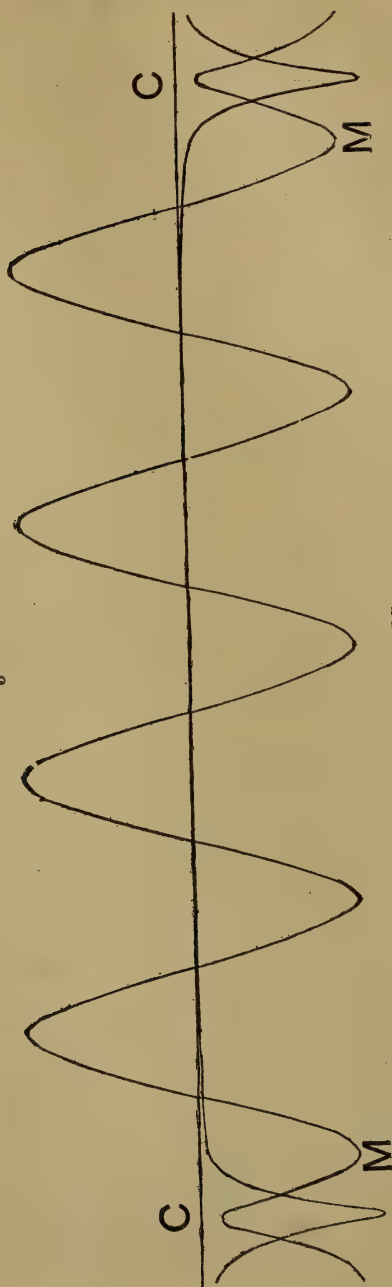
Fɪɢ. 15.
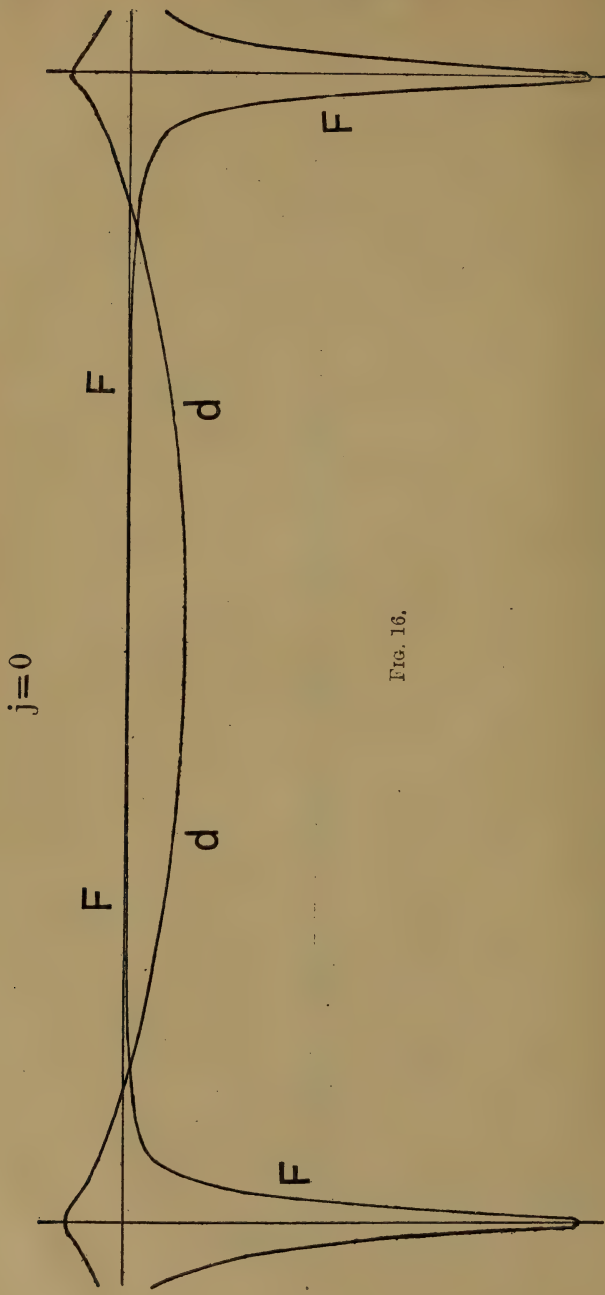
Fig. 16.

between C and C. Imagine the curve to be exactly sinusoidal throughout, and continued sinusoidally to cut the zero line at C C.

We should thus have in C C a train of $4\frac{1}{2}$ sinusoidal waves; and if the same is continued throughout the infinite procession .... C C C .... we have a discontinuous periodic curve made up of continuous portions each $4\frac{1}{2}$ periods of sinusoidal curve beginning and ending with zero. The change at each point of discontinuity C is merely a half-period change of phase. A slight alteration of this discontinuous curve within 60° on each side of each C, converts it into the continuous wavy curve of fig. 15, which represents the water-surface due to motion at speed $\sqrt{ga/9\pi}$ of the pressural forcive represented by the other continuous curve of fig. 15.

§ 49. Every word of § 48 is applicable to figs. 14 and 13 except references to *speed* of the forcive, which is $\sqrt{ga/19\pi}$ for fig. 14 and $\sqrt{ga/41\pi}$ for fig. 13; and other statements requiring modification as follows :—

For $4\frac{1}{2}$ " periods " or " waves," in respect to fig. 15; substitute $9\frac{1}{2}$ in respect to fig. 14, and $20\frac{1}{2}$ in respect to fig. 13.

For " depression " in defining M M in respect to figs. 15, 14 ; substitute *elevation* in the case of fig. 13.

§ 50. How do we know that, as said in § 48, the formula $\{(83), (86), (87)\}$ gives for a wide range of about 120° on each side of $\theta = 180°$,

$$\mathbf{d}(\theta) \doteq (-1)^j \mathbf{d}(180°) . \sin(j+\tfrac{1}{2})\theta \quad . \quad . \quad . \quad (88),$$

which is merely §§ 48, 49 in symbols? *it being understood that* j *is any integer not* < 4 ; *and that* e *is* ·9, *or any numeric between* ·9 *and* 1 ? I wish I could give a *short* answer to this question without help of hydrokinetic ideas ! Here is the only answer I can give at present.

§ 51. Look at figs. 12–16, and see how, in the forcive defined by $e = ·9$, the pressure is almost wholly confined to the spaces $\theta < 60°$ on each side of each of its maximums, and is very nearly null from $\theta = 60°$ to $\theta = 300°$. It is obvious that if the pressure were perfectly annulled in these last-mentioned spaces, while in the spaces within 60° on each side of each maximum the pressure is that expressed by (74), the resulting motion would be sensibly the same as if the pressure were throughout the whole space C C ($\theta = 0°$ to $\theta = 360°$), exactly that given by (74). Hence we must expect to find through nearly the whole space of 240°, from 60° to 300°, an almost exactly sinusoidal displacement of water-

surface, having the wave-length $360°/(j+\frac{1}{2})$ due to the translational speed of the forcive.

§ 52. I confess that I did *not* expect so small a difference from sinusoidality through the *whole* 240°, as calculation by {(83), (86), (87)} has proved ; and as is shown in figs. 18, 19, 20, by the D-curve on the right-hand side of C, which represents in each case the value of

$$D(\theta) = d(\theta) - (-1)^j d(180°). \sin (j+\tfrac{1}{2})\theta \quad . \quad (89),$$

being the difference of $d(\theta)$ from one continuous sinusoidal curve. The exceeding smallness of this difference for distances from C exceeding 20° or 30°, and therefore through a range between C C of 320°, or 300°, is very remarkable in each case.

§ 53. The dynamical interpretation of (88), and figs. 18, 19, 20, is this :—Superimpose on the solution {(83), (86), (87)} a " free wave " solution according to (73), taken as

$$-(-1)^j d(180°). \sin (j+\tfrac{1}{2})\theta \quad . \quad . \quad (90).$$

This approximately annuls the approximately sinusoidal portion between C and C shown in figs. (13), (14), (15) ; and approximately doubles the approximately sinusoidal displacement in the corresponding portions of the spaces C C, and C C on the two sides of C C. This is a very interesting solution of our problem § 41 ; and, though it is curiously artificial, it leads direct and short to *the* determinate solution of the following general problem of canal ship-waves :—

§ 54. Given, as forcive, the isolated distribution of pressure defined in fig. 12, travelling at a given constant speed ; required *the* steady distribution of displacement of the water in the place of the forcive, and before it and behind it ; which becomes established after the motion of the forcive has been kept steady for a sufficiently long time. Pure synthesis of the special solution given in §§ 1–10 above, solves not only the problem now proposed, but gives the whole motion from the instant of the application of the moving forcive. This synthesis, though easily put into formula, is not easily worked out to any practical conclusion. On the other hand, here is my present short but complete solution of the problem of steady motion for which we have been preparing, and working out illustrations in §§ 32–53.

Continue leftward, indefinitely, as a curve of sines, the D curve of each of figs. 18, 19, 20 ; leaving the forcive curve, F, isolated, as shown already in these diagrams. Or, analytically stated :—in (89) calculate the equal values of $d(\theta)$ for

Abscissas represent values of f
The line AB is asymptotic to $d(\theta)$

$d(\theta)$

$(-1)^j d(180°)$

$j = 20$ $q_{\jmath} = 4°\cdot39024$

Fig. 17.

Fig. 18.

$j = 9^\circ$
$v = q^\circ 4; 368$

Fig. 19.

$j = 4$
$q = 20^\circ$
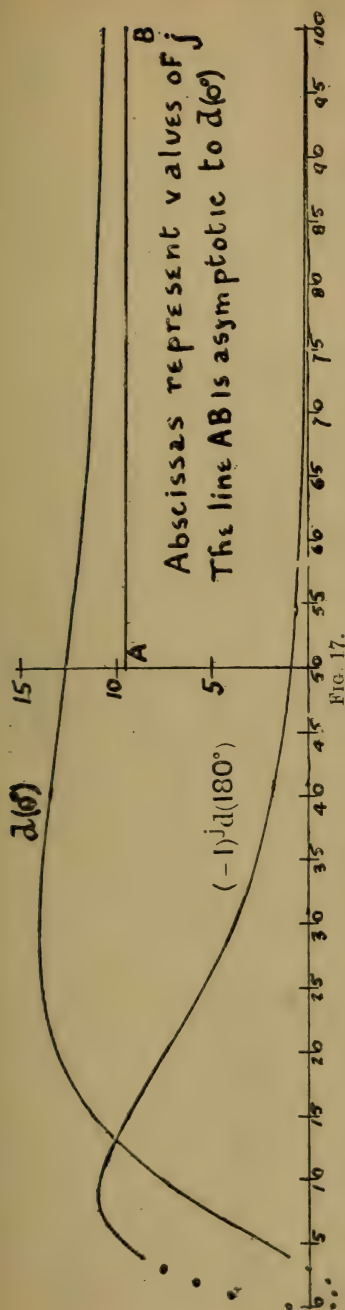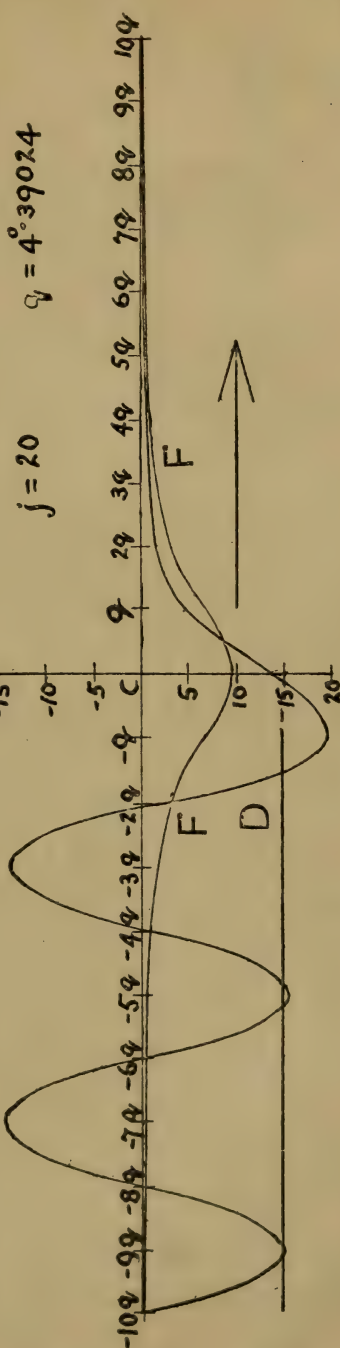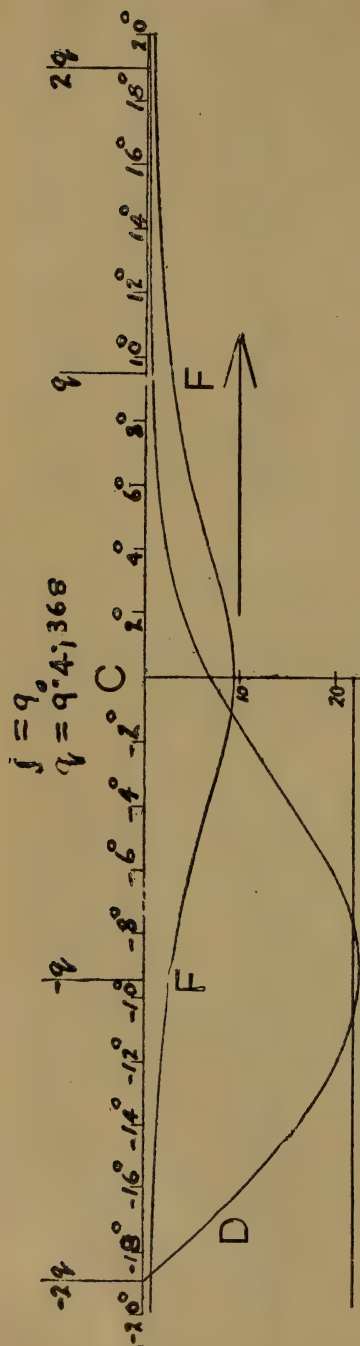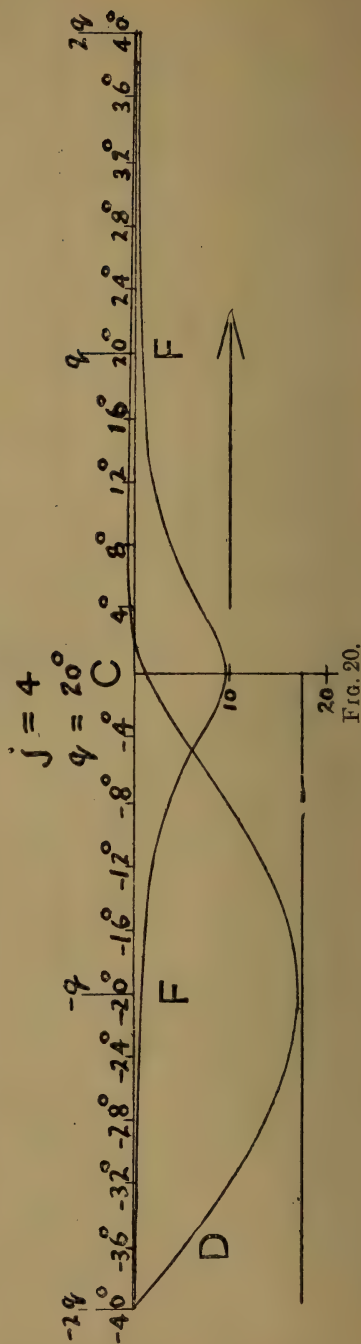
Fig. 20.

equal positive and negative values of θ from $0°$ to $40°$ or $50°$ by $\{(83), (86), (87)\}$; and for all larger values of θ take

$$\mathbf{d}(\theta) = (-1)^j \, \mathbf{d}(180°) \sin (j + \tfrac{1}{2})\theta \quad . \quad . \quad (91),$$

where $\mathbf{d}(180°)$ is calculated by $\{(83), (86), 87)\}$. This used in (89), makes $\mathbf{D}(\theta) = 0$ for all positive values of θ greater than $40°$ or $50°$; and makes it the double of (91) for all negative values of θ beyond $-40°$ or $-50°$.

§§ 55, 56. *Rigid Covers or Pontoons, introduced to apply the given forcive (pressure on the water-surface).*

§ 55. In any one of our diagrams showing a water-surface imagine a rigid cover to be fixed, fitting close to the whole water-surface. Now look at the forcive curve, F, on the same diagram, and wherever it shows no sensible pressure remove the cover. The motion (non-motion in some parts) of the whole water remains unchanged. Thus, for example, in figs. 13, 14, 15, 16, let the water be covered by stiff covers fitting it to $60°$ on each side of each C ; and let the surface be free from $60°$ to $300°$ in each of the spaces between these covers. The motion remains unchanged under the covers, and under the free portions of the surface. The pressure Π constituting the given forcive, and represented by the F curve in each case, is now automatically applied by the covers.

§ 56. Do the same in figs. 18, 19, 20 with reference to the isolated forcives which they show. Thus we have three different cases in which a single rigid cover, which we may construct as the bottom of a floating pontoon, kept moving at a stated velocity relatively to the still water before it, leaves a train of sinusoidal waves in its rear. The D curve represents the bottom of the pontoon in each case. The arrow shows the direction of the motion of the pontoon. The F curve shows the pressure on the bottom of the pontoon. In fig. 20 this pressure is so small at $-2q$ that the pontoon may be supposed to end there; and it will leave the water with free surface almost exactly sinusoidal to an indefinite distance behind it (infinite distance if the motion has been uniform for an infinite time). The F curve shows that in fig. 19 the water wants guidance as far back as $-3q$, and in fig. 18 as far back as $-8q$ to keep it sinusoidal when left free ; q being in each case the quarter wave-length.

§§ 57–60. *Shapes for Waveless Pontoons, and their Forcives.*

§ 57. Taking any case such as those represented in figs. 18,
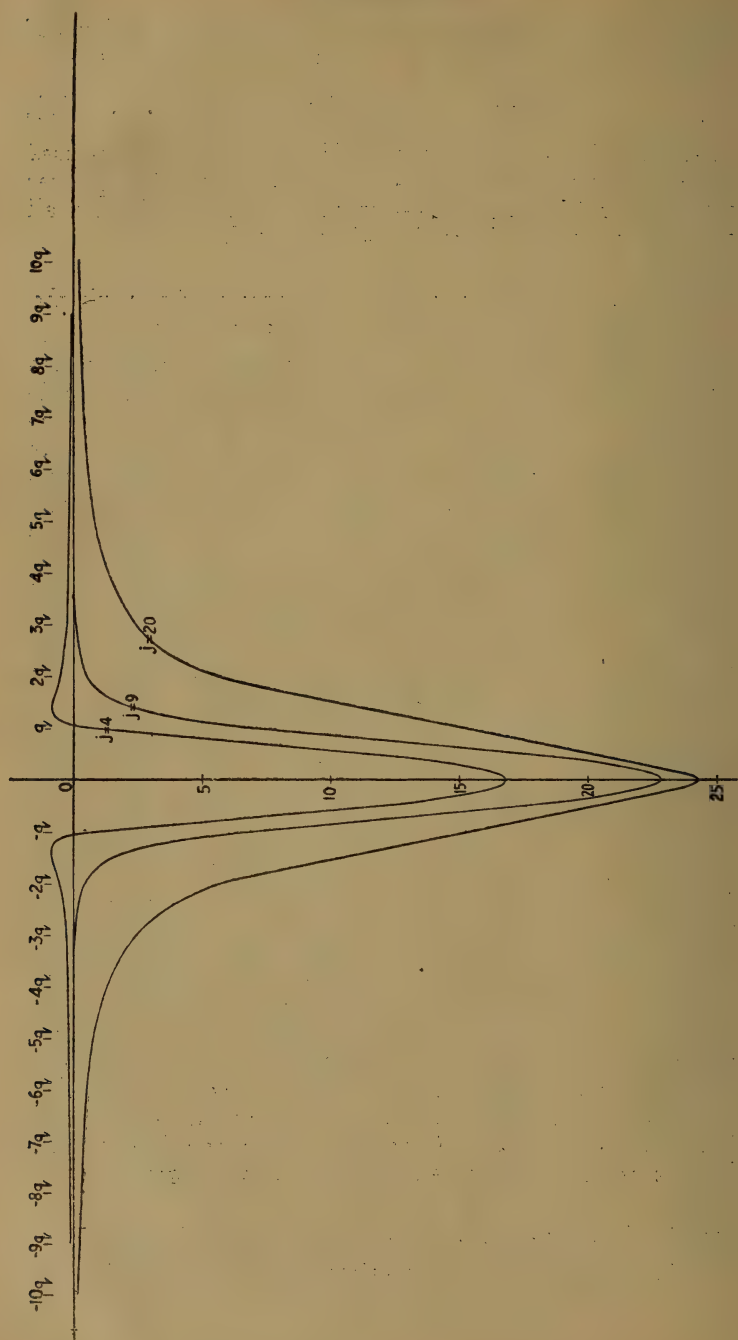
FIG. 21.

19, 20; we see obviously that if any two equal and similar forcives are applied, with a distance $\frac{1}{2}\lambda$ between corresponding points, and if the forcive thus constituted is caused to travel at speed equal to $\sqrt{g\lambda/2\pi}$, being, according to (77) above, the velocity of free waves of length λ, the water will be left waveless (at rest) behind the travelling forcive.

§ 58. Taking for example the forcives and speeds of figs. 18, 19, 20, and duplicating each forcive in the manner defined in § 57, we find (by proper additions of two numbers, taken from our tables of numbers calculated for figs. 18, 19, 20), the numbers which give the depressions of the water in the three corresponding waveless motions. These results are shown graphically in fig. 21, on scales arranged for a common velocity. The free wave-length for this velocity is shown as $4q$ in the diagram.

§ 59. The three forcives, and the three waveless water-shapes produced by them, are shown in figs. 22, 23, 24 on different scales, of wave-length, and pressure, chosen for the convenience of each case.

§ 60. As most interesting of the three cases take that derived from $j = 9$ of our original investigation. By looking at fig. 23 we see that a pontoon having its bottom shaped according to the D curve from $-3q$ to $+3q$, $1\frac{1}{2}$ free wave-lengths, will leave the water sensibly flat and at rest if it moves along the canal at the velocity for which the free-wave-length is $4q$. And the pressure of the water on the bottom of the pontoon is that represented hydrostatically by the F curve.

§ 61. Imagine the scale of abscissas in each of the four diagrams, figs. 21–24, to be enlarged tenfold. The greatest steepnesses of the D curve in each case are rendered sufficiently moderate to allow it to fairly represent a real water-surface under the given forcive. The same may be said of figs. 15, 16, 18, 19, 20; and of figs. 13, 14 with abscissas enlarged twentyfold. In respect to mathematical hydrokinetics generally; it is interesting to remark that a very liberal interpretation of the condition of infinitesimality (§ 36 above) is practically allowable. Inclinations to the horizon of as much as $1/10$ of a radian ($5°\!\cdot\!7$; or, say, $6°$), in any real case of water-waves or disturbances, will not seriously vitiate the mathematical result.

§ 62. Fig. 17 represents the calculations of $\mathbf{d}(0°)$ and $(-1)^j\mathbf{d}(180°)$ for twenty-nine integral values of j; 0, 1, 2, 3, ... 19, 20, 30, 40, ... 90, 100; from the following formulas, found by putting $\theta = 0°$ and $\theta = 180°$; and with $e = \cdot 9$ in each case, and $c = 1$

$$\mathbf{d}^j(0°) = (2j+1)e^j\left[-\tfrac{1}{2}e^{\frac{1}{2}}\log\frac{1+\sqrt{e}}{1-\sqrt{e}}+1+\frac{e^{-1}}{3}+\frac{e^{-2}}{5}+\ldots+\frac{e^{-j+1}}{2j-1}\right.$$
$$\left.+\tfrac{1}{2}\frac{e^{-j}}{2j+1}\right] \; (92),$$

$$\mathbf{d}^j(180°) = (-1)^j(2j+1)e^j\left[\tfrac{1}{2}e^{\frac{1}{2}}\tan^{-1}\frac{2\sqrt{e}}{1-e}+1-\frac{e^{-1}}{3}+\frac{e^{-2}}{5}+\ldots\right.$$
$$\left.+(-1)^{j-1}\frac{e^{-j+1}}{2j-1}+(-1)^j\tfrac{1}{2}\frac{e^{-j}}{2j+1}\right] \; (93).$$

The asymptote of $\mathbf{d}(0°)$ shown in the diagram is explained by remarking that when j is infinitely great, the travelling velocity of the forcive is infinitely small; and therefore, by end of § 41, the depression is that hydrostatically due to the forcive pressure. This, at $\theta=0°$, is equal to

$$\tfrac{1}{2}\frac{1+e}{1-e}c = \frac{1\cdot9}{2}c = 9\cdot5.c.$$

§ 63. The interpretation of the curves of fig. 17 for points between those corresponding to integral values of j is exceedingly interesting. We shall be led by it into an investigation of the disturbance produced by the motion of a single forcive, expressed by

$$\Pi = \frac{gcb}{b^2+x^2} \quad \ldots \quad \ldots \quad (94);$$

but this must be left for a future communication, when it will be taken up as a preliminary to sea ship-waves.

§ 64. The plan of solving by aid of periodic functions the two-dimensional ship-wave problem for infinitely deep water, adopted in the present communication, was given in Part IV. of a series of papers on Stationary Waves in Flowing Water, published in the Philosophical Magazine, October 1886 to January 1887, with analytical methods suited for water of finite depths. The annulment of sinusoidal waves in front of the source of disturbance (a bar across the bottom of the canal), by the superposition of a train of free sinusoidal waves which double the sinusoidal waves in the rear, was illustrated (December 1886) by a diagram on a scale too small to show the residual disturbance of the water in front, described in § 53 above, and represented in figs. 18, 19, 20.

In conclusion, I desire to thank Mr. J. de Graaff Hunter for his interested and zealous co-operation with me in all the work of the present communication, and for the great labour he has given in the calculation of results, and their representation by diagrams.
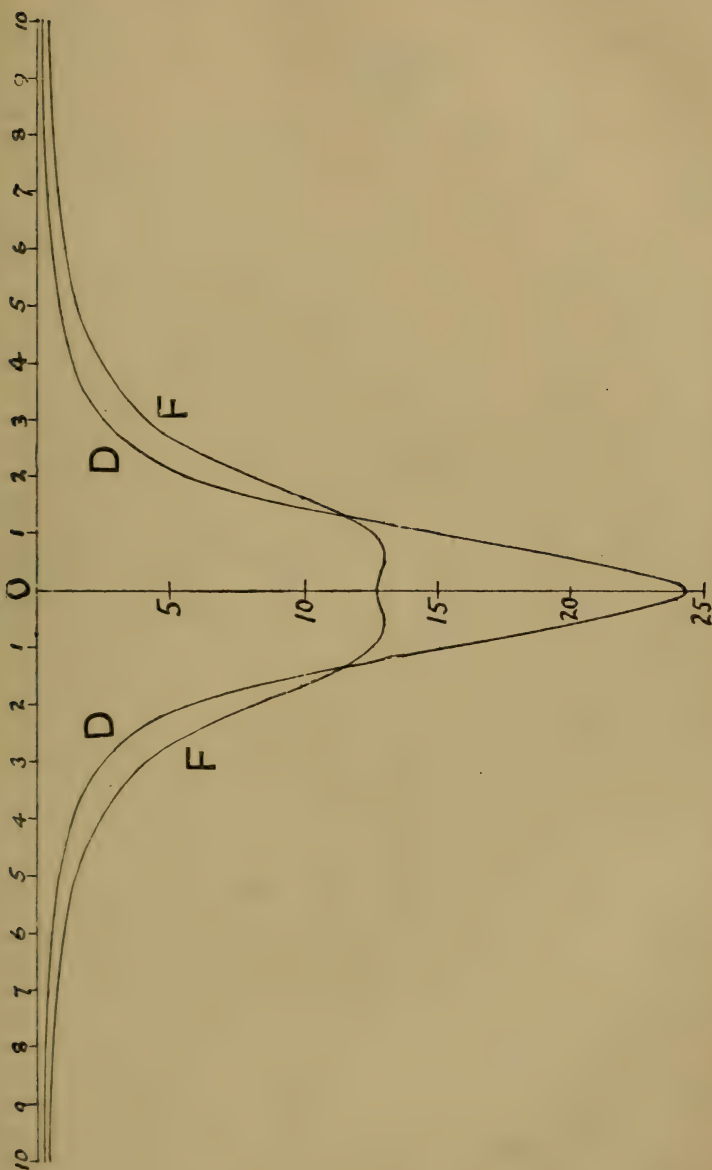
Fig. 22 ; j = 20.—Scale of abscissas is quarter-wave-lengths.
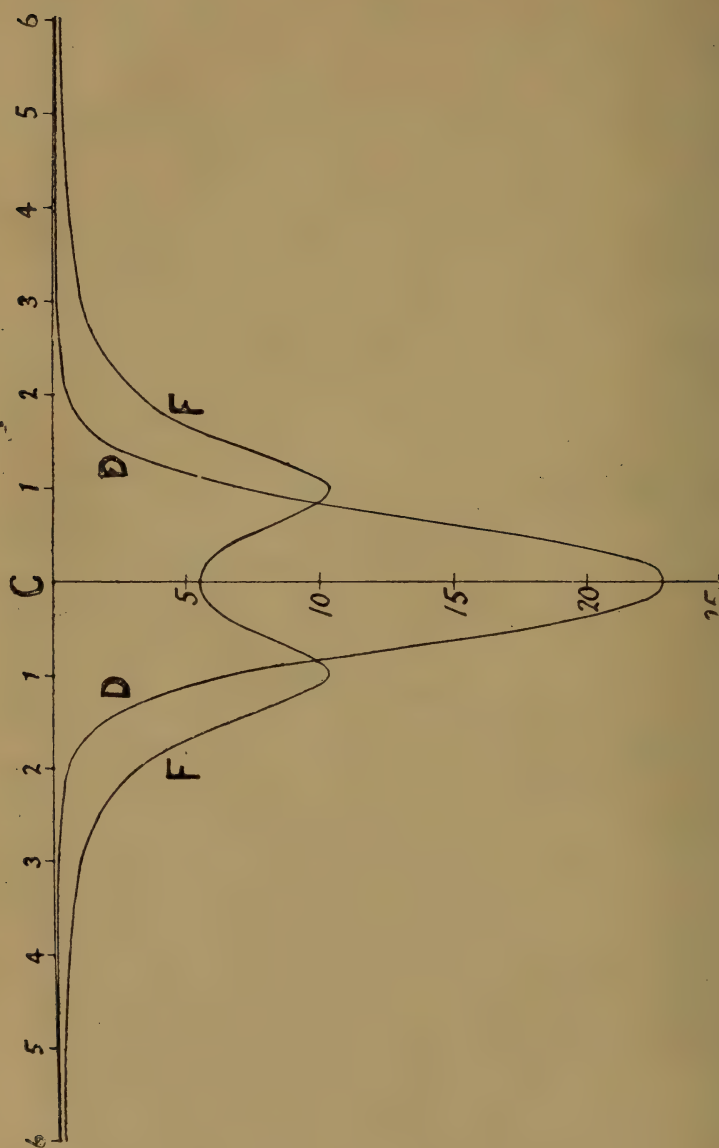
Fig. 23; j = 9.
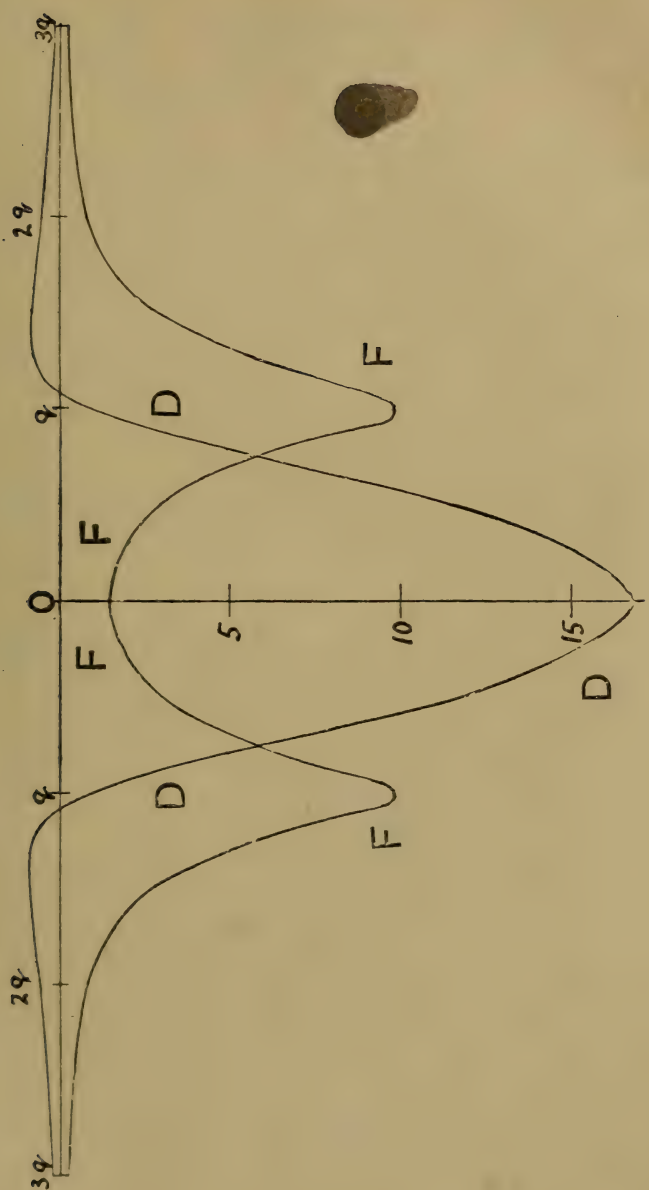
FIG. 24; j = 4.—Scale of abscissas is quarter-wave-lengths.

LXXII. *The Application of the Cymometer to the Determination of the Coefficient of Coupling of Oscillation Transformers. By* J. A. FLEMING, *M.A., D.Sc., F.R.S., Professor of Electrical Engineering in University College, London* [*].

[Plate XI.]

THE author has already described one form of direct-reading cymometer for the determination of wave-lengths and frequencies in connexion with electric-wave telegraphy[†]. Since that publication another more compact form of instrument has been designed by him, adapted for measuring a larger range of oscillation constant.

The principle on which the above-mentioned appliance works is as follows :—If there be any circuit, open or closed, in which electrical oscillations are taking place, we can set up oscillations of identical frequency in another inductively coupled and adjustable circuit, provided that the mutual inductance of the two circuits is very small.

If this last circuit can have its inductance (L) and capacity (C) varied, and if the values of these quantities are known, the maximum resonance will be excited when

$$p = 2\pi n = 1/\sqrt{\overline{CL}},$$

where n is the frequency in the primary or tested circuit. This simple relation holds good, however, only when the mutual inductance of the two circuits is sufficiently small.

For the sake of brevity the author calls the quantity $\sqrt{\overline{CL}}$ the *oscillation constant* of the circuit. Let us assume then that we have a circuit consisting of an inductance-coil and a condenser in series with each other, the circuit being closed on itself. Let part of this circuit consist of a straight wire or rod which can be placed near to any circuit in which oscillations are taking place. Then these last induce others in the first-named circuit. Suppose that means exist for varying together both the inductance (L) and the capacity (C) of this closed circuit ; we can bring it into a condition in which its oscillation constant, viz. $\sqrt{\overline{CL}}$, has such a relation to the frequency of the oscillations in the primary circuit that $n = 1/2\pi\sqrt{\overline{CL}}$, and then the maximum potential difference will exist between the plates of the condenser forming the closed circuit. This critical condition may be ascertained

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[*] Communicated by the Physical Society : read March 24, 1905.
[†] See J. A. Fleming, "On an Instrument for the Measurement of the Length of Long Electric Waves, and also Small Inductances and Capacities," Proc. Roy. Soc. Lond. vol. lxxiv. p. 488 (1905).

by the employment of a vacuum-tube of the spectroscopic type preferably containing rarefied Neon, which is connected to the plates of the condenser.

Such an arrangement constitutes the instrument called by the author a cymometer *.

In its practical and most recent form it is constructed as follows :— On a mahogany baseboard about 136 centims. long and 30 wide is mounted an inductance-coil H, H (see Plate XI.). This consists of an ebonite tube 108 centims. long and 3 cms. diameter, having a helical groove cut on its surface, with 4 turns to the centimetre (10 to the inch exactly). In this groove is wound a bare copper wire 1·22 mm. in diameter, the ends of which are secured to brass collars clamped on the ebonite tube. The inductance L of such a spiral can be calculated very approximately by the formula

$$L = (\pi DN)^2 \, l,$$

where D is the mean diameter of one circular turn of the helix and N the number of turns of wire per centimetre or otherwise, πDN represents the length of wire which is wound on one unit of length (1 cm.) of the helix, and l is the length of the helix.

This helix is supported by wood brackets, a, b, on the base-board.

Parallel to it is fixed a sliding tubular condenser. This consists of a brass tube I, having a solid pin fixed in one end by means of which it is clamped in ebonite clips fixed to a wooden block c. This tube has a length of 104 centims. and an outside diameter of 2·49 centims. Over this brass tube is fixed tightly a thin ebonite tube E, the sides of which are only 1·6 mm. thick, and this extends for a length of 5 centims. beyond the inner brass tube. Over this ebonite tube slides easily another brass tube O, 100 centims. long and 3·05 centims. outside diameter. This tube has a heavy collar, k, at one end, to which is attached an ebonite handle, h, by which to displace the tube, and also a projecting pin, l, which carries a semicircular collar, K, resting on the inductance spiral HH. Also this pin or rod carries an index P, which moves over a divided scale SS.

The instrument is completed by a copper bar, $L_1 \, L_2 \, L_3$, of square section, 6 mm. in side, which joins one end of the inductance spiral to the inner tube of the condenser. The outer tube of the condenser moves through an insulated metal ring, g, and from this ring and from a terminal T at

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* From κῦμα, a wave. The author is indebted to his colleague Prof. Platt, M.A., for the correct form of this word.

the end of the inner condenser-tube stout wires are brought to the ends of a vacuum-tube V.

It will then be seen that if the handle is moved along, the outer condenser-tube slides off the inner one, thus reducing the capacity of the condenser by an amount which is almost exactly proportional to the displacement of the handle.

Experiment shows that the measured capacities of the tubular condenser with the handle in various positions plot out into a nearly straight line in terms of the displacement of the outer tube. Again, the same movement of the handle reduces proportionately the amount of inductance included in the closed circuit, because the inductance of that portion of the spiral included between the collar at the end b and the semicircular clip K is almost exactly proportional to the length of the spiral included. Therefore the oscillation constant of the circuit or the quantity $\sqrt{CL}$ for various positions of the handle is proportional to the displacement, and can be marked on the scale.

In the instrument here described the scale covers a range of oscillation constant from 0 to 12, and it is almost exactly equidivisional.

The scale values have been determined experimentally by measuring the capacity of the sliding condenser and the inductance of the spiral for various positions of the handle, by methods already described to the Physical Society *.

If, then, we have any circuit, open or closed, in which oscillations are taking place, we can determine their frequency if we place the copper bar forming part of the cymometer circuit near and parallel to the first-named circuit, and move the handle until the vacuum-tube shines most brilliantly. The scale-reading will then show us the oscillation constant of the cymometer circuit in that position. Since the frequency corresponding to that reading is related to the constant as the abscissæ and ordinates of a rectangular hyperbola, the scale of the cymometer can also be graduated to show frequencies as well. Again, if the circuit tested is an open circuit which is radiating electric waves, there is a definite relation between the length of the radiated waves and their frequency expressed by $V = n\lambda$, where V is the velocity of electromagnetic radiation. Hence the scale of the cymometer can also be graduated to show wave-lengths directly, and it becomes a direct-reading electric-wave meter for measuring wave-lengths as used in wireless telegraphy.

* See J. A. Fleming and W. C. Clinton, " On the Measurement of Small Inductances and Capacities," Phil. Mag. ser. 6, vol. v. p. 493 (1903) ; also Proc. Phys. Soc. Lond. vol. xviii. p. 386 (1903).

The instrument has also uses in the laboratory such as determining capacities, inductances, and coefficients of coupling of induction-coils.

For this purpose, the box in which the instrument is packed has a rectangular-shaped circuit of insulated wire ABCD (No. 16 S.W.G.) fixed to the lid. This rectangular-shaped circuit is interrupted in two places, and tails of wire left as in fig. 1,

Fig. 1.
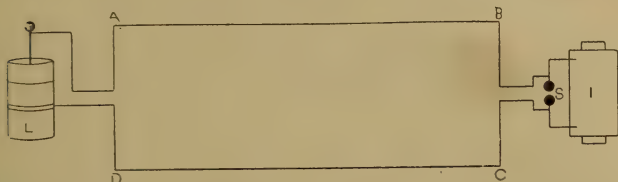
This circuit is given a form such that its inductance for high frequency currents can be calculated. It is intended to be used as follows :—Suppose we desire to determine the capacity of a small leyden-jar for high frequency currents. The jar is connected by outside and inside to one pair of ends of the rectangle, and the other pair of ends are connected to the secondary spark-balls of an induction-coil. We then place the cymometer-bar near and parallel to one long side of the rectangle, and set up oscillations in the leyden-jar circuit with the induction-coil. Next, the cymometer handle is moved until the vacuum-tube (preferably a Neon tube) glows most brightly, and the value of the oscillation constant read off on the scale. We then know that this also must be the oscilllation constant of the jar circuit.

The value of the high frequency inductance of the rect-angular circuit is given with the instrument, and hence if we call this value L_1 centimetres and the capacity of the jar C_1 microfarads, and the scale-reading of the oscillation constant of the cymometer corresponding therewith O_1, we have

$$C_1 = \frac{O_1^2}{L_1}.$$

The calculated value of the inductance of the rectangular circuit is obtained by the following formula.

We consider in the first place two equal rectangular circuits the sides of which are respectively S and S' in length, placed with planes parallel and at a distance b apart. If these circuits are filamentary circuits, then, starting with the Neumann formula for the mutual potential energy

M of such circuits, viz.,

$$M = \iint \frac{ds\,ds'}{r}\cos\theta,$$

where r is the distance of any pair of elements ds and ds' of the two circuits and θ their inclination, we extend the integration over every possible pair of elements in two rectangular circuits. It is not difficult to arrive at the following result for the mutual potential energy of the two rectangular circuits conveying unit currents, viz.[*]

$$M = 4\left[S\log\frac{(S + \sqrt{S^2 + b^2})\,\sqrt{S'^2 + b^2}}{(S + \sqrt{S^2 + S'^2 + b^2})b} + S'\log\frac{(S' + \sqrt{S'^2 + b^2})\,\sqrt{S^2 + b^2}}{S' + \sqrt{S'^2 + S^2 + b^2})b} \right.$$
$$\left. + 2\sqrt{S^2 + S'^2 + b^2} - 2\sqrt{S'^2 + b^2} - 2\sqrt{S^2 + b^2} + 2b \right].$$

To obtain the inductance of a rectangular-shaped circuit, we have then to obtain the mutual potential energy of two rectangular circuits conveying unit currents placed at a distance apart equal to the *geometric mean distance* of all the filamentary currents into which we can suppose the real current in the real circuit divided. If we consider that circuit to be a circular-sectioned wire and the currents to be high frequency, then, as Maxwell shows, the geometric mean distance of all points lying on the circumference of a circle is equal to its radius[†].

Hence, in the above formula, if we make b small compared with S or S', and substitute for b the value $d/2$, where d is the diameter of the round wire, we shall have an expression for the inductance L_1 of a rectangular-shaped circuit of such wire for currents of infinite frequency. The expression is as follows:—

$$L_1 = 4\left[(S + S')\log\frac{4SS'}{d} - S\log(S + \sqrt{S^2 + S'^2}) \right.$$
$$\left. - S'\log(S' + \sqrt{S^2 + S'^2} + 2\sqrt{S^2 + S'^2} - 2(S + S') \right] (1)$$

The above formula is a strictly accurate one for infinite frequency, and can easily be applied to any case of a real rectangular circuit. The logarithms are of course Napierian.

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[*] There is no need to give the various steps of the integration as substantially the above formula is given in Mascart and Joubert's Treatise on Electricity and Magnetism, Atkinson's English Translation, vol. ii. p. 477.

[†] See Clerk Maxwell's Treatise on Electricity and Magnetism, 2nd ed. vol. ii. p. 298.

We can therefore construct a rectangular circuit of wire attached to the lid of the box of the cymometer, which has a known predetermined inductance of say 5000 centims. Strictly speaking there is a small correction for the tails of parallel wire which connect the rectangle to the jar at one end and to the coil at the other. If considered necessary, this may be taken into account by employing a reduced case of the above formula for the inductance of the rectangle. If there be a pair of round wires of diameter d placed at a distance D apart, the inductance for a length l of the parallel wires is given by the formula

$$L' = 4l \left(\log \frac{2D}{d} \right) \quad . \quad . \quad . \quad . \quad (2)$$

Hence, if the length of the tails of wire at each end of the rectangle is the same and equal to l, the inductance of the whole circuit is equal to $L_1 + 2L'$, where L_1 and L' have the values given by the formulæ above.

If we wish to determine the high frequency inductance of a short length of wire, say a loop of copper wire of one or two turns, we proceed as follows:—We insert this loop in series with the rectangular circuit and with a condenser of which the capacity has been determined, and employ the cymometer as above described to determine the oscillation constant of the circuit. Then let L_2 be the inductance of the loop of wire, and L_1 that of the rectangle, and O_1 the observed oscillation constant when L_1 is used alone, and O_2 when L_1 and L_2 are in series. Then we have

$$O_1{}^2 = CL_1$$
$$O_2{}^2 = C(L_1 + L_2);$$
whence
$$L_2 = \frac{O_2{}^2 - O_1{}^2}{C}.$$

We can always check the result by using as a loop some form of circuit, of which the inductance can be calculated.

Thus if we bend a bare round-sectioned copper wire into a square with the ends brought quite near together, we can predetermine its inductance.

We have here a reduced case of the general formula for a rectangular circuit. In equation (1) above put $S = S'$ and put $4S = l$, then the formula reduces to

$$L = 2l \left(\log \frac{4l}{d} - 2 \cdot 853 \right) \quad . \quad . \quad . \quad . \quad (3)$$

Strictly speaking we should add to the value of the expressions (1) and (3) for the inductance of a rectangle and a square a term equal to $R'/2\pi n$, where R' is the high frequency resistance corresponding to a frequency n. The formulæ (1) and (3) as they stand give the inductance for infinite frequency. The value of $R'/2\pi n$ is, however, generally negligible compared with the other term, and the expressions given may be taken to be the inductances for any frequency of the order of 10^6.

In the next place, we may employ the same instrument to determine the coefficients of coupling of the circuits of an air core-transformer such as an oscillation transformer used in wireless telegraphy. Suppose the inductance of the primary circuit to be denoted by L, that of the secondary by N, and the mutual inductance by M. Then $M/\sqrt{LN}$ is called the *coefficient of coupling*, and is a quantity of importance in the theory of high-frequency transformers.

We may join the two circuits of the oscillation transformer into one circuit so that they assist or oppose each other in creating co-linked flux. In one case the effective inductance is equal to $L+2M+N$, and in the other case it is $L-2M+N$.

Hence if we treat the oscillation transformer so joined up in the two ways, and measure as above its effective inductances, and call them L_1 and L_2, we have

$$L_1 = L + 2M + N,$$
$$L_2 = L - 2M + N.$$

Hence
$$M = \frac{L_1 - L_2}{4},$$

and
$$L + N = \frac{L_1 + L_2}{2}.$$

We can then determine directly and independently the larger of the two inductances L or N, and hence we can calculate the value of $M/\sqrt{LN}$ or the coefficient of coupling of the circuits. As an instance of such a determination we may give the measurements made with a form of oscillation transformer used in wireless telegraphy. The primary circuit consisted of one single turn composed of 8 turns of 7/22 insulated copper wire in parallel wound round a square wooden frame. The secondary circuit consisted of turns of the same stranded wire wound over the primary circuit. The resultant inductances were measured by the cymometer with the circuits joined up to add and oppose each other.

The measured values were as follows :—

$$L_1 = L + 2M + N = 62576 \text{ centims.}$$
$$L_2 = L - 2M + N = 49621 \quad \text{,,}$$
$$N = 55445 \quad \text{,,}$$

Whence we have $\qquad M = 3239 \quad$,,

and $\qquad L + N = 56098 \quad$,,

therefore $\qquad L = 653 \quad$,,

and $\qquad N = 55445 \quad$,,

Therefore $\qquad M/\sqrt{LN} = 0\cdot54$

The coupling would therefore be called " close," as it is usual to call the coupling " close " or " tight " when the coefficient exceeds 0·5, and " loose " when it is smaller.

The theory of the instrument is involved in that of oscillation transformers generally. The case of two inductively coupled circuits, each consisting of an inductance of negligible resistance placed in series with a condenser has been fully treated by A. Overbeck *. It has also been discussed by G. Seibt. If we assume, as we may do, that all the currents and electromotive forces vary in simple sinoidal manner, we may employ complex quantities to represent the vectors with which we are concerned. Suppose then that V represents the potential-difference of the condenser-plates, I the current in the circuit, whilst C, L, and M denote the capacity, inductance, and mutual inductance. The symbols stand for the maximum values of the periodic quantities, and suffixes 1 or 2 have reference to the primary or secondary circuits. Then neglecting the energy dissipation by resistance we can write the vector equations connecting these variables and constants for the two circuits in the form

$$V_1 + jpL_1I_1 + jpMI_2 = 0,$$
$$V_2 + jpL_2I_2 + jpMI_1 = 0,$$
$$I_1 = jpC_1V_1,$$
$$I_2 = jpC_2V_2,$$

where j has the usual significance of the sign of perpendicularity, and $p = 2\pi$ times the frequency.

Then, eliminating I_1, I_2, V_1, V_2, we have

$$p^4 - p^2 \frac{C_1 L_1 + C_2 L_2}{C_1 C_2 (L_1 L_2 - M_2)} + \frac{1}{C_1 C_2 (L_1 L_2 - M_2)} = 0.$$

Hence

$$p^2 = \frac{1}{2 C_1 C_2 (L_1 L_2 - M^2)} \left\{ (C_1 L_1 + C_2 L_2) \mp \sqrt{(C_1 L_1 - C_2 L_2)^2 + 4 C_1 C_2 M^2} \right\}$$

Suppose now that the two circuits have independently and separately the same oscillation constant or same natural time-period T_0. Then $C_1 L_1 = C_2 L_2 = CL$, and if we call k the co-efficient of coupling so that $M^2 = k^2 L_1 L_2$, we have for the value of p^2 the expression

$$p^2 = \frac{1}{CL} \cdot \frac{1 \pm k}{1 - k^2}.$$

Hence if k has a value different from zero, there will be two oscillations of different frequency induced which have frequencies given by the equations

$$n_1 = n_0 \frac{1}{\sqrt{1 - k}},$$

$$n_2 = n_0 \frac{1}{\sqrt{1 + k}},$$

where $n_0 = 1/2\pi \sqrt{CL}$ is the natural frequency of each circuit separately. Consequently

$$n_1^2 (1 - k) = n_0^2, \quad n_2^2 (1 + k) = n_0^2$$

are two equations which determine the relation of the component frequencies of the complex oscillation set up when two circuits of equal natural time-period are inductively connected. It is evident therefore, that if a circuit has in it oscillations of a certain frequency n_0, and we couple it inductively with another circuit which can be adjusted to have the same oscillation constant ($\sqrt{CL}$); in order that oscillations of only one single frequency equal to n_0 should be induced in this adjustable circuit, it is essential that the coefficient of coupling k of the two circuits should be very small. Otherwise two oscillations of different frequency are excited, the frequency of one being greater and that of the other being less than that of the free independent original frequency n_0 it is desired to determine.

In the form of cymometer here described, this necessary

result is obtained by making the mutual inductance between the cymometer and the circuit being tested very small. We have then to employ a sensitive detector for the condition of resonance, viz., a Neon vacuum-tube. This condition is not fulfilled if the whole inductance of the cymometer takes part in creating mutual inductance. The necessary condition cannot be easily satisfied if the detecting arrangement consists of any form of hot-wire ammeter. In using a cymometer for measuring the frequency of the oscillation in any circuit, we have to be on our guard against disturbing the very quantity we wish to measure, or setting up in the cymometer circuit some oscillation of a different frequency. It is an obvious deduction from the above investigation, that in using the cymometer we should place the bar of the cymometer as far away as possible from the circuit being tested. Suppose, however, that we have two circuits of the same time-period when separated and we couple them together inductively. Then, if we investigate with the cymometer the oscillations set up in the secondary circuit, we find it to be a complex oscillation resolvable into components of different periods. The cymometer therefore acts just like an electrical spectroscope. It resolves the complex vibrations in a circuit into their simple components and shows us what they are. This effect is very marked in the case of inductively coupled aerials in wireless telegraphy. If we have a nearly closed condenser circuit with spark-gap in which oscillations are set up, which is inductively coupled to an aerial or antenna, then, even if the two circuits are in common language "tuned" to each other so that they have the same independent time-period, yet when coupled, if coupled at all tightly, there are two oscillations set up in the aerial of different frequencies, and two waves radiated of different wave-length, which may differ in length by 15 or 20 per cent.

The cymometer in the above described form can be used to measure not merely the length of the outgoing wave from a sending aerial, but also the length of the wave being received. To do this a coil of a few turns is wrapped around the long helical inductance-coil, and this is connected between the earth and the receiving aerial. The vacuum-tube is then replaced by any form of electric-wave detector such as a coherer or Marconi magnetic detector, and the cymometer circuit adjusted by moving the handle until the maximum effect is produced, or until the receiver indicates when this is the case, the scale-reading of the cymometer will indicate the length of the arriving waves.

LXXIII. *The Production of Radium from Uranium.*
By FREDERICK SODDY, *M.A.* *

SINCE the theory was brought forward that radioactivity is accompanied by the change of the atom of the radio-element (Rutherford & Soddy, Phil. Mag. [6] v. 1903, p. 576), it has been recognized that the rate of change of radium is sufficiently rapid to cause the practically complete disappearance of any initial quantity of radium during the course of at most a few tens of thousands of years, so that the theory, if correct, must be able to account for the continued existence of this element. The rate of change of an element, like its atomic weight, seems to be a completely invariable and unvarying constant of nature, and there is no evidence to justify the view that the processes which accompany radioactivity have commenced at a late and recent period of the earth's history. On the other hand, the case of radium is essentially similar to that of all the other types of radioactive matter. It is changing much more rapidly than the elements uranium and thorium, but far slower than polonium. Any initial quantity of polonium would, according to the recent accurate determinations of the rate of change of this element (*Ber. d. D. Chem. Gesell.* 1905, p. 592) be reduced to less than one-tenth per cent. after the lapse of only five years, while the times for the same to occur in the case of the emanations of radium, thorium, and actinium respectively are 40 days, 10 minutes, and 37 seconds. To these cases, the idea of the comparatively recent origin of radioactive processes is of course inapplicable. From the first it was obvious that the conception of the continuous reproduction of the more rapidly changing types from the slower-changing so-called "parent-elements," which was arrived at by the study of the radioactive emanations and other similar types, embraced also the cases of radium and polonium. If, for example, radium and polonium are themselves intermediate or transition forms of elementary matter, which are being reproduced as fast as they themselves change, by the excessively slow disintegration of some parent element of higher atomic weight, not only the continued existence, but also the extremely minute quantity in which radium and polonium occur in nature is at once explained. As only two elements, thorium and uranium, are known with atomic weight greater than that of radium, the choice of a possible parent was

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* Communicated by the Author.

practically restricted to these two elements. Considerations analogous to those which indicated helium as one of the ultimate products of radioactive change, pointed strongly to uranium being the parent element of radium and polonium. The latter elements are only to be found in the uranium minerals, and only those richest in the latter contain sufficient of the new elements to make the extraction profitable. Since radium and polonium are transition forms, they do not, like helium, accumulate to an indefinite extent with the lapse of time. In any disintegration series equilibrium is attained, and the relative quantities of the various members of the series do not further alter when certain definite proportions of the transition forms have accumulated. These proportions expressed in terms of the quantity of the parent element are the inverse ratios of the rates of change of the transition forms and parent elements respectively. From the relative radioactivity of uranium and radium, it followed that the rate of change of the former is about one-millionth of that of the latter, so that the quantity of radium accumulating in a mineral should be about one-millionth of the quantity of uranium present, and this ratio should be independent of the age of the mineral. This is approximately the value found experimentally by M. and Mme. Curie, and by Giesel, so that from the first there was a strong presumption in favour of the view that uranium in its own disintegration is continually producing radium. Rutherford has recently (Phil. Mag. [6] viii. 1904, p. 636) brought forward direct experimental evidence that polonium is one of the disintegration products of radium ; and in this paper similar direct experimental evidence is given to show that the radium itself results from the disintegration of uranium. The continuous disintegration series thus revealed, starting with the heaviest element known, embraces the majority of the known types of radioactive matter, and although direct experimental evidence is still lacking, probably ends, so far as the manifestation of radioactive phenomena is concerned, in the ultimate production of one of the heaviest non-radioactive elements, bismuth or lead. The time occupied in the passage, after the initial change of the uranium until the final change of polonium, during which the changing matter is present as intermediate forms, is a very considerable one, the average time being probably of the order of ten thousand years.

On this account experiments by a direct method, in which a reproduction of radium is looked for in uranium originally freed from that element, must be continued over a long

period before a decisive result is attained. The experiments here described were started in May 1903. At first only negative results were obtained, but later, after a kilogram of uranium nitrate had been kept under observation for a period of eighteen months, undeniable evidence of a positive character was obtained, although a much slower rate of production of radium from uranium was indicated than was to be expected if a direct change of uranium or uranium X into radium occurred. These two results were briefly announced in letters to ' Nature ' (May 12th, 1904, and January 26th, 1905). Whetham (' Nature,' May 5th, 1904, and Feb. 2nd, 1905) has given the result of some experiments undertaken on the same problem, and was able to confirm the positive result announced in the second letter.

During the progress of these investigations, the problem has been attacked by a different method by other investigators, to whose work some reference will be made later in the paper. The method is based upon the accurate determination of the ratio of relative quantities of uranium and radium in a large number of minerals, to see whether the requirement that the two elements always exist in a fixed relative proportion is fulfilled.

Radium is distinguished from uranium in giving a relatively short-lived gaseous disintegration product or emanation, which is characterized by certain well-marked peculiarities. The emanation may therefore be used as a qualitative test for the presence of radium. Since the laws regulating the rate of production and accumulation of this emanation are well known (Rutherford & Soddy, Phil. Mag. [6] v. 1903, p. 450), the amount of emanation accumulating in a known time can be used as a quantitative measure of the amount of radium present. The research divides itself into four sections. In the first place it will be shown that a quantity of uranium originally practically free from radium, and possessing therefore no power of producing an emanation, after being kept a certain time in a closed bottle develops an unmistakeable though still small emanating power. Secondly, the characteristics of the emanation produced will be shown to conform to those of the radium emanation. Thirdly, experiments will be given in which the quantity of emanation produced from the uranium was compared with the quantity produced from a known weight of radium, and the amount of radium finally present in the uranium deduced. Lastly, the probable cause of the slow rate of production will be considered.

I.—Preliminary experiments with a hundred grams of commercial uranium nitrate showed evidence of the presence of radium in noticeable amount. It was found that the radium could be removed practically completely by repeated precipitation in its aqueous solution of small quantities of barium in the form of sulphate. The general method of testing for the presence of radium may be briefly described. The solution of uranium nitrate in water was kept for a period of at least some days in a closed bottle, into the mouth of which inlet and outlet tubes provided with taps had been ground. At the end of the period, which was accurately noted, a current of air was blown through the uranium solution, through a drying-tube containing calcium chloride between plugs of cotton-wool, into the electroscope. A volume of air sufficient to displace the whole of the air in the bottle, but smaller than the volume of the electroscope, was used. The electroscope was of the ordinary single-leaf type insulated by a sulphur bead, first described by C. T. R. Wilson, and the rate of discharge of the leaf was read by a microscope provided with a scale in the eyepiece. The natural leak of the instrument was always determined at the beginning of each experiment. The somewhat large natural rates of leak in the final series of measurements are due to the gradual accumulation of the later products of slow change left behind by the emanation. Rutherford has shown that this effect increases rather than decays with lapse of time. As, however, in every case the natural leak was small compared with the rate of leak measured, and as it remained constant throughout the day's experiments, it did not introduce any error into the measurements.

A kilogram of commercial uranium nitrate was dissolved in water, sulphuric acid was added and then successive small quantities of barium-nitrate solution, the liquid being left to stand, and filtered from the precipitated barium sulphate after each addition. The most effective conditions for the removal of the radium appeared to be to carry out the addition of barium drop by drop to the cold dilute solution of uranium nitrate, with constant stirring. Preliminary experiments indicated that the radium had been practically all removed, and on June 13th, after a final precipitation, the uranium solution was closed up in the bottle, and except for occasional tests, in which air was blown through, the solution remained undisturbed throughout the whole series of experiments. The initial observation was taken on June 20th, 1903, seven days after the closing of the bottle. The reading-microscope

employed was one of home manufacture, and was only one-eighth of the magnifying power of that used in the final series of observations. These rates of leak have therefore been multiplied by eight to make them comparable with the later measurements. After blowing the air from the uranium solution into the electroscope, the rate of leak was increased from 0·048 division per minute (natural leak) to 0·136 division per minute. This is so small compared with the rates of leak ultimately obtained that it may be neglected. It represents less than 10^{-11} gram of radium.

Observations were taken occasionally over a period of eighteen months, and indicated a gradual growth of emanating power of the uranium solution. But, owing to the disturbing effects of the radium employed in other researches in the laboratory, the intermediate series of observations only possess a qualitative value. The final series was carried out in December 1904 and January 1905, in the newly erected chemical laboratory at the Glasgow University, before any radium had been introduced into the building. The first test was made on December 17th, 1904, the air not having been blown out of the bottle for nine months. An increase of the rate of leak of the instrument from 1·56 divisions per minute (natural leak) to 16·7 divisions per minute was obtained. Thus, after a period of about 500 days, the rate of leak due to accumulation of the emanation in the uranium solution was over a hundred times greater than that initially observed.

II.—The emanation from the uranium solution comported itself in all respects like the radium emanation, and there can be no doubt that the emanation resulted from radium produced from the uranium. The radium emanation when newly admitted into an electroscope, owing to the production of the excited activity, gives a rate of leak which steadily rises over the first five minutes and then remains comparatively steady during the next five minutes. If the emanation is then blown or sucked out and the rate of leak immediately tested, it will be found to be roughly equal to the amount by which the initial rate of leak increased. In the course of a few minutes after the removal of the emanation the rate of leak of the electroscope becomes very small, if the emanation has remained in the instrument only a few minutes. The actual measurements recorded in the test of the air from the uranium-nitrate solution on Dec. 17th 1904, are here given as a fairly typical example of what was always observed.

Natural leak of electroscope 1·56 divisions per minute.   Leak after the introduction of the air from the uranium solution:—

<div style="text-align:right">Corrected rates of leak
(divisions per minute).</div>

0 to 10 divisions in 42 seconds	....	12·74
15 to 25	,,	35·3	,,	....	15·44
30 to 40	,,	35·8	,,	....	15·20
45 to 55	,,	37·0	,,	....	14·65

Electroscope recharged.

0 to 10 divisions in 38·4 seconds	....	14·06
15 to 25	,,	36·0	,,	....	15·1
30 to 40	,,	37·0	,,	....	14·6

Microscope reset.

0 to 20 divisions in 63·2 seconds	....	17·4
25 to 35	,,	28·4	,,	....	19·5

Microscope reset.

0 to 10 divisions in 32 seconds	....	17·2

The air in the electroscope was then sucked out:

1 minute afterwards	0 to 5 divisions in 30 secs.	8·4
4 minutes afterwards	0 to 5	,,	57·3 ,,	3·7
6	,,	,,	0 to 5	,,	91 ,,	1·8
18	,,	,,	37 divisions in 13 minutes	1·3
10	,,	,,	11	,,	5	,,	0·6

Similar results were always obtained when the emanation from the uranium was tested, and also when minute quantities of the radium emanation itself were introduced into the electroscope for the purpose of calibration.

A further test of the identity of the emanation from the uranium with that from radium was obtained by taking four series of observations in which the period of accumulation of the emanation from the uranium was varied.   For the case of radium the quantity of emanation accumulating attains a practical maximum after three or four weeks, and generally if I_∞ represents the rate of leak due to this maximum, and I_t that due to the quantity of emanation accumulating in t seconds

$$\frac{I_t}{I_\infty} = 1 - \epsilon^{-\lambda t}$$

where $\lambda = 2.10^{-6}$.   The following results were obtained:—

Date.	Time of accumulation.	Rate of Leak.	Natural Leak.	Corrected Rate of Leak.
Dec. 17, 1904	9 months	16·7	1·56	15·1
Dec. 21, 1904	4 days	12·0	1·8	10·2
Jan. 7, 1905	17 days	18·0	2·35	15·65
Jan. 9, 1905	53·5 hours	9·5	2·8	6·7

The rates of leak are given in divisions of the scale in seconds, and were those obtained after the emanation had been left in the electroscope about 5 minutes, that is, at the end of the initial period of *rapid* increase.   It may be pointed out that it is impossible to obtain strictly comparable measurements of the quantity of radium emanation with the electroscope.   No attempt was made to arrive at more than the order of magnitudes measured.   If the observation of January 7 is selected as the most accurate of the series, and the rates of leak theoretically to be expected for the other periods of accumulation calculated from it by means of the equation given, the observed results are found to be in fair agreement with the theoretical.

	Time of accumulation.	Observed.	Calculated.
I.	9 months	15·1	16·5
II.	4 days	10·2	8·2
III.	17 days	*15·65*	*15·65*
IV.	52·5 hours	6·7	5·3

The agreement is as close as can be expected and, taken in conjunction with the whole behaviour of the emanation, leaves no doubt as to its identity with that of radium.

III.—The quantity of emanation observed in the experiments recorded after a period of eighteen months' production of radium is far below that which is to be expected on the assumption that the radium atom results directly by the change of the uranium X atom.   The diagram (fig. 1) represents the course of change in a quantity of uranium represented by 10^{11} atoms for 150 days.   Time in days is plotted on the horizontal axis, while the figures on the vertical axis denote numbers of atoms.   The straight line AB represents the number of atoms of uranium undergoing change, primarily into uranium X atoms.   The curve ACD represents the number of atoms of radium that would be formed if the uranium X atom changed directly into the radium atom.   The area ABDC gives the number of uranium X atoms present at any instant.   AE is the average life of the uranium X atom.   The rate of change or radioactive constant of uranium is taken to be 6×10^{-17} (secs.$^{-1}$), and is derived from that of radium by comparing the relative

α radioactivity of the two elements and the number of changes in which α rays are expelled in the two cases. The quantity of radium *actually* produced even after 500 days is far too

Fig. 1.
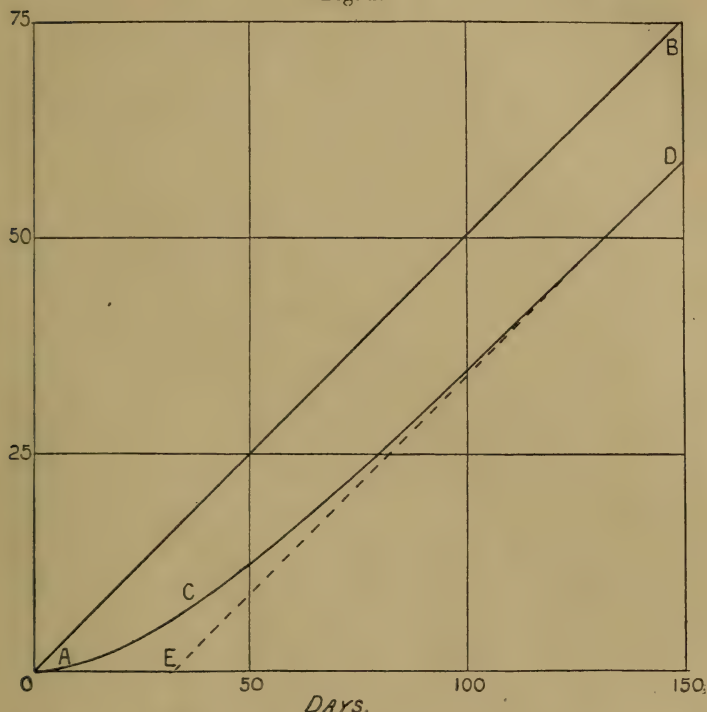
small to be represented on the diagram, and as a matter of fact amounts to about one atom of radium for every 3×10^{11} atoms of uranium.

This result was arrived at in the following manner :—A quantity of Giesel's purest radium bromide weighing 1·1 milligrams was dissolved in water in a small closed test-tube, into which inlet and outlet tubes provided with taps were sealed. The emanation accumulating for a known time was swept out by a current of air into a gas-burette, and the volume measured. After thorough mixing of the contents of the gas-burette a small known volume was delivered into a second gas-burette, again diluted to a known volume, mixed, and a small known fraction delivered into a third burette, The dilution was performed so that about one-millionth of the original emanation was finally obtained and delivered

into the electroscope.   Care was taken to use fresh burettes, rubber tubing, mercury, &c. for each dilution, as it was found that otherwise the results were quite untrustworthy. In one experiment, the emanation had been allowed to accumulate for 16 days in the radium solution.   In this time about 94 per cent. of the maximum or equilibrium quantity accumulates.   It was diluted with air to 73 c.c. and 0·71 c.c. taken.   This was diluted to 92 c.c. and 0·042 taken.   The last quantity was made up to 50 c.c. and 20 c.c. introduced into the electroscope.   After 5 minutes the corrected rate of leak was 24 divisions per minute.

Several determinations gave results of the same order of magnitude, although the individual determinations varied over an extreme range of 1 to 4.   From the mean of the results it was deduced that the equilibrium quantity of emanation from about 10^{-10} gram of radium (element) caused a leak of 1 division of the scale per minute.   16·5 divisions per minute may be taken as the measure of the equilibrium amount of emanation accumulating in the uranium solution after 567 days.   This corresponds to the presence of $1·6 \times 10^{-9}$ gram radium.   If the uranium nitrate is assumed to contain 50 per cent. of uranium, and the radioactive constant of the latter is taken as 6×10^{-17} (sec.$^{-1}$), then, on the assumption that there is a direct change of uranium into radium, the quantity of the latter that would be produced from a kilogram of uranium nitrate in 567 days is $1·5 \times 10^{-6}$ gram.   The effect of the uranium X stage would be practically inappreciable after a period of one and a half years.   So that the result shows that the production of radium from uranium during the first one and a half years is only about one thousandth of the quantity calculated on the assumption of a direct change.

IV.—From the figure already given it will readily be seen that the effect of the uranium X stage is to reduce the initial rate of production of radium below the final rate during an initial period corresponding in order of magnitude to the period of average life of the uranium X atom.   The same consideration would hold true for any other unknown intermediate products between uranium X and radium.   So that the low result obtained would be explained if one or more intermediate forms existed between uranium X and radium, for which the period of average life was large compared with the time of experiment.   Before considering some independent evidence in favour of this view, another possibility has first to be considered.   It seemed not unlikely that the radium, under the conditions in which it was produced,

was in the non-emanating form, so that the emanation was retained and did not escape freely from the solution. In the purification process an excess of sulphuric acid had been added, and some still remained in the solution after the final precipitations with barium. The radium therefore would be present as sulphate and it is well known that this compound obstinately retains the emanation. Mme. Curie has stated in her Thesis that the activity of radium sulphate is unchanged by shaking with water for a whole day.

To test this possibility the uranium solution was removed from the bottle, a small quantity of barium nitrate solution was added drop by drop, and the precipitated sulphate filtered off. This precipitate should have contained the greater part of the produced radium. It was ignited, and any emanation evolved sent into the electroscope. No great quantity of emanation was, however, given off. The barium was then got into solution by first boiling with sodium carbonate solution, washing the barium carbonate produced, and dissolving it in hydrochloric acid. The process was repeated until the whole of the barium (and radium) was obtained in solution. The uranium solution was treated with excess of barium to free it completely from sulphuric acid, concentrated and returned to the bottle. Tests were then carried out on the amount of emanation accumulating both in the uranium and in the barium solutions. That accumulating in the former amounted to about one-third, and that accumulating in the latter to about two-thirds of the amount obtained from the uranium before treatment, and no rapid increase of the emanating power of the uranium solution was observed for the first few days. There are therefore no grounds for believing that the low rate of production was due to the radium not giving up its emanation, or that more radium was present than the original experiments indicated.

With regard to the hypothesis of intermediate forms of relatively slow rates of change, it is possible, owing to some recent results, to approach the question on more general grounds than formerly. It is clear that as polonium and radium are to be regarded as products of uranium, the other radio-elements, and especially actinium, present in pitchblende probably have a similar origin, and may in fact be the missing intermediate forms. From Rutherford's recent investigations (Phil. Mag. [6] viii. 1904, p. 636 and 'Nature,' Feb. 9th 1905) it appears probable that polonium is the product of the seventh change of the radium atom, and that four out of the seven changes are accompanied by the expulsion of the heavy α particle. In a recent paper by Marckwald on radio-

tellurium (*Ber. d. D. Chem. Gesell.* 1905, p. 591) fairly
conclusive evidence of a chemical nature is brought forward
that polonium (radio-tellurium) stands to tellurium in the
periodic table in a similar relation to that in which radium
stands to barium.   It therefore immediately follows bismuth,
the atomic weight of which is 208·5 (O = 16), and the atomic
weight of polonium may be estimated with fair probability
as being in the neighbourhood of 212.   Thus in the change
of the radium atom (225) into polonium, four α particles are
expelled and a reduction of atomic weight amounting to
about 13 units occurs, so that the view that the α particle is
or becomes an atom of helium seems to be fairly well borne
out.   In the change of the atom of uranium (238) into
radium a similar reduction of atomic weight of 13 units takes
place, and it may therefore be expected that about four α
particles are expelled in this case also.   By analogy with
other disintegration series, it is to be expected that in
addition to the α-ray changes several rayless and β-ray
changes occur also, and of the latter one, the change of
uranium into uranium X, is in fact known.   The hypothesis
that there are several slow-period intermediate changes
between uranium and radium gains in definiteness by these
considerations.

The results obtained in the investigations of the ratio in
which radium and uranium are contained in the natural
minerals, which have been recently published, bear out this
hypothesis in the strongest possible manner.   Since the
drafting of the present paper Boltwood (Phil. Mag. [6] ix.
1905, p. 599) has published analyses of 22 minerals con-
taining very varying proportions of uranium, and finds in
each case the expected proportionality to exist between the
quantities of the two elements.   McCoy (*Ber. d. D. Chem.
Gesell.* 1904, p. 2641) has examined the radioactivity of
several minerals containing uranium, and has shown that in
those in which thorium is not present the activity is pro-
portional to the quantity of uranium present.   This supports
the view that radium, actinium, and polonium are all inter-
mediate products in the disintegration of uranium.   An
interesting speculation arises as to the nature of the final
product in the disintegration series.   After the change of
polonium, in which α rays only are expelled, no further changes
accompanied by the expulsion of radiant particles occur.   If
the atomic weight of polonium is in fact 212, the expulsion
of one α particle from the atom would lower the atomic
weight further to about 208.   If this change is in fact the
last of the series, the ultimate product must accumulate

indefinitely in the minerals, and therefore be a known element. The choice it would seem must rest between bismuth (208·5) and lead (206·9). Boltwood calls attention to the persistent appearance of lead as a constituent of the uranium-radium minerals, and cites certain evidence in favour of the view that it is one of the final products. Owing to the rapid rate of change of polonium, and to the refined methods of separating it from the mineral on a large scale introduced by Marckwald, the experimental identification of the ultimate product would seem to be now mainly a matter of cost.

LXXIV. *An Optical Paradox.*
By Lord Rayleigh, *O.M., F.R.S.*[*]

CONSIDER the following combination :—A point source A of approximately homogeneous light (λ) is focused by the lens LL upon the object-glass of a telescope T. In its
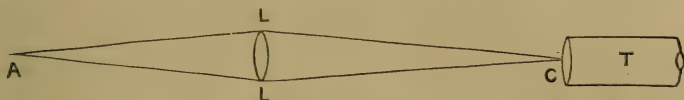
turn the telescope is focused upon L. According to geometrical optics the margin of the lens L should be seen sharp by an eye applied to the telescope ; but when we consider the limitation of aperture at the object-glass of the telescope, we come to the conclusion that the definition must be very bad. The image of A at C constitutes the usual diffraction pattern of which most of the light is concentrated in the central disk. The diameter of this disk is of the order λ . LC/LL. If this be regarded as the effective aperture of T, the angular resolving power will be found by dividing λ by the above quantity, giving LL/LC; so that the entire angular magnitude of the lens LL is on the limits of resolving power.

If this be admitted, we may consider next the effect of enlarging the source A, hitherto supposed to be infinitely small. If the process be carried far enough, the object-glass of T will become filled with light, and we may expect the natural resolving power to be recovered. But here we must distinguish. If the enlarged source at A be a self-luminous body, such as a piece of white-hot metal or the carbon of an electric arc, no such conclusion will follow. There is no phase-relation between the lights which act at

different parts of the object-glass, and therefore no possibility of bringing into play the interferences upon which the advantage of a large aperture depends. It appears, therefore, that however large the self-luminous source at A may be, the definition is not improved, but remains at the miserably low level already specified. If, however, the source at A be not a real one, but merely an aperture through which light from real sources passes, the case may be altered.

Returning to the extended self-luminous source, we see that the inefficiency depends upon the action of the lens L. If the glass be removed from its seat, so that A is no longer focused upon the object-glass, the definition must recover.

I do not know how far the above reasoning will seem plausible to the reader, but I may confess that I was at first puzzled by it. I doubt whether any experimenter would willingly accept the suggested conclusion, though he might be unable to point out a weak place in the argument. He would probably wish to try the experiment; and this is easily done. The lens L may be the collimating-lens of an ordinary spectroscope whose slit is backed by a flame. The telescope is removed from its usual place to a distance of say 10 feet and is focused upon L. The slit is at the same time focused upon the object-glass of the telescope. Although the image of the slit is very narrow, the definition of L as seen in the telescope does not appear to suffer, the vertical parts of the circular edge (parallel to the slit) being as well defined as the horizontal parts. If, however, at the object-glass a material screen be interposed provided with a slit through which the image of the first slit can pass, the definition at the expected places falls off greatly, even although a considerable margin be allowed in the width of the second slit.

This experiment gives the clue to the weak place in the theoretical argument. It is true that the greater part of the light ultimately reaching the eye passes through a very small area of the object-glass; but it does not follow that the remainder may be blocked out without prejudice to the definition of the boundary of the field. In fact, a closer theoretical discussion of the diffraction phenomena leads to conclusions in harmony with experiment.

In the case of a point-source and the complete circular aperture LL, the question turns upon the integral

$$\int_0^\infty J_0(\alpha x) J_1(\beta x)\, dx,$$

J_0, J_1 being the Bessel's functions usually so denoted. The

integral passes from 0 to $1/\beta$, as α passes through the value β *.

If the aperture of LL be reduced to a narrow annulus, the integral to be considered is

$$\int_0^\infty J_0(\alpha x)\, J_0(\beta x)\, x\, dx.$$

This assumes an infinite value when $\alpha = \beta$ †.

If the apertures be rectangular, the integrals take still simpler forms.

LXXV. *A Dynamical Theory of Diffusion for Non-Electro-lytes and the Molecular Mass of Albumin.* By WILLIAM SUTHERLAND ‡.

IN a paper communicated to the Australian Association for the Advancement of Science at Dunedin, 1904, on the Measurement of Large Molecular Masses, a purely dynamical theory of diffusion was outlined, with the aim of getting a formula for calculating from the data of diffusion those large molecular masses for which the ordinary methods fail. The formula obtained made the velocity of diffusion of a substance through a liquid vary inversely as the radius a of its molecule and inversely as the viscosity of the liquid. On applying it to the best data for coefficients of diffusion D it was found that the products aD, instead of being constant, diminished with increasing a in a manner which made extra-polation with the formula for substances like albumin seem precarious. After looking a little more closely into the dynamical conditions of the problem, it seems to me that the diminution of aD can be accounted for, and can be expressed by an empirical formula which enables us to extrapolate with confidence for a value of a for albumin, and so to assign for the molecular mass of albumin a value whose accuracy depends on that with which D is measured.

The theory is very similar to that of "Ionization, Ionic Velocities and Atomic Sizes" (Phil. Mag. Feb. 1902). Let a molecule of solute of radius a move with velocity V parallel to an x axis through the dilute solution of viscosity η. Then the resistance F to its motion is given by Stokes's formula

$$F = 6\pi V \eta a \, \frac{1 + 2\eta/\beta a}{1 + 3\eta/\beta a} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

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* A theorem attributed to Weber. See Gray and Matthews' 'Bessel's Functions,' p. 228.

† See 'Theory of Sound,' § 203, equations (14), (16).

‡ Communicated by the Author.

where β is the coefficient of sliding friction if there is slip between the diffusing molecule and the solution. For N molecules of solute per c.c. of solution the total resistance will be N times this, and in the steady state of diffusion will equilibrate the driving force due to variation of the osmotic pressure of the solute, namely dp/dx, which by the osmotic laws is $RT dc/dx$, if c is the concentration of the solute at x and R is the gas constant. Hence

$$RT \frac{dc}{dx} = 6\pi V \eta a N \frac{1 + 2\eta/\beta a}{1 + 3\eta/\beta a} ; \quad \cdot \quad \cdot \quad \cdot \quad (2)$$

and the required formula for the coefficient of diffusion with C for the number of molecules in a gramme-molecule is

$$D = \frac{RT}{6\pi\eta a C} \frac{1 + 3\eta/\beta a}{1 + 2\eta/\beta a} \cdot \quad \cdot \quad \cdot \quad \cdot \quad (3)$$

If $\beta = \infty$, that is, if there is no slipping of solution at surface of molecule, aD is the same for all molecules diffusing through a given solvent at a given temperature. Now for a large molecule of solute moving amongst smaller ones of solvent, we can see that the slipping is probably small. But in the other extreme case of a small molecule of solute moving amongst larger ones of solvent, an effect analogous to slipping will occur, since the small molecule will travel a good deal in the gaps which would be left if the molecules of solvent were forced almost into permanent contact. We have thus two extreme cases of the formula.

When $\beta = 0$, $\qquad D = \dfrac{RT}{4\pi\eta a C}$ $\left.\vphantom{\dfrac{RT}{4\pi\eta a C}}\right]$

and when $\beta = \infty$, $\qquad D = \dfrac{RT}{6\pi\eta a C}$ $\left.\vphantom{\dfrac{RT}{6\pi\eta a C}}\right\}$ $\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (4)$

Thus with increasing values of a we should have aD diminishing from the upper limit $RT/4\pi\eta C$, when a is small, to the lower limit $RT/6\pi\eta C$, when a is large. This is analogous to the actual behaviour of $B^{\frac{1}{3}}D$ obtained from experiment, B being the volume of the molecules in a gramme-molecule of solute. The first of the following tables contains the coefficients of diffusion for various gases through water determined by Hüfner [*]. I have reduced these all to a temperature of $16°$ C., and expressed them with the second as unit of time instead of the day. The values of B are taken mostly from " Further Studies on Molecular Force " (Phil. Mag. [6] xxxix.). In the second last row are given the values of

* Wied. *Ann.* 1897, vol. xl., and *Zeit. f. Phys. Chem.* xxvii.

$10^6 B^{\frac{1}{3}}D$ which we are studying. H. Euler's values for the diffusion coefficients of Cl_2 and Br_2 are added at the right hand of the table:—

	CO_2.	NH_3.	O_2.	N_2	N_2O.	Cl_2.	H_2.	Cl_2.	Br_2.
10^7D	166	132	167	178	156	127	474	154	101
B	30	21	19·3	22·7	29	38	8·6	38	52
$10^6 B^{\frac{1}{3}}D$...	51	36	44	50	48	43	97	52	38
10^6 calcul.	44	50	51	48	44	40	73	40	37

In the next table the coefficients of diffusion are those found by Thovert for a number of non-electrolytes in water (*Comptes Rendus*, cxxxv.).

Alcohols.

	Methyl.	Ethyl.	Allyl.	Propyl.	Butyl.	Amyl.
10^7D	137	111	99	98	88	88
B	26	42·5	52	59	75·5	92
$10^6 B^{\frac{1}{3}}D$...	41	39	37	38	37	40
10^6 calcul.	46	39	37	36	33	32

Other Substances.

	Urea.	Urethane.	Glycerol.	Phenol.	Hydro-quinone.	Resorcin.
10^7D	98	87	79	80	73	75
B	51	75	72	80	85	85
$10^6 B^{\frac{1}{3}}D$...	36	37	33	34	32	33
10^6 calcul.	37	33	34	33	32	32

	Pyrogallol.	Glucose.	Mannite.	Antipyrin.	Maltose.	Raffinose.
10^7D	66	57	55	57	41	35·5
B	90	134	141	166	254	374
$10^6 B^{\frac{1}{3}}D$...	30	29	29	31	26	26
10^6 calcul.	32	29	29	28	26	25

In spite of irregularities it can be seen that the chief tendency is for the product $10^6 B^{\frac{1}{3}}D$ to diminish with increasing value of B and to seemingly converge to a lower limit, as is required by the theory. But the case of H_2 makes it impossible to recognize convergence to the upper limit required by the theory. The other gases in an irregular way fall in with the idea of convergence to an upper limit.

Under the circumstances it seems to me best to express the connexion between D and $B^{\frac{1}{3}}$ by an empirical equation, namely:—

$$10^6 B^{\frac{1}{3}} D = b + k/B^{\frac{2}{3}}, \quad \cdots \quad (5)$$

where b and k are constants for a given solvent at a given temperature. For water at $16°$ C. $b = 21$, and $k = 220$, and with these values the last row of numbers in the previous tables marked "calcul." have been obtained. This empirical formula gives a connexion between velocity of diffusion and molecular radius for substances ranging from hydrogen with a molecular mass 2 to raffinose with a molecular mass 500. It makes $10^6 B^{\frac{1}{3}} D = 21$ when $B^{\frac{1}{3}}$ is large. For molecules of large radius we have thus good reason to rely on the relation

$$10^6 B^{\frac{1}{3}} D = 21 \cdot \quad \cdots \quad (6)$$

as approximately true. To illustrate its application we can use it to calculate the molecular mass of egg albumin from the value of $10^7 D$ calculated by Stefan from Graham's data, namely 7. This makes $B^{\frac{1}{3}} = 30$, and $B = 27,000$. This is the volume of a gramme-molecule of egg albumin.

Representative minimum molecular formulæ for albumin obtained from chemical considerations are given in Cohnheim's *Chemie der Eiweisskörper.* Hofmeister for serum albumin gives $C_{450}H_{720}N_{116}O_{140}S_6$ with a molecular weight (mass) 10166. For egg albumin he gives a minimum of 5378. For hæmoglobin Hüfner and Jaquet assign 16669 for the minimum molecular mass. From the lowering of the freezing-point in albumin solutions Sabanajew and Alexandrow found for egg albumin the molecular mass 14270, but the experimental error due to impurities allows little reliance to be placed on this estimate. Now to pass from the gramme-molecular volume 27000 to the molecular mass we may proceed thus. From the values of B published in previous papers of mine we can estimate the limiting volume of a gramme-atom of C as 8, of H as 4, of N as 8, of O as 6, and of S as 18. The average percentage composition of egg albumin is C 52·5, H 7·2, N 15·3, O 23·5, and S 1·5. Hence if x is the number of carbon atoms in the molecule of egg albumin we have the equation

$$x \left\{ 8 + \frac{12}{52·5} \left(4 \cdot \frac{7·2}{1} + 8 \cdot \frac{15·3}{14} + 6 \cdot \frac{23·5}{16} + 18 \cdot \frac{1·5}{32} \right) \right\} = 27000$$

and $x = 1436$. Thus the Graham-Stefan coefficient of diffusion leads to $C_{1436}H_{2364}N_{359}O_{482}S_{15}$ as the formula for egg albumin, the molecular mass being 32814, which is a small multiple of

the above minima. But as the preparation of pure crystallized albumins was not known in Graham's day, it is necessary that his diffusion experiments should be repeated with various pure albumins and with their products of disintegration, such as peptones, if we are to arrive at accurate estimates of the size of albumin molecules and their parts.

Melbourne, March 1905.

LXXVI. *On the Stresses in the Earth's Crust before and after the Sinking of a Bore-hole. By* C. CHREE, *Sc.D., LL.D., F.R.S.* [*]

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§ 1. IN ' Nature,' October 20, 1904, p. 602, there appeared letters by Mr. G. Martin and the Hon. C. A. Parsons dealing with the size of the stresses in the Earth's crust and speculating as to what would happen if a hole were bored to a depth of twelve miles.

The letters indicate that some interest attaches to the problem, but its true character seems to stand in need of careful consideration. We know at present so little of the nature of the earth's material, even at such small depths as twelve miles, and have such scanty knowledge of the combined effects of high temperature and high pressure, that there are no data for making an *exact* calculation. Rocks as we know them at the earth's surface are not isotropic or even homogeneous solids, and they are not perfectly elastic for any considerable stresses [†]; but the crust of the earth seems to behave as a solid so far as can be inferred from earthquake-waves, and nothing that is positively known forbids the hypothesis that the material a few miles down is elastic for moderate changes

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[*] Communicated by the Physical Society : read March 10, 1905.
[†] Japanese Earthquake Investigation Committee, Reports No. 17.

of stress.   Further, if any calculation is to be made of the internal stresses, there seems no alternative to the application of elastic solid theory.   The following results based on the ordinary mathematical theory may thus be worth the consideration of engineers and others interested in the practical problem.

I propose first to present the results applicable to a series of mathematical problems, only one or two of which are absolutely novel, and then to consider what their bearing is on the problem now under discussion.   Only the general character of the mathematical steps will be indicated.

§ 2. The two best known theories as to rupture, or more strictly as to the limits of application of the elastic solid theory, are :—

(i.) The maximum stress difference theory, according to which $\bar{S}$ the greatest value of S, where S is the *difference* between the algebraically greatest and least of the three principal stresses at a point, must not exceed a certain experimental limit ;

(ii.) The greatest strain theory, according to which the limiting value attaches to the largest strain in any part of the structure, assuming that to be an extension.

Theory (i.) seems that least favourable to the permanence of the structure in such problems as those to be considered here, and attention is almost exclusively devoted to it.   It has the recommendation that there seems considerable experimental evidence in favour of the view that it is the maximum stress difference on which depends the tendency to flow in solids under severe non-uniform pressure *.

Notation.

§ 3. In isotropic material m and n represent the two elastic constants in Thomson and Tait's notation.   These are connected with Young's modulus E, and Poisson's ratio η, by the relations

$$m/1 = n/(1-2\eta) = E \div \{2(1+\eta)(1-2\eta)\}. \quad . \quad . \quad (1)$$

In applications of spherical coordinates, r, θ, ϕ, the displacement along the radius vector is denoted by u ; in most of the cases treated here the three principal stresses are the radial $\widehat{rr}$, and the two orthogonal stresses $\widehat{\theta\theta}$ and $\widehat{\phi\phi}$.   Also $\widehat{\phi\phi}$ in the cases treated, with the exception of § 6, is equal to $\widehat{\theta\theta}$.   The three principal strains are then du/dr, u/r, and u/r.

* Todhunter and Pearson's 'History of Elasticity,' vol. ii. art. 247, &c.

In cylindrical coordinates the displacements are u along r, the outward drawn perpendicular from the cylindrical axis, or axis of z, and w parallel to this axis. The principal stresses in the problems treated here are $\widehat{rr}$ and $\widehat{zz}$, parallel respectively to r and z, and $\widehat{\phi\phi}$ perpendicular to these two directions. The corresponding principal strains are du/dr, dw/dz, and u/r.

The dilatation Δ is given in spherical and cylindrical coordinates respectively, in the cases of symmetry here considered, by

$$\Delta = du/dr + 2u/r, \quad \ldots \ldots \ldots \quad (2)$$

$$\Delta = du/dr + u/r + dw/dz. \quad \ldots \ldots \quad (3)$$

Homogeneous isotropic gravitating sphere.

§ 4. If a be the radius, ρ the density, and g 'gravity' at the surface, then, the centre being origin,

$$
\left.
\begin{aligned}
u &= \frac{g\rho r}{10a(m+n)}\left\{r^2 - a^2\frac{5m+n}{3m-n}\right\}, \\
\widehat{rr} &= -g\rho(a^2 - r^2)(5m+n) \div \{10a(m+n)\}, \\
\widehat{\theta\theta} &= -g\rho\{a^2(5m+n) - r^2(5m-3n)\} \div \{10a(m+n)\}, \\
S &= 2g\rho r^2 n \div \{5a(m+n)\}, \\
\overline{S}\,\text{(at surface)} &= 2g\rho a n \div \{5(m+n)\} \equiv \tfrac{1}{5}g\rho a(1-2\eta)/(1-\eta).
\end{aligned}
\right\} \quad (4)^*
$$

At points near the surface for which

$$h \equiv a - r \text{ is small,}$$

we have, retaining only the lowest power of h, as first approximations,

$$
\left.
\begin{aligned}
\widehat{rr} &= -g\rho h(3-\eta) \div \{5(1-\eta)\}, \\
\widehat{\theta\theta} &= -S = -g\rho a(1-2\eta) \div \{5(1-\eta)\}.
\end{aligned}
\right\} \quad \cdot \quad (5)
$$

In the special case when the material is incompressible,

$$\text{i. e.} \qquad n/m = 0, \quad \text{or} \quad \eta = 0\cdot5,$$

we have throughout the mass of the sphere

$$
\left.
\begin{aligned}
u &= 0, \\
\widehat{rr} &= \widehat{\theta\theta} = -g\rho(a^2 - r^2)/2a, \\
S &= \overline{S} = 0;
\end{aligned}
\right\} \quad \ldots \ldots \quad (6)
$$

* Camb. Phil. Soc. Trans. vol. xiv. p. 281.

and close to the surface

$$\widehat{rr} = \widehat{\theta\theta} = -g\rho h. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

§ 5. Gravitating spherical " Earth" consisting of a core of radius b, density $\rho + \rho'$ and elastic constants m', n', and of a crust or layer of density ρ, and elastic constants m, n, resting on the core and bounded externally by a spherical surface of radius a.

The expressions for the displacements and stresses are as follows :—

In the core :

$$\left.\begin{aligned}
u &= \tfrac{1}{3}r\mathrm{A} + \tfrac{2}{15}\pi\frac{(\rho+\rho')^2 r^3}{m'+n'}, \\
\widehat{rr} &= \tfrac{1}{3}(3m'-n')\mathrm{A} + \frac{5m'+n'}{15(m'+n')}\,2\pi(\rho+\rho')^2 r^2, \\
\widehat{\theta\theta} &= \tfrac{1}{3}(3m'-n')\mathrm{A} + \frac{5m'-3n'}{15(m'+n')}\,2\pi(\rho+\rho')^2 r^2.
\end{aligned}\right\} \quad . \quad (8)$$

In the layer :

$$\left.\begin{aligned}
u &= \tfrac{1}{3}r\mathrm{B} + r^{-2}\mathrm{C} + \frac{2\pi\rho^2 r^3}{15(m+n)} - \frac{2}{3}\frac{\pi\rho\rho' b^3}{m+n}, \\
\widehat{rr} &= \tfrac{1}{3}(3m-n)\,\mathrm{B} - 4nr^{-3}\mathrm{C} + \frac{5m+n}{15(m+n)}\,2\pi\rho^2 r^2 - \frac{m-n}{m+n}\frac{4}{3}\frac{\pi\rho\rho' b^3}{r}, \\
\widehat{\theta\theta} &= \tfrac{1}{3}(3m-n)\,\mathrm{B} + 2nr^{-3}\mathrm{C} + \frac{5m-3n}{15(m+n)}\,2\pi\rho^2 r^2 - \frac{m}{m+n}\frac{4}{3}\frac{\pi\rho\rho' b^3}{r}.
\end{aligned}\right\} \quad (9)$$

Here A, B, C are three arbitrary constants to be determined by the surface conditions, and the gravitational force between two masses μ and μ' at distance R is taken as $(\mu\mu'/\mathrm{R}^2) \times 1$.

We shall suppose the outer surface $r=a$ to be free from force. If uniform normal pressure acts its effects are most easily obtained separately.

Supposing the outer surface $r=a$ free from force, $\widehat{rr}$ must vanish over it, whence

$$\tfrac{1}{3}(3m-n)\mathrm{B} - 4na^{-3}\mathrm{C} = -\frac{5m+n}{15(m+n)}\,2\pi\rho^2 a^2 + \frac{m-n}{m+n}\frac{4}{3}\,\pi\rho\rho'\frac{b^3}{a}. \quad (10)$$

Over the common surface $r=b$, the radial displacements must be the same for the core and layer, whence

$$\mathrm{B} + 3b^{-3}\mathrm{C} = \mathrm{A} + \frac{2}{5}\frac{\pi(\rho+\rho')^2 b^2}{m'+n'} - \frac{2}{5}\frac{\pi\rho^2 b^2}{m+n} + \frac{2\pi\rho\rho' b^2}{m+n}. \quad (11)$$

Finally the values of $\widehat{rr}$ in the core and layer must be equal over the common surface, and so

$$\tfrac{1}{3}(3m-n)\,\mathrm{B}-4nb^{-3}\mathrm{C}=\tfrac{1}{3}(3m'-n')\,\mathrm{A}+\frac{5m'+n'}{15(m'+n')}\,2\pi(\rho+\rho')^2b^2$$
$$-\frac{5m+n}{15(m+n)}\,2\pi\rho^2b^2+\frac{m-n}{m+n}\frac{4}{3}\,\pi\rho\rho'b^2. \quad . \quad (12)$$

These three equations determine A, B, and C without ambiguity, and there is no difficulty, except in the length of the expressions, in obtaining the solution for the general case.

As we shall see, however, physical interest is mainly if not entirely restricted to the case when the material of the core is incompressible or very nearly so, and I shall thus limit myself to the case when n'/m' is negligible, n' being finite.

From (12) we see that when n'/m' is negligible A must vanish. For n' being finite, m' must be infinite, and so the one term $\tfrac{1}{3}(3m'-n')\mathrm{A}$ unless A vanished would become infinite, whilst the other terms in the equation would remain finite. We must thus treat A as negligible, except when multiplied by m', and regard (12) as simply determining the (finite) value of $(3m'-n')\mathrm{A}/3$.

Our surface equations thus reduce to two, viz. (10) and

$$\mathrm{B}+3b^{-3}\mathrm{C}=-\frac{2}{5}\frac{\pi\rho^2b^2}{m+n}+\frac{2\pi\rho\rho'b^2}{m+n}, \quad . \quad . \quad (11')$$

this being the form taken by (11) when A is negligible and m' is infinite.

The values of B and C may be at once written down from (10) and (11'). Confining our attention to the case where the thickness $a-b=t$ of the crust is so small that $(t/a)^2$ is negligible, we find

$$\left.\begin{array}{l}\mathrm{B}=2\pi\rho a^2\{-\rho+2\rho'(1-3t/a)\}\div\{3(m+n)\},\\[4pt]\mathrm{C}=2\pi\rho a^5\{\tfrac{2}{5}\rho+\rho'(1-3t/a)\}\div\{9(m+n)\}.\end{array}\right\} \quad (13)$$

Substituting these values of B and C in (9), then putting $r=a-h$, and neglecting $(h/a)^2$, we find after reduction

$$\left.\begin{array}{l}\widehat{rr}=-\tfrac{4}{3}\pi\rho ah\{\rho+\rho'-3(t/a)\rho'\},\\[4pt]\widehat{\theta\theta}=-\tfrac{4}{3}\pi\rho ah\dfrac{m-n}{m+n}\left\{\rho+\rho'-3(t/a)\rho'\right\}.\end{array}\right\} \quad . \quad (14)$$

But if g be the acceleration of gravity at the surface

$$g=(4\pi/3)\{\rho a^3+\rho'b^3\}/a^2,$$
$$=\tfrac{4}{3}\pi a\{\rho+\rho'-(3t/a)\rho'\},$$

when $(t/a)^2$ is neglected.

Thus we have in the crust :

$$
\left.
\begin{aligned}
\widehat{rr} &= -g\rho h, \\
\widehat{\theta\theta} &= -g\rho h(m-n)/(m+n) \equiv -g\rho h\eta/(1-\eta), \\
S &= g\rho h(1-2\eta)/(1-\eta).
\end{aligned}
\right\} \quad (15)
$$

§ 6. *Slightly spheroidal, homogeneous, gravitating and rotating earth.*

The surface values of the stresses, which are by no means very complicated, will suffice for our immediate object.

Let $2a$, $2c$ represent the equatorial and polar diameters, r the perpendicular from any point on the polar axis, p the perpendicular from the centre on the tangent plane, ρ the density, G the gravitational force between two unit masses at unit distance, and ω the angular velocity.

At the surface the principal stresses are $\widehat{nn}$ along the normal, $\widehat{tt}$ along the tangent in the meridian plane, and $\widehat{\phi\phi}$ perpendicular to the meridian plane.

Neglecting terms of order $(1-c^2/a^2)^2$ we find *

$$
\left.
\begin{aligned}
\widehat{nn} &= 0, \\
\widehat{tt} &= -\frac{4\pi G\rho^2 a^2}{15(1-\eta)}\left\{\frac{c^2}{p^2}(1-2\eta)+\frac{a^2-c^2}{a^2}\frac{1-\eta-4\eta^2}{7+5\eta}\right\} \\
&\quad +\frac{\omega^2\rho a^2(c/p)^2}{5(1-\eta)(7+5\eta)^2}\left\{(7+5\eta)(3-6\eta-5\eta^2)+4\frac{a^2-c^2}{a^2}(1+\eta)(6-5\eta-5\eta^2)\right\}, \\
\widehat{\phi\phi} &= -\frac{4\pi G\rho^2 a^2}{15(1-\eta)}\left\{1-2\eta+\frac{a^2-c^2}{a^2}\left(1-3\frac{r^2}{a^2}\right)\frac{1-\eta-4\eta^2}{7+5\eta}\right\} \\
&\quad +\frac{\omega^2\rho a^2}{5(1-\eta)(7+5\eta)^2}\left[(7+5\eta)(3-6\eta-5\eta^2)+4\frac{a^2-c^2}{a^2}(1+\eta)(6-5\eta-5\eta^2)\right. \\
&\quad \left.+\frac{r^2}{a^2}\left\{5(7+5\eta)(1-\eta^2)-\frac{a^2-c^2}{a^2}(30-4\eta-82\eta^2-41\eta^3)\right\}\right]
\end{aligned}
\right\} (16)
$$

When $c=a$, $4\pi b G a/3 = g$ the gravitational acceleration at the surface. Thus in a perfect sphere the contributions from the gravitational terms to $\widehat{tt}$ and $\widehat{\phi\phi}$ are equal, and agree with the value $-g\rho a(1-2\eta)\div\{5(1-\eta)\}$ given in (5) of § 4 for $\widehat{\theta\theta}$; they vanish when the material is incompressible.

The contributions from the "centrifugal force" terms to the surface stresses are in a perfect sphere, λ denoting the

* Roy. Soc. Proc. vol. lviii. eqns. (15), p. 43, and eqns. (23) to (26), pp. 44 and 45.

latitude,

$$\widehat{tt} = \widehat{\theta\theta} = \omega^2\rho a^2(3-6\eta-5\eta^2) \div \{5(1-\eta)(7+5\eta)\}, \left.\begin{array}{r}\\ \\ \end{array}\right\} (17)$$
$$\widehat{\phi\phi} = \frac{\omega^2\rho a^2}{5(1-\eta)(7+5\eta)}\left\{3-6\eta-5\eta^2+5\cos^2\lambda(1-\eta^2)\right\}\cdot$$

The value of $\widehat{\theta\theta}$ is thus constant over the surface. Its sign depends on that of $3-6\eta-5\eta^2$. This expression vanishes when $\eta = \cdot3798$, or $n/m = \cdot2404$. It also vanishes when $m/n = \cdot2404$, but this latter value is physically impossible. Thus $\widehat{\theta\theta}$ is positive or negative, *i. e.* a tension or a pressure, according as η is less or greater than 0·38. $\widehat{\phi\phi}$ is always positive in the equator ; at the poles it is equal to $\widehat{\theta\theta}$.

When $\widehat{\phi\phi}$ is positive and $\widehat{\theta\theta}$ negative (which implies η being $>\cdot3798$) $\widehat{\phi\phi}-\widehat{\theta\theta}$ is the maximum stress difference, and the greatest value occurs in the equator where

$$S=\omega^2\rho a^2(1+\eta)/(7+5\eta). \quad . \quad . \quad . \quad (18)$$

This increases slightly with η, varying from $0\cdot155\omega^2\rho a^2$ when $\eta=\cdot3798$ to $0\cdot158\omega^2\rho a^2$ when $\eta=0\cdot5$.

Supposing the earth an incompressible sphere in which $\rho=5\cdot5$, $a=3963$ miles, and $\omega^2a=g/293$, we find for the maximum *surface* value of S

$$\bar{S} = 12\cdot0 \text{ tons weight per square inch.} \quad . \quad . \quad (19)$$

Except for incompressible material, the principal terms depending on ω^2 in $\widehat{tt}$ and $\widehat{\phi\phi}$ are small compared with the principal gravitational terms ; and when the material is incompressible the secondary terms depending on ω^2 which contain a^2-c^2 are small compared with the secondary gravitational terms. Thus we may in general neglect the secondary terms in ω^2 for a first approximation. We may also neglect the differences between a, c, and p in all terms depending on ω or containing a^2-c^2 as a factor, and may replace $4\pi G\rho a/3$ by g, where g is the mean value of gravity over the surface.

Doing so, we find as first approximations to the principal stresses in the surface of an incompressible nearly spherical s, heroid

$$\widehat{nn} = 0, \left.\begin{array}{r}\\ \\ \\ \\ \end{array}\right\} \quad . \quad (20)$$
$$\widehat{tt} = \frac{1}{19}g\rho a\left(\frac{4}{5}\frac{a-c}{a}-\frac{\omega^2a}{g}\right),$$
$$\widehat{\phi\phi} = \frac{1}{19}g\rho a\left(\frac{4}{5}\frac{a-c}{a}-\frac{\omega^2a}{g}\right)\left(1-3\frac{r^2}{a^2}\right).$$

Thus $\widehat{tt}$ and $\widehat{\phi\phi}$ would both vanish all over the surface if

$$\frac{4}{5}\frac{a-c}{a} = \frac{\omega^2 a}{g}$$

In the actual earth

$$\omega^2 a/g = 1/290 \quad \text{approximately.}$$

Bessel's value for $(a-c)/a$ is 1/299, while Clarke's is 1/293·5. Taking the former value, and employing as before λ to denote the latitude, we find in the surface

$$\left.\begin{aligned}\widehat{tt} &= -0\cdot90 \quad . \quad . \quad . \quad \text{tons weight per square inch}\\\widehat{\phi\phi} &= -0\cdot90(1-3\cos^2\lambda) \quad\quad \text{,,} \quad\quad\quad \text{,,} \quad\quad\quad \text{,,}\\\overline{S} &= 2\cdot7 \quad . \quad . \quad . \quad . \quad \text{,,} \quad\quad\quad \text{,,} \quad\quad\quad \text{,,}\end{aligned}\right\} (21)$$

The following particular values of $\widehat{\phi\phi}$ (all in tons weight per square inch) should be noted :—

Equator.	Lat. 35° 16'.	Lat. 54° 44'.	Poles.
+ 1·8	+ 0·9	0·0	− 0·9

The stress in the meridian plane is always a pressure when the material is incompressible ; but that perpendicular to the meridian is a tension in latitudes below $54\frac{3}{4}°$. The numerical values of these surface stresses would be reduced by assigning to the earth's compression $\underline{a-c/a}$ any value, such as Clarke's, which is larger than Bessel's.

§ 7. Right vertical prism of density ρ acted on by gravity, the prismatic surface being exposed to normal pressure $p-Cz$, where p and C are constants, and the upper plane surface $z=h$ being free from force.

To give definiteness to the problem we shall suppose the vertical displacement nil at the C.G. of the base. The shape of the cross section does not matter.

The solution is obtainable from one on p. 545 of the Phil. Mag. for November 1901, by suitably altering the notation.

Taking rectangular axes of x, y, z, the axis of z being drawn vertically upwards, and the origin being at the C.G. of the base of the prism, the displacements α, β, γ are given by

$$\left.\begin{aligned}\alpha/x &= \beta/y = (1/\mathrm{E})\{\eta g\rho(h-z) - (1-\eta)(p-Cz)\},\\\gamma &= -(z/\mathrm{E})\{g\rho(h-\tfrac{1}{2}z) - \eta(2p-Cz)\} + (1/2\mathrm{E})(x^2+y^2)\{g\rho\eta - (1-\eta)C\}\cdot\end{aligned}\right\} (22)$$

The principal stresses and the stress-difference are given by

$$\left.\begin{aligned}\widehat{xx} &= \widehat{yy} = -(p-Cz) = -p_{\mathrm{H}},\\\widehat{zz} &= -g\rho(h-z) = -p_{\mathrm{V}},\\\mathrm{S} &= p_{\mathrm{V}} \sim p_{\mathrm{H}},\end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

where p_V and p_H are the pressures at the depths in question in the vertical and horizontal directions respectively.

Assuming $p - Cz$ everywhere positive, the stresses are all numerically greatest over the base $z=0$; also

$$S_{z=0} = g\rho h \smallfrown p \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

§ 8. Hollow, vertical, circular cylinder of density ρ acted on by gravity, the outer cylindrical surface $r=a$ being exposed to normal pressure $p - Cz$, where p and C are constants, the inner cylindrical surface $r=a'$ and the upper plane surface $z=h$ being free from force.

The solution is obtainable from one on p. 550 of the Phil. Mag. for November 1901 by suitably altering the notation. It is, in cylindrical coordinates,

$$\left. \begin{aligned} u &= (r/E)\left[g\rho\eta(h-z) - (1-\eta)\frac{a^2(p-Cz)}{a^2-a'^2}\right] - \frac{a^2 a'^2(p-Cz)}{2nr(a^2-a'^2)}, \\ w &= (z/E)\left[-g\rho h + \frac{2\eta a^2 p}{a^2-a'^2}\right] + \tfrac{1}{2}\frac{g\rho}{E}(z^2+\eta r^2) \\ &\quad - \frac{Ca^2}{2(a^2-a'^2)}\left\{ \frac{(1-\eta)r^2 + 2\eta z^2}{E} + \frac{a'^2}{n}\log(r/r_1)\right\}. \end{aligned} \right\} \quad (25)$$

Here r_1 is a constant which can be determined so that w has any assigned value at any given point.

For the stresses we have

$$\left. \begin{aligned} \widehat{rr} &= -\frac{a^2}{a^2-a'^2}\left(1-\frac{a'^2}{r^2}\right)\left(p-Cz\right), \\ \widehat{\phi\phi} &= -\frac{a^2}{a^2-a'^2}\left(1+\frac{a'^2}{r^2}\right)\left(p-Cz\right), \\ \widehat{zz} &= -g\rho(h-z). \end{aligned} \right\} \quad . \quad . \quad (26)$$

Assuming p, C, and $p-Ch$ positive, the stresses are all negative and numerically greatest at the base of the cylinder, where z vanishes; and the greatest value of the maximum stress difference—which is met with over the inner surface—is the greater of the two quantities :—

$$\left. \begin{aligned} \bar{S}_1 &= 2pa^2/(a^2-a'^2), \\ \bar{S}_2 &= g\rho h. \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (27)$$

When a'/a is very small, then very approximately

$$\bar{S}_1 = 2p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

§ 9. Right prism exposed to uniform pressures P and p over its flat ends and prismatic surface respectively. The

axis of z being along the prismatic axis, the stresses and stress-difference are given by

$$\left.\begin{array}{l} \widehat{xx} = \widehat{yy} = -p, \\ \widehat{zz} = -\text{P}, \\ \text{S} = \bar{\text{S}} = \text{P} \sim p. \end{array}\right\} \quad \cdots \quad \cdots \quad (29)$$

§ 10. Hollow right circular cylinder outer radius a, inner a', exposed to pressures p and P over its external cylindrical surface and flat ends respectively.

The stresses are

$$\left.\begin{array}{l} \widehat{rr} = -pa^2(1-a'^2/r^2)/(a^2-a'^2), \\ \widehat{\phi\phi} = -pa^2(1+a'^2/r^2)/(a^2-a'^2), \\ \widehat{zz} = -\text{P}. \end{array}\right\} \quad \cdots \quad (30)$$

At a given distance r from the axis the maximum stress difference may according to circumstances be any one of the three following :—

$$\left.\begin{array}{l} \text{S}_1 = \widehat{rr} - \widehat{\phi\phi} = 2pa^2a'^2 \div r^2(a^2-a'^2), \\ \text{S}_2 = \widehat{rr} - \widehat{zz} = \text{P} - pa'^2(1-a'^2/r^2)/(a^2-a'^2), \\ \text{S}_3 = \widehat{zz} - \widehat{\phi\phi} = -\text{P} + pa^2(1+a'^2/r^2)/(a^2-a'^2). \end{array}\right\} \cdot \quad (31)$$

But for the greatest value of S there are only two options, both referring to the inner surface of the cylinder, viz. :—

$$\left.\begin{array}{l} \bar{\text{S}}_1 = 2pa^2/(a^2-a'^2), \\ \bar{\text{S}}_2 = \text{P}. \end{array}\right\} \quad \cdots \quad \cdots \quad (32)$$

When a'/a is very small, a close approximation is

$$\bar{\text{S}}_1 = 2p. \quad \cdots \quad \cdots \quad \cdots \quad (33)$$

§ 11. Solid sphere of radius a exposed to uniform surface-pressure p.

The solution is

$$\left.\begin{array}{l} u/r = du/dr = -p/(3m-n), \\ \widehat{rr} = \widehat{\theta\theta} = -p, \\ \text{S} = 0. \end{array}\right\} \quad \cdots \quad (34)$$

§ 12. Spherical shell outer radius a, inner a', exposed to uniform pressure p over the outer surface.

The stresses and stress-difference are given by

$$\begin{cases} \widehat{rr} = -pa^3(1-a'^3/r^3)/(a^3-a'^3), \\ \widehat{\theta\theta} = -pa^3(1+a'^3/2r^3)/(a^3-a'^3), \\ S = \tfrac{3}{2}\,p(aa')^3 \div \{r^3(a^3-a'^3)\}, \\ \bar{S} \text{ (at inner surface)} = \tfrac{3}{2}pa^3/(a^3-a'^3). \end{cases} \quad . \quad (35)$$

When a'/a is very small a closely approximate value is

$$\bar{S} = 3p/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

§ 13. Material bounded by an infinite plane,—on one side of which it extends to infinity,—acted on by a surface-tension uniformly distributed over a circular area of radius a'.

For clearness suppose the material to be on the lower side of the plane $z=0$, the tension T, per unit area, acting vertically upwards. Let z be measured positively downwards from an origin at the centre of the stressed area, and let R denote the distance between an element $d\sigma$ of the stressed area and a point r, z in the solid r being the perpendicular on the axis of z.

The solution, as obtained by Boussinesq and Cerruti *, is

$$\begin{cases} u=(\mathrm{T}/4\pi n)\dfrac{d}{dr}\left\{\dfrac{d}{dz}\iint \mathrm{R}d\sigma + (1-2\eta)\iint \log(z+\mathrm{R})d\sigma\right\}, \\[2mm] w=(\mathrm{T}/4\pi n)\left\{\dfrac{d^2}{dz^2}\iint \mathrm{R}d\sigma - (3-2\eta)\iint \dfrac{1}{\mathrm{R}}\,d\sigma\right\}, \\[2mm] \Delta=-(\mathrm{T}/2\pi n)(1-2\eta)\dfrac{d}{dz}\iint \dfrac{1}{\mathrm{R}}\,d\sigma. \end{cases} \quad . \quad (37)$$

The integrals extend over the whole of the stressed area, *i. e.* over the area enclosed by the circle $r=a'$.

The above integrals in their general form are somewhat unmanageable. At a point whose distance from the stressed area is such that a'/R is always small, closely approximate values for the displacements are

$$\begin{cases} u=-\dfrac{\mathrm{T}a'^2 r}{4n\mathrm{R}}\left\{\dfrac{z}{\mathrm{R}^2}-\dfrac{(1-2\eta)}{\mathrm{R}+z}\right\}, \\[3mm] w=-\dfrac{\mathrm{T}a'^2}{4n\mathrm{R}}\left\{\left(\dfrac{z}{\mathrm{R}}\right)^2+2(1-\eta)\right\} \end{cases} ; \quad . \quad (38)$$

where R now denotes $\sqrt{r^2+z^2}$.

Along a given radius the above displacements vary inversely as the distance from the origin, *i. e.* from the

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* Todhunter and Pearson's 'History of Elasticity,' vol. ii. art. 1492, equations (ix.).

centre of the stressed area; consequently the strains and stresses vary inversely as the square of this distance. For instance, at a point at a distance z vertically below the centre of the stressed area, a'/z being small,

$$\left.\begin{array}{l} \widehat{rr} = \widehat{\phi\phi} = -(1-2\eta)\mathrm{T}a'^2/(4z^2), \\ \widehat{zz} = 3\mathrm{T}a'^2/(2z^2) \end{array}\right\}; \quad . \quad . \quad . \quad (39)$$

while in the plane of the loaded area, but outside it, a'/r being small,

$$\left.\begin{array}{l} -\widehat{rr} = \widehat{\phi\phi} = (1-2\eta)\mathrm{T}a'^2/2r^2, \\ \widehat{zz} = 0. \end{array}\right\} \quad . \quad . \quad . \quad (40)$$

Thus the effects diminish very rapidly as the distance from the loaded area increases, and so far as rupture is concerned interest centres in the material close to the loaded area. The determination of what happens close to the loaded area is in general rather a delicate operation, especially at points situated close to its boundary.

At any point on the axis of z

$$\mathrm{R}^2 = z^2 + r'^2,$$

where r' is the distance of the element $d\sigma$ from the centre of the loaded area, and so

$$d\mathrm{R}/dz = z/\mathrm{R}.$$

Thus from the last of equations (37)

$$\frac{2\pi n\Delta}{(1-2\eta)\mathrm{T}} = \iint \frac{z}{\mathrm{R}^3}\,d\sigma = \int_0^{a'} \frac{z \cdot 2\pi r'dr'}{(z^2+r'^2)^{\frac{3}{2}}}$$
$$= 2\pi\left\{1 - z/(a'^2+z^2)^{\frac{1}{2}}\right\}.$$

Thus in the limit when $z=0$, *i. e.* at the centre of the loaded area,

$$\Delta = \mathrm{T}(1-2\eta)/n. \quad . \quad . \quad . \quad . \quad (41)$$

Now by symmetry it is clear that $\widehat{rr}$, $\widehat{\phi\phi}$, $\widehat{zz}$ are the principal stresses at the centre, and also that

$$\widehat{rr} = \widehat{\phi\phi}.$$

Thus

$$2\widehat{rr} + \widehat{zz} = \widehat{rr} + \widehat{\phi\phi} + \widehat{zz}$$
$$= (3m-n)\Delta = 2(1+\eta)\mathrm{T}.$$

But by the surface conditions

$$\widehat{zz} = \mathrm{T}.$$

Hence

$$\left.\begin{array}{l} \widehat{rr} = (1+2\eta)\mathrm{T}/2, \\ \mathrm{S} = \widehat{zz} - \widehat{rr} = (1-2\eta)\mathrm{T}/2. \end{array}\right\} \quad . \quad . \quad . \quad (42)$$

and

As this method of obtaining the stresses is very artificial, I may add that I have deduced directly from the second of equations (37) for points in the axis of z

$$w = (\mathrm{T}/2n)[(1-2\eta)z - 2(1-\eta)(a'^2 + z^2)^{\frac{1}{2}} + z^2(a'^2 + z^2)^{-\frac{1}{2}}].$$

Thus in the limit when $z = 0$, *i. e.* at the centre of the loaded area,

$$dw/dz = \mathrm{T}(1-2\eta)/2n.$$

I have also succeeded in deducing from the first of equations (37) as limiting values when $z = 0$

$$\alpha/x = \beta/y = u/r = du/dr = \mathrm{T}(1-2\eta)/4n.$$

These values for the strains are in harmony with the values given above for the stresses.

In the case of tension or pressure over a very small area it is practically immaterial whether the bounding surface is a plane or a sphere of large radius. Thus in practice we can regard (15) and (37) as applicable to an elastic " Earth " so long as h is a very small fraction of the radius. Superposing the two stress systems, and replacing T by $g\,\rho\,h$, we have a small area $\pi a'^2$ at a depth h from the surface free from pressure, whilst other elements at the same depth experience a pressure $g\rho h$. The solution thus answers closely to the conditions existing below a bore-hole of depth h. At the base of the hole, at its centre, we find combining (15) and (42), using the notation of (15),

$$\left. \begin{aligned} \widehat{rr} &= 0, \\ \mathrm{S} = \widehat{\theta\theta} &= g\rho h\, \frac{(1-2\eta)}{1-\eta}\frac{1+\eta}{2} \end{aligned} \right\} \quad \cdot \quad \cdots \quad (43)$$

This value of S it will be noticed bears to that in (15) the ratio

$$1 + \eta : 2.$$

Applications to the Earth.

§ 14. The first question to consider is the state of matters prior to the existence of the bore-hole. The Earth is spheroidal and rotates about its axis of figure, circumstances which introduce variety into the conditions in different latitudes. Results such as (17), (18), and (19) show that an elastic sphere of the Earth's size rotating in a day must suffer stresses from the " centrifugal forces " which are very considerable even for material such as steel. The results (20) and (21) show,

however, that the stresses due to centrifugal force in a homogeneous Earth are at least in large measure neutralized by the compensating action of the spheroidal form on the gravitational stresses. In our ignorance of the distribution of density and elasticity throughout the Earth, there is necessarily uncertainty as to the extent of this compensating action. The compensation may be less exact than according to (21), but it is at least as likely to be more exact. Further, whilst the pure centrifugal stresses are very considerable at the surface, their rate of variation with the depth is small. There is thus every reason to believe that whatever may be the combined effect of rotation and ellipticity, it will not produce throughout a bore-hole only ten or twelve miles deep effects differing in any essential respect from those observed in ordinary mines. Owing to centrifugal forces there may be a slightly greater tendency for a bore-hole to collapse in a north-south than in an east-west direction. Rotation may slightly facilitate collapse; but everything points to the conclusion that its effects, whether direct or indirect, are only of secondary importance.

§ 15. A more serious source of uncertainty arises in connexion with the value (5) of the horizontal stress $\widehat{\theta\theta}$ in a homogeneous gravitating sphere. If, for example, we suppose η equal $0 \cdot 25$, which is approximately true of glass or iron, the value of $-\widehat{\theta\theta}$ is $(2/15)g\rho a$; this means a pressure answering to the weight of a column of density ρ some 500 miles high. Near the Earth's surface this enormous horizontal pressure would exist in company with only a relatively small vertical pressure; and under such conditions no known material would continue an elastic solid. There is further the observed fact that near the Earth's surface large horizontal stresses are normally at least non-existent. If they occurred, mines could not be constructed. No theory, in short, of the Earth's condition can well be entertained which is incompatible with horizontal stresses remaining small at moderate depths.

The most obvious way out of the difficulty is to assume the homogeneous "Earth" incompressible. We then, as appears from (6) and (7), have the horizontal stresses vanishing at the surface equally with the vertical stresses. To this hypothesis there is, however, the objection that no known material is incompressible, and that the materials of which the Earth's crust is composed do not, so far as is known, show any close approach to such a condition.

The hypothesis § 5, which limits the incompressibility to

the deep-seated material, is less open to criticism. At great depths, according to any view yet suggested, the material must be exposed to very severe pressure; and it is difficult to imagine material exposed to pressure of hundreds of tons weight on the square inch being anything but nearly incompressible under variations of that pressure. The hypothesis of an incompressible nucleus and compressible crust is of course at best only an approximation to the truth. A theory which treated the material as varying continuously with the depth, or as consisting of a large number of homogeneous layers, would presumably be more exact. Its complication, however, would be great; and in the absence of data the simplest consistent hypothesis seems the best.

§ 16. The above considerations indicate that in the absence of a bore-hole the stresses at depths of a few miles in reasonably homogeneous strata are probably not widely different from those appearing in (15). According to this equation the pressure on a horizontal plane equals the weight of the material between it and the surface. This pressure, in short, is the same as if the superincumbent material were liquid. The result implies that the superincumbent material is so much dead weight, and does not act to an appreciable extent as a protecting arch. In reality, some slight action of this kind is likely to exist. In not a few cases, the bending visible in rocks is suggestive of large horizontal thrusts. The condition, however, of the material when the bending occurred may have been plastic.

An idea of the probable diminution in the vertical pressure due to the horizontal thrust in superincumbent material may be derived from (5). If in this case we suppose η equal 1/4 the vertical pressure, $-\widehat{rr}$, is only 11/15 of its value when the material is incompressible; but the horizontal or arching pressures to which this reduction is due (*i. e.* the value of $-\widehat{\theta\theta}$ in (5)) if we suppose the density to be 5·5, amounts at the surface to nearly 3000 tons weight on the square inch. The natural inference is that whilst $g\rho h$ is a maximum estimate for the vertical pressure on a horizontal plane, it is unlikely to be many per cent. in error.

As regards horizontal pressure, the largest value given by (15) for $-\widehat{\theta\theta}$ is $g\rho h$, answering to $\eta = 0\cdot 5$. It follows from our previous considerations that if shallow borings show no considerable horizontal pressure, the value of this pressure at depth h is unlikely to be much in excess of $g\rho h$. On the other hand, according to (15), $-\widehat{\theta\theta}$ vanishes when $\eta = 0$, and is only $g\rho h/3$ when $\eta = \frac{1}{4}$. It is thus by no means improbable

that the horizontal pressure may be very considerably less than $g\rho h$, and so less than the vertical pressure.

Noticing that $\widehat{rr}$ in (15) is really a vertical stress, it is obvious that the stress systems (15) and (23) are of the same type. The system (23) is the more general, in so far as it does not assume any relationship between the magnitudes of the vertical and horizontal pressures, and does not make the latter necessarily vanish at the upper surface. The material in the Earth's crust thus presents the same elastic conditions as a vertical prism of similar density acted on by gravity, provided the latter be exposed to suitable horizontal pressures whose intensity depends only on the level.

§ 17. As a numerical example, let us take Mr. Martin's case, where h is twelve miles and ρ thrice the density of water. Employing these values in (15) or (23), we find for the vertical pressure at this depth

$$-\widehat{zz}=36\cdot8 \text{ tons weight per square inch.}$$

Mr. Martin himself speaks of 440 tons, which suggests the omission of 12 in some divisor. Mr. Parsons gives 40 tons as a rough approximation, which agrees substantially with the above estimate. If the prism existed as an isolated pillar, it would, neglecting atmospheric pressures, be free from horizontal pressure, and the maximum stress-difference at its base would be simply the above value of $-\widehat{zz}$, or $36\cdot8$ tons weight on the square inch. But when the prism forms part of the Earth's crust, horizontal pressures exist; and so long as they are less than $g\rho h$, the larger they are the smaller is S. If, for example, we supposed the horizontal pressure to be given by $-\widehat{\theta\theta}$ in (15) with $\eta=1/4$ we should find S reduced to $24\cdot4$ tons weight per square inch.

Comparing (15) and (23) with (29), we see that our reasoning points to the conclusion that the material at any given depth in the crust, like the material in a gravitating prism, is under the same condition as a short non-gravitating prism over whose flat ends and curved surface there act *uniform* pressures, equal respectively to the vertical and horizontal pressures actually existent at the point in the crust. Thus experiments on a solid prism might supply definite information as to the elastic state of the Earth's crust if rocks were available of the same composition as the more deep-seated material, and the temperature at which the experiments were made was sufficiently high.

Effects of a Bore-hole.

§ 18. The existence of a bore-hole may profoundly influence the conditions in its immediate neighbourhood; but clearly it will not produce an appreciable effect at distances from the hole which are a large multiple of its diameter. Over the cylindrical surface of the hole the normal stress must vanish, *i. e.* the horizontal pressure must be *nil*, supposing the hole vertical. Thus if we suppose an imaginary cylindrical surface coaxial with the bore-hole, but of relatively large diameter, the material within it will be practically in the state of the hollow cylinder of § 8 when we suppose a'/a very small. From this conclusion we must exclude the material immediately surrounding the extreme foot of the hole, whose condition will be somewhat modified. Excluding this very limited volume, we infer from (27) and (28) that the maximum stress-difference S at depth h, which was $g\rho h \sim p$ before the hole was bored, is, subsequent to the boring, the greater of the two quantities $2p$ and $g\rho h$. Here p denotes the horizontal pressure at depth h at a distance from the hole.

If p vanishes, *i. e.* if the material is in the condition of an isolated pillar, the presence of the bore-hole makes no difference.

On the hypothesis of a compressible crust on an incompressible nucleus, after the hole is formed S equals $g\rho h$ so long as η does not exceed $1/3$; thereafter it increases with η to a maximum of $2g\rho h$ when η equals $0\cdot5$. Particulars on this hypothesis are as follows :—

Value of $S/g\rho h$.

	$\eta = 0$	0·2	·25	·3	·3̈	·4	·45	·5
Before hole made	1	0·75	0·6̇	0·57	0·5	0·3̈	0·18	·00
After „ „	1	1	1	1	1	1·3̇	1·64	2·00

When a bore-hole is formed the engineer may thus anticipate over its surface a stress-difference varying from p to $2p$ according to circumstances, where p represents the hydrostatic pressure at the same depth in a liquid of the same mean density as the superincumbent material.

From the identity in type of the results in §§ 8 and 10, the probable behaviour of the tube-walls may be deduced from experiments in which a circular cylinder of the material with a coaxial hole of small bore is exposed to the combined effects of uniform pressures over its flat ends and curved surface. The former should equal the pressure in a liquid,

whose density equals that of the crust, at a depth equal to
that of the projected boring ; the latter may be as large as
the former or have any smaller value.   Under the conditions
supposed by Mr. Martin the pressures, in tons weight on the
square inch, would be

<div align="center">

36·8 on the flat ends,

from 0 to 36·8 on the curved surface.
</div>

§ 19. Near the bottom of the hole the conditions are com-
plicated.   An approximate idea of the conditions immediately
below the hole may be derived from § 13.   At the centre of
the bottom of the hole the tendency to rupture according to
(43) is actually less than before the hole was bored, unless
the material be incompressible when it vanishes in either
case.   This diminution is almost certainly confined to the
central part of the base of the hole.   When a core is hollowed
out of a cylinder exposed to uniform pressure p the stress-
difference—as shown in § 10—rises to $2p$.   Under the same
circumstances, when a core is taken out of a sphere the stress-
difference—as shown in (36)—becomes $3p/2$.   The conditions
at the edge of the base of the hole seem more likely to
approach the conditions in the sphere than those in the
cylinder.   Thus while uncertainty prevails as to the condi-
tions in the base—a good deal depending in practice on the
borer—this seems on the whole not so weak a spot as the
walls of the boring a few diameters of the bore above the
base.

The above considerations unite in indicating that the
largest value to be anticipated for the maximum stress-
difference anywhere over the surface of the bore-hole is $2p$,
where p is the hydrostatic pressure that would exist at the
bottom of the hole if filled with a liquid whose density equals
the mean density of the core extracted.

§ 20. There are, however, two contingencies to be borne in
mind.   Whilst large horizontal pressures of the order of
hundreds of tons weight on the square inch may be regarded
as practically impossible near the surface, it is quite con-
ceivable that smaller but still large horizontal pressures may
exist in strata which differ notably from adjacent material.
If a bore-hole should pierce such a stratum a nipping action
may well ensue.   Again, whilst no one is likely to select the
immediate neighbourhood of an active volcano as a site for
a boring, it might be well in any case to be prepared for the
possibility of piercing material in so unstable a condition
that sudden relief of pressure may lead to semi-volcanic
action.

LXXVII. *The Parallel Motion of* Sarrut *and some Allied Mechanisms. By* G. T. Bennett, *Emmanuel College, Cambridge**.

§ 1. ANY student of mechanism who has gained his knowledge of the subject from the current text-books might be pardoned for supposing that the cardinal problem of obtaining exact rectilinear motion by the use of linkwork was first solved in the year 1864 by Peaucellier. Most of the works which deal with the subject, after describing the approximate " parallel motions " of Watt and others, place first in their account of the exact rectilinear motions now known a discussion of the inverting mechanism of Peaucellier. The *Encyclopædia Britannica* (vol. xxii. p. 512), more explicit, and expressing certainly the common view, states† that " it was for long believed that the production of an exact straight-line motion was impossible until the problem was solved by the invention of the Peaucellier cell." When this discovery had become known, the subject of linkworks attracted much fruitful attention ; and other solutions of the problem, some requiring a smaller number of pieces, were soon found. These mechanisms, however, gave rectilinear motion only to a single point : (or, more strictly, to all points of a line, in one of the pieces, parallel to the hinge-lines ; or else to all points of a hinge-line connecting two pieces). There remained yet to be solved the more complete problem, embracing and including the other, of giving rectilinear translational motion to a whole piece. This also has since been contrived in several ways ; all, however, more or less complicated. It seems therefore very desirable to point out what appears to have escaped notice or to have been entirely forgotten : namely, that this last problem itself was solved exceedingly simply by Sarrut in the year 1853, eleven years before Peaucellier's announcement of his inverting cell.

§ 2. Sarrut's description of his mechanism appears in the *Comptes Rendus* of the Académie des Sciences for 1853 (vol. xxxvi. p. 1036) ; and later (p. 1125) is given the report of Poncelet upon Sarrut's mechanism. The Academy agreed " de lui accorder son entière approbation et de decider que le court mémoire qui en contient l'explication sera inséré dans le Recueil des Savants étrangers, avec la description du modèle que l'auteur y a joint." But it seems never to have appeared.

The mechanism may be thus briefly described :—A moving

† Quoting Kempe's " How to draw a straight line."

piece A is to have rectilinear motion, say vertically up and down, relatively to a frame or base B (figs. 1, 2, 3).   To

Fig. 1.
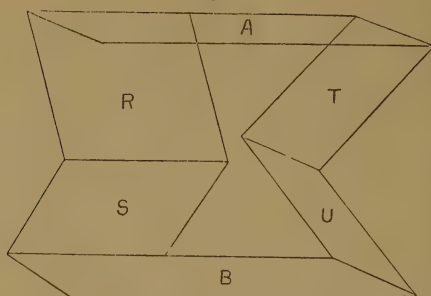
A perspective outline of fig. 2.

Fig. 2.
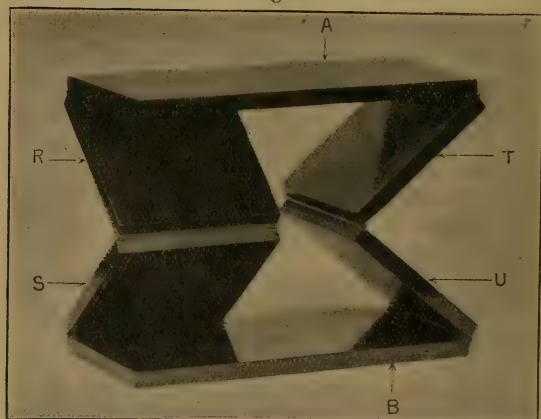
A model of one form of the Sarrut parallel motion.

effect this, four connecting pieces R S T U are used.   The pieces A R S B are consecutively hinged by three parallel horizontal hinges * ; and, again, the pieces A T U B are consecutively hinged by three parallel horizontal hinges ; the two directions being different.   Connected thus with B the piece A has a movement which is rectilinear and vertical.

* By the term " hinge," here and throughout, is meant, quite generally, any form of connexion which ensures a permanent axis of pure rotation for the relative movement of two bodies.   " Axis," " turning pair," " cylinder pair," " pivot," " pin-joint," are other terms to be found in use with the same theoretic meaning.

The whole forms a closed kinematic chain of six pieces. Seven being the normal number of pieces in a closed kinematic chain necessary to ensure freedom of one degree, it

Fig. 3.
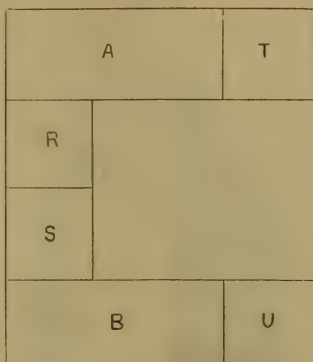
The hinge TU of fig. 1 being disconnected, the mechanism flattens out as shown here in plan.

follows that the mechanism of Sarrut belongs to a special class of mechanisms, with only six pieces, possessing singularly and exceptionally one degree of freedom.

Fig. 4.
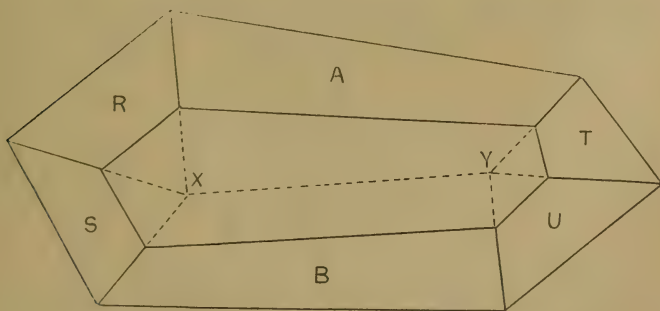
The general mechanism of class (*a*). The hinges of A are non-intersecting, as are also the hinges of B.

§ 3. It is instructive to compare the mechanism of Sarrut with a certain generalized form to which it stands in the relation of a degenerate case. Retaining the same number of pieces and hinges, let it be supposed (figs 4, 5, 6) that

the hinges consecutively connecting A R S B are concurrent in a finite point X instead of being parallel; and, further,

Fig. 5.
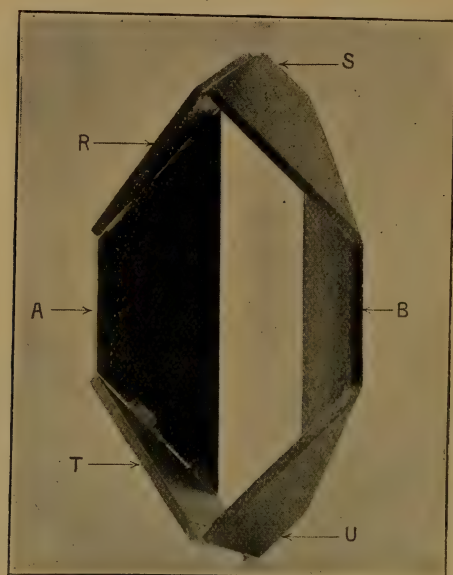
A model, of a special form, of the mechanism (*a*). The two vertical plates have relative rotation about a virtual hinge.

Fig. 6.

The hinge TU of the model of fig. 5 being disconnected, the mechanism flattens out as shown here in plan.

that the hinges consecutively connecting A T U B are concurrent in a finite point Y. The piece A has then, relatively to B, a motion of pure rotation about the line X Y. Like the Sarrut mechanism, this is a closed kinematic chain of six pieces singular in possessing one degree of freedom.

Another view of this last mechanism is useful. It may

be regarded as composed of two spherical* mechanisms, each of four pieces, placed in tandem with different centres. One consists of a closed chain of four pieces ARSB consecutively hinged along four lines concurrent in X ; the other similarly of four pieces ATUB with hinges concurrent in Y; the pieces AB and also their hinge-line XY being common to the two mechanisms. But the singular six-piece mechanism described above arises only after the removal or omission of the hinge XY, which is superfluous or redundant. The line XY is then no longer an actual and mechanical hinge, but yet remains kinematically as a *virtual* hinge-line, in respect of the relative movement of A and B.

This type of mechanism shall be referred to as (*a*).

§ 4. From the type (*a*) two special forms are derivable by taking one or both of the points XY at infinity. If one only, say Y, is at infinity, a case (*b*) is obtained. The three hinges connecting ARSB meet still in a finite point X ; but the three hinges connecting ATUB are parallel ; and the motion of A relative to B is a movement of pure rotation about a virtual hinge-line parallel to the hinges last named and passing through X (figs. 7, 8). This mechanism may be regarded as composed of a plane mechanism and a spherical mechanism in tandem, with the common hinge of the two common pieces omitted.

If both X and Y are at infinity a case (*c*) arises. Each set of three hinges forms a parallel system ; the movement of A relative to B is a rotation about the line at infinity which meets all the hinges, a rectilinear motion therefore in the direction perpendicular to all the hinges. It is the type of mechanism described by Sarrut. It may be regarded as composed of two plane mechanisms in tandem ; these mechanisms having parallel sets of hinges in two different directions and having the virtual common hinge of the two common pieces at infinity. In technical terms the two mechanisms would be called crossed slider-crank chains,

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* The term is perhaps not well enough established to pass without explanation. Mechanisms composed of pieces connected by hinges which are all parallel, being sufficiently represented by any plane section perpendicular to the hinges, are loosely and commonly spoken of as " plane " mechanisms. And by analogy mechanisms of which all the hinge-lines are concurrent in one point, being sufficiently represented by the section in which they are cut by any sphere centred at that point, may be called " spherical " mechanisms. Custom, however, is variable. Reuleaux uses the terms " cylindric " and " conic "; the relative motion of any two pieces being representable by the rolling of cylinder on cylinder in the one case and cone upon cone in the other.

having the two sliding pieces in common and the slide-
connexion itself entirely omitted.

Fig. 7.
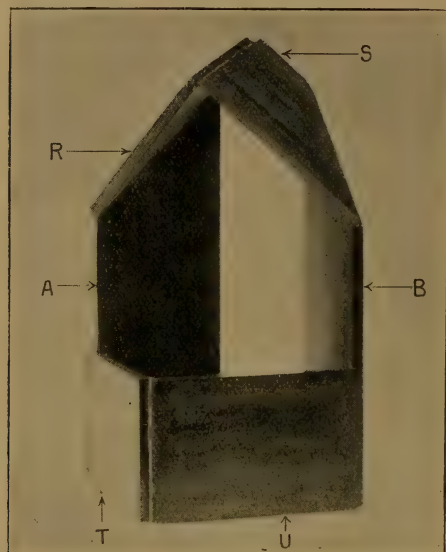
A model, of a special form, of the mechanism (*b*).   The four upper
pieces all move as in fig. 5.

Fig. 8.
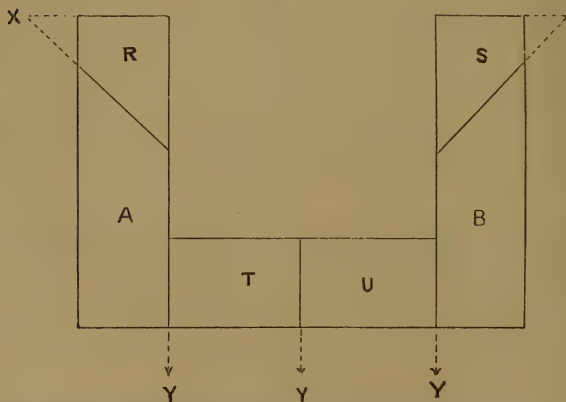
The hinge RS of the model of fig. 7 being disconnected, the
mechanism flattens out as shown here in plan.

§ 5. After deriving the mechanisms (*b*) and (*c*) from the class (*a*), I was interested to try to discover any references to them. A brief account of what I have been able to find out may be set down here ; and may perhaps be the means of eliciting further information.

Among the collection of models left by Professor Robert Willis occurs a specimen of class (*a*). It departs slightly from generality (see fig. 4) in that the two hinges connecting the pieces here called R A T intersect each other : a non-essential peculiarity. There is also a specimen which would belong to class (*b*) if the hinge X Y were made virtual instead of actual. Of class (*c*) there appears to be no specimen. It was in searching*, however, through the voluminous MS. lecture-notes of the Professor, that, though finding no mention of either (*a*) or (*b*), I first came upon a reference to the article of Sarrut ; together with references to a quotation of his result in two French text-books † of slightly later date, and a rough sketch, marked with dimensions, of a simple form of the mechanism. It seems quite likely that a model was made, but that it was subsequently dislocated and the parts adapted to other uses ; a frequent custom (so Mr. J. Willis Clark tells me) with the Professor. In any case the parallel motion failed to become known ; and twice at least (not to count the present article) it has been reinvented and republished apparently with but little more success. In 1880 a patent (Specification 5492, Dec. 30) was taken out by H. M. Brunel ; and in 1891 it was again invented by Professor Archibald Barr (Proc. Phil. Soc. Glasgow, March 18, 1891)‡. Yet to this day it appears to remain practically unknown to mechanicians. Its obvious merits and its long neglect of fifty years seem therefore to be worth insisting upon with some emphasis. Compared indeed with any of its later rivals in the presentment of rectilinear motion, it has obviously two points of great superiority. In the first place, there is an entire absence of any special restrictive metrical relations to be satisfied by the dimensions of the parts as a condition of the movement. In the second place, the connecting pieces are but four in number, with six hinges ; whereas rectilinear motion of a piece seems not to have been obtained otherwise §

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* By kind permission obtained at the Engineering Laboratory, Cambridge.

† Laboulaye, *Traité de Cinématique*, 1854, p. 634 ; Girault, *Transformation du Mouvement*, 1858, p. 267.

‡ I have to thank Professor F. G. Baily for referring me to Professor Barr : the latter, in his turn, had heard of Brunel's patent from Sir Frederick Bramwell, to whom Brunel had once shown it.

§ By Kempe, Hart, and Darboux.

with less than eight pieces of connexion and eight or more hinges (some of them multiple) ; or six pieces of connexion and ten hinges.   The simplest rectilinear point-movement uses five moving pieces and seven hinges.

§ 6. It may be not amiss to mention certain very familiar objects in which the mechanisms here discussed are fore-shadowed and, in a sense, are even potentially contained. The bellows used in blowing organs * have, since about the year 1419 †, been made with each of the four collapsible sides or walls composed of two wooden plates (ribs so-called), hinged together along the horizontal median line, as an improvement on mere folds of leather.   A horizontal lid and base, hinged to the ribs, with suitable gussets of leather at the eight corners, complete the apparatus.   The step, which took so long in making, from this mechanism to the parallel motion of Sarrut, consisted in removing two of the adjacent walls, and realizing that the rectilinear up-and-down motion of the lid was then not only still a possible movement but the only movement possible.   (Removal of *opposite* walls would of course at once leave the mechanism with *two* degrees of freedom.)   Again, too, in the folding flaps which connect the edges of the two boards of some portfolios we may see, in an approximate form, the details of the mechanism (a). Some purses, pocket-books, and card-cases show the same arrangement.   In all such instances, however, the very crucial and distinctive omission of the actual and principal hinge remains to be made before the singular mechanism (a) is realized.

§ 7. Some of the foregoing explanations may appear need-lessly explicit ; but the difficulties of picturing and describing adequately mechanisms of a specifically three-dimensional character (a large class too little studied so far) may be sufficient excuse.   Mistakes and misconceptions arise only too easily in this region.   Sarrut himself, for example, appears to have thought that any closed chain of six pieces would be normally free ; and Brunel, desiring to convert his parallel piece-motion into a rectilinear point-motion, figures and describes a spurious mechanism which is nothing but a stiff framework.

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* Willis has the suggestion "organ-bellows" written against his sketch.

† Grove's Dictionary of Music : Article " Bellows."

LXXVIII. *On the Depression due to a Load at the Centre of an Elastic Chain tightly stretched between Two Points in the same Horizontal Plane.* By CHARLES H. LEES, *D.Sc.**

THE ease and convenience of determining Young's Modulus for a material by stretching a wire between two supports in the same horizontal plane, and measuring the depression at its centre produced by a load suspended there †, has led me to enquire more carefully into the theory of the method, with a view to defining the limits within which the method is applicable. In the theoretical work known to me, the weight and flexural rigidity of the wire are neglected, so that it may be taken as two straight lines extending from the point at which the load is applied to the two supports.

When the weight of the wire is taken into account but its flexural rigidity still neglected, the case becomes that of a chain, the links of which increase in length under longitudinal stress by stretching of their longitudinal parts without bending of their transverse parts. The curve of equilibrium of the chain on each side of the load is then a catenary, the lowest point of which is beyond the point of application of the load. When the load is withdrawn the whole chain becomes a catenary with its central lowest point at a distance below its ends which depends on the weight of the chain and the stretching force applied.

Let AB (fig. 1) be the two points of support at a distance $2l$ apart, C_1 the centre of the chain, M the mass suspended at it, D the lowest point of the catenary of which BC_1 forms an arc. Let the axis EO of the catenary be at a distance c below D, and let $EO = e$.

The equation to the catenary in terms of rectangular coordinates with O as origin is

$$y = c \cosh \frac{e + x}{c}.$$

Hence if y_1 is the ordinate of C_1 and $y_1 + b$ that of B, we have

$$b = c\left(\cosh\frac{e+l}{c} - \cosh\frac{e}{c}\right)$$

$$= 2c\sinh\frac{l}{2c}\sinh\frac{2e+l}{2c}. \quad . \quad . \quad . \quad . \quad (1)$$

Fig. 1.
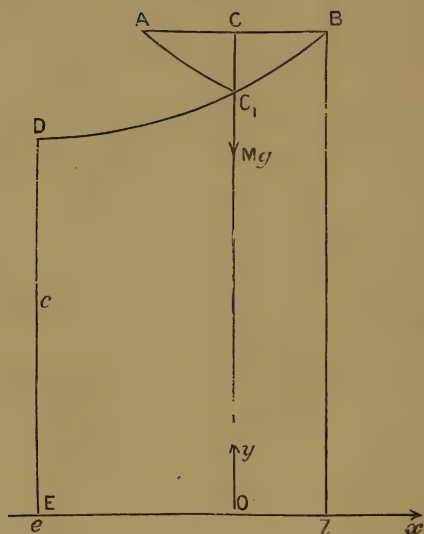
If s is the length of the curve from C_1 to B

$$s = c\left(\sinh\frac{e+l}{c} - \sinh\frac{e}{c}\right) = 2c\sinh\frac{l}{2c}\cosh\frac{2e+l}{2c} \quad . \quad (2)$$

If θ is the angle of inclination of the catenary at any point to the x axis

$$\tan\theta = \frac{dy}{dx} = \sinh\frac{e+x}{c}.$$

Also if T is the stretching force at any point

$$T = \sigma gy = \sigma gc\cosh\frac{e+x}{c}, \quad . \quad . \quad . \quad . \quad (3)$$

where σ is the mass of the chain per unit length.

When the chain is unloaded the centre of the catenary is

at C_0, a point at a distance b_0 below C, and

$$b_0 = c_0 \left(\cosh \frac{l}{c_0} - 1 \right) \quad . \quad . \quad . \quad . \quad . \quad (1')$$

$$s_0 = c_0 \sinh \frac{l}{c_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2')$$

and $\qquad T_0 = \sigma g c_0 \cosh \frac{x}{c_0}. \quad . \quad . \quad . \quad . \quad . \quad (3')$

Since the object of the experimental method is to secure a comparatively large depression CC_1 for a small increase in the length of AC_1B, the initial stretching force T_0 should be such as to make the chain nearly straight. The maximum value of T must be within the elastic limit of the material of the chain. In most cases this will mean that the extension cannot exceed 1/1000 of the original length. The value of CC_1 cannot therefore exceed $1/\sqrt{1000} = 1/32$ of CB, *i. e.* the upper limit to $\tan \theta$ will be 1/32.

Hence the upper limit of $\sinh \frac{e}{c}$ and therefore of $\frac{e}{c} = 1/32$.

Since the difference in height of C_1 and B is small we may take the stretching force T to be constant throughout the chain, and equal to its value T' at C', a point half way between B and C, *i. e.* for which $x = l/2$.

Hence $\qquad T = T' = \sigma g c \cosh \dfrac{2e + l}{2c}.$

Since there is equilibrium at C' we must have
$$2T' \sin \theta' = (M + \sigma l)g,$$
where θ' is the inclination of the curve at C' to the axis of X.

But $\qquad \tan \theta' = \sinh \dfrac{2e + l}{2c}.$

Hence $\qquad \sin \theta' = \tanh \dfrac{2e + l}{2c}.$

Therefore $\qquad 2T' \tanh \dfrac{2e + l}{2c} = (M + \sigma l)g, \quad . \quad . \quad . \quad . \quad (4)$

and $\qquad c \sinh \dfrac{2e + l}{2c} = \dfrac{M + \sigma l}{2\sigma}. \quad . \quad . \quad . \quad . \quad (5)$

When the chain is unloaded these equations become

$$2T'_0 \tanh \frac{l}{2c} = \sigma l g, \quad . \quad . \quad . \quad . \quad (4')$$

and $\qquad c_0 \sinh \dfrac{l}{2c} = \dfrac{l}{2}. \quad . \quad . \quad . \quad . \quad . \quad (5')$

If ϵ is Young's Modulus for the material of the chain, a the total area of the transverse section of the longitudinal part of a link, T_0' the initial stretching force, T' that when a mass M is suspended from the centre of the chain,

$$\frac{s}{1+\dfrac{T'}{a\epsilon}} = \frac{s_0}{1+\dfrac{T_0'}{a\epsilon}},$$

where s_0 and s are the half-lengths of the chain in the two cases.

Since $\dfrac{T'}{a\epsilon}$ never exceeds 1/1000 we may write this equation

$$s = s_0\left(1+\frac{T'-T_0'}{a\epsilon}\right).$$

Or

$$T'-T_0' = a\epsilon\left(\frac{s}{s_0}-1\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence substituting from equations (4), (4′), (2), and (2′), we have

$$\frac{(M+\sigma l)g}{2}\coth\frac{2e+l}{2c} - \frac{\sigma l g}{2}\coth\frac{l}{2c_0} = a\epsilon\left\{\frac{c\sinh\dfrac{l}{2c}\cosh\dfrac{2e+l}{2c}}{c_0\sinh\dfrac{l}{2c_0}\cosh\dfrac{l}{2c_0}}-1\right\}.$$

Or, expanding the hyperbolic functions,

$$\frac{(M+\sigma l)g}{2}\frac{2c}{2e+l}\left(1+\frac{1}{3}\left(\frac{2e+l}{2c}\right)^2\ldots\right) - \frac{\sigma l g}{2}\cdot\frac{2c_0}{l}\left(1+\frac{1}{3}\left(\frac{l}{2c_0}\right)^2\ldots\right)$$

$$= a\epsilon\left\{\frac{1+\frac{1}{2}\left(\dfrac{2e+l}{2c}\right)^2\ldots}{1+\frac{1}{2}\left(\dfrac{l}{2c_0}\right)^2\ldots}-1\right\}.$$

Hence, since $\left(\dfrac{2e+l}{2e}\right)^2$ never exceeds 1/1000,

$$\frac{(M+\sigma l)g}{2}\frac{2c}{2e+l} - \sigma g c_0 = \frac{a\epsilon}{2}\left\{\left(\frac{2e+l}{2c}\right)^2-\left(\frac{l}{2c_0}\right)^2\right\}. \quad . \quad (7)$$

Expanding similarly equations (5) and (1), we have

$$c\frac{2e+l}{2c}\left(1+\frac{1}{3}\left(\frac{2e+l}{2c}\right)^2\right) = \frac{M+\sigma l}{2\sigma}, \quad \text{i. e.} \quad \frac{2e+l}{2}\left(1+\frac{1}{3}\left(\frac{2e+l}{2c}\right)^2\right) = \frac{M\, r}{2}$$

and $$2c\cdot\frac{l}{2c}\left(1+\frac{1}{3}\left(\frac{l}{2c}\right)^2\ldots\right)\cdot\frac{2e+l}{2c}\left(1+\frac{1}{3}\left(\frac{2e+l}{2c}\right)^2\ldots\right) = b,$$

i. e. $$\frac{2e+l}{2c}\left(1+\frac{1}{3}\frac{(2e+l)^2+l^2}{4c^2}+\ldots\right) = \frac{b}{l}.$$

Hence so long as $\frac{b}{l}$ does not exceed $1/30$ we may write

$$e = \frac{M}{2\sigma},$$

and

$$\frac{2e+l}{2c} = \frac{b}{l},$$

i. e.

$$\frac{e}{c} = \frac{b}{l} \frac{2e}{2e+l} = \frac{b}{l} \cdot \frac{\dfrac{M}{\sigma}}{\dfrac{M}{\sigma}+l} = \frac{b}{l} \cdot \frac{M}{M+\sigma l},$$

From equation (1′) we have similarly

$$c_0 = \frac{l^2}{2b_0}.$$

Hence equation (7) becomes

$$\frac{(M+\sigma l)g}{2} \frac{l}{b} - \sigma g \frac{l^2}{2b_0} = \frac{a\epsilon}{2} \left\{ \frac{b^2}{l^2} - \frac{b_0^2}{l^2} \right\}.$$

Let $\sigma l = m$ half the mass of the chain, then

$$\frac{(M+m)}{b} - \frac{m}{b_0} = \frac{a\epsilon}{gl^3}(b^2 - b_0^2), \quad . \quad . \quad . \quad . \quad (8)$$

a cubic equation to determine the depression b if M, m, a, ϵ, l, and b_0 are known. From it we have :—

$$\epsilon = \frac{gl^3}{a} \frac{\dfrac{M+m}{b} - \dfrac{m}{b_0}}{b^2 - b_0^2} . \quad . \quad . \quad . \quad . \quad (9)$$

Or if M_1 and M_2 are any two masses suspended, b_1, b_2, the depressions produced

$$\epsilon = \frac{gl^3}{a} \frac{\dfrac{M_1+m}{b_1} - \dfrac{M_2+m}{b_2}}{b_1^2 - b_2^2} , \quad . \quad . \quad . \quad (10)$$

an equation to determine ϵ from observation of b.

From equation (10) it is evident that the simple theory which considers the chain to be without mass will give values for ϵ correct to about 1 part in 300 for depressions which do not exceed $\frac{1}{60}$ of the length of the chain, if the masses actually suspended from the centre of the chain be supposed increased by half the mass of the chain itself, and are not too nearly alike to make the difference between the values of $\frac{M+m}{b}$ less than one-third that of either.

The distances b in the above equation are measured vertically from the horizontal straight line joining the points of support. The position of this zero point on the scale on which the depressions are measured is best determined by tilting the frame holding the chain or wire till the chain or wire is vertical, clamping it in position under a tension sufficient to keep it straight, then taking the measurements while the chain or wire is still vertical. If the clamping and measurement cannot be done with the wire in a vertical position, the zero may be determined with the chain or wire, the frame and the scale on which the measurements are taken all horizontal, so that the plane of bending of the wire is at right angles to the line of measurement.

In the case of the thin wires used by Mr. Grime and myself[*] the mass of the wire was neglected. If taken into account it makes a difference in the final value of ϵ which in most cases is less than 1 per cent. The experiment with the thickest wire used, no. 27 S.W.G. copper, and the lowest initial tension $T_0 = 380$ g. [†] is recalculated below, the zero from which the depressions are measured being calculated from the value of the initial tension T_0.

Copper wire, no. 27 S.W.G., half length 26·50 cm.
Mean diameter ·0412 cm.   Area of cross-section ·00133 sq. cm.
Density of copper 8·95.   Mass per cm. $\sigma = $ ·0119.
Mass of half the wire $m = $ ·315 grams.

$$c_0 = \frac{T_0}{\sigma g} = 31930 \text{ cms.}$$

$$b_0 = c_0 \left(\cosh \frac{l}{c_0} - 1 \right) = \frac{l^2}{2c_0} = \text{·0110 cm.}$$

M. grs.	M + m. grs.	b cms.	$\frac{M+m}{b}$	$\frac{M_1+m}{b_1} - \frac{M_2+m}{b_2}$	b^2.	$b_1^2 - b_2^2$.	$\frac{\frac{M_1+m}{b_1} - \frac{M_2+m}{b_2}}{b_1^2 - b_2^2}$	ϵ.
0	·315	·0110	28·64	......	·0001			
15	15·32	·3740	40·96	21·64	·1399	·2534	85·4	$1·17 \times 10^{12}$
25	25·32	·5035	50·28	18·05	·2535	·2183	82·7	1·13 ,,
35	35·32	·5985	59·01	......	·3582			

Mean ϵ ... $1·15 \times 10^{12}$.

The result obtained by neglecting the mass of the wire was $1·17 \times 10^{12}$. In this case the more accurate gives a result 2 per cent. less than that obtained by the simpler theory.

[*] Phil. Mag. ix. p. 258 (1905).
[†] *Ibid.* p. 264.

LXXIX. *Notices respecting New Books.*

Physikalisches Praktikum. Von EILHARD WIEDEMANN und HERMANN EBERT. Fünfte Verbesserte und Vermehrte Auflage. Mit 366 eingedruckten Abbildungen. Braunschweig : F. Vieweg und Sohn. 1904. Pp. xxx+590.

THE fifth edition of this well-known text-book differs but little from its predecessor. Among the additions, we may note the determination of the critical temperature, the measurement of specific heats by means of the ice-calorimeter, the determination of the calorific value of a fuel, the efficiency test of a shunt-wound motor and dynamo, and the determination of their characteristic curves. Very few elementary text-books on practical physics can rival in completeness and clearness the excellent laboratory guide provided by Messrs. Wiedemann and Ebert, and we can heartily commend it to teachers and students alike.

Elektrizität und Materie. Von Dr. J. J. THOMSON, *F.R.S., Professor der Experimentalphysik an der Universität in Cambridge.* Autorisierte Übersetzung von G. SIEBERT. Mit 19 eingedruckten Abbildungen. Braunschweig : F. Vieweg und Sohn. 1904. Pp. viii+100.

THE latest addition to the timely series of monographs at present being issued by the well-known publishing firm of Messrs. Vieweg, and entitled ' Die Wissenschaft,' is a translation of Prof. J. J. Thomson's well-known Silliman Lectures on " Electricity and Matter." The absorbing interest of the bold and profoundly original speculations contained in this volume is sure to secure for it a wide and appreciative circle of German readers. The translation has been carried out with great care, and is in every way a faithful rendering of the original.

LXXX. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 620.]

March 8th, 1905.—J. E. Marr, Sc.D., F.R.S., President, in the Chair.

THE following communications were read :—

1. 'Observations on some of the Loxonematidæ, with Descriptions of two New Species.' By Miss Jane Donald.

2. ' On some Gasteropoda from the Silurian Rocks of Llangadock (Caermarthenshire).' By Miss Jane Donald.

March 22nd.—J. E. Marr, Sc.D., F.R.S., President,
in the Chair.

The following communications were read :—

1. ' An Experiment in Mountain-Building : Part II.' By the
Right Hon. the Lord Avebury, P.C., F.R.S., P.S.A., F.G.S.

In this paper some experiments are described, which were
conducted by an apparatus by means of which pressures could
be applied in two directions at right angles to one another, a
space of 2 feet square being reduced to one 22 inches square.
In the first series, plastic materials such as cloth and thin oil-
cloth were used, with layers of sand between them. Two main
folds crossing at right angles were formed, the upper one shifted over
the lower. The use of two layers of linoleum produced a different
type of folding, and the lower layers of the linoleum were broken
along the principal ridges. In the second series, a layer of plaster
was introduced : this was found to be fractured, tilted up into a
' writing-desk ' form, and forced irregularly into the sandy layers.
Overthrusts were thus produced, so that in some cases a boring
would have passed through two or even four layers of the rigid
substance. In other cases, the edges of the primary fracture broke
off more or less regularly, and the detached pieces were pushed up,
assuming gradually a very steep angle. The remainder of the edges
of the plate of plaster, having now room, were able to approach
each other. Pliable material above the plaster was thrown into
one or a few extensive folds, while that beneath assumed a greater
number of small folds.

2. ' The Rhætic Rocks of Monmouthshire.' By Linsdall
Richardson, F.G.S.

The Rhætic rocks occur only in the neighbourhood of Newport ;
and the present paper describes three new sections and four new
exposures. They are the following :—Goldcliff, Bishton, Llan-
martin, Llanwern, Milton, Bishpool, and Lis-Werry. Measured
sections are given at each locality, that at Goldcliff being especially
full in the middle portion of the series, because, when visited by
the author, it was exposed in consequence of a breach in the
sea-wall. The plane of separation between the ' Tea-Green Marls '
(Keuper) and the Black Shales of the Rhætic is very definite, and is
not infrequently accompanied by an inch or two of conglomerate.
The Rhætic ocean appears to have spread with comparative rapidity
over the flats of the ' Tea-Green Marl ' in this neighbourhood.
The most complete sequence from Keuper to Rhætic is around
Cardiff, where the ' Sully Beds ' (Etheridge's ' Grey Marls ' : not the
'Tea-Green Marls,' which belong to the Keuper) constitute transition-
beds, formed where deposition proceeded continuously. Farther
north the 'Black Shales' overstep on to lower and lower portions
of the ' Tea-Green Marls.' A non-sequence appears to occur at
the base of the Paper-Shales at Goldcliff, where the upper surface
of the Cotham-Marble-equivalent is conspicuously waterworn.

INDEX to VOL. IX.

END OF THE NINTH VOLUME.